Second-Order-Optimal Filter on Lie Groups for Planar Rigid Bodies

Damiano Rigo, Chiara Segala, Nicola Sansonetto, and Riccardo Muradore, *Member, IEEE*

***Abstract*—Attitude estimation is a core problem in many mobile robotic systems, such as unmanned aerial and ground vehicles. The configuration space of these systems is properly modeled by exploiting the theory of Lie groups. In this article, we propose a second-order-optimal minimum-energy filter on the matrix Lie group TSE(2), the tangent bundle of the special Euclidean group SE(2), where the optimality is with respect to a cost function in the unknown input and output error measurements. In this article, the measurement equation consists of a global positioning system-like device with two antennas attached to the planar rigid body and an inertial measurement unit-like device. Even though the mathematics is quite complicated, the accuracy of the filter justifies this approach. Simulations show the effectiveness of the proposed filter.**

***Index Terms*—Geometric modeling, nonlinear filters, nonlinear dynamical systems, optimization.**

I. Introduction

Attitude estimation is a key problem in aerospace and robotic applications since in many cases the sensing system does not provide direct measurements about the pose (and its rate) of mechanical systems. In other situations, measurements are noisy and require a preprocessing to filter out disturbances and biases. The design of pose (position and orientation) estimators for robotic systems is of paramount importance to guarantee effective regulation and tracking [1]. Moreover, the pose of the robot is crucial for unmanned aerial vehicle (UAV) and unmanned ground vehicle (UGV) that exploit simultaneous localization and mapping algorithms for computing their position and for planning their trajectory [2].

In the last decades, many linear and nonlinear, deterministic and stochastic observers have been proposed in the literature. The most famous approach is based on Kalman filtering [3] and its many extensions. The recursive formulation uses measurements over time, and produces more accurate estimates of unknown variables by computing at runtime the joint probability distribution. The Kalman filter is optimal only for linear and Gaussian systems, such assumption is usually too strong in UAV and UGV, where the model is nonlinear and where nonholonomic constraints may arise and must be taken into account. The unscented Kalman filter [4] and the particle filter [5] overcome the Gaussian assumption by approximating the probability density of the state function with a certain amount of sampling points around the mean value and updating their values according to the past state and current measurements. Nonlinear maps can then be considered to get more accurate estimations of the mean and covariance of the state vector.

Other approaches exploit the theory of Lie groups to provide the proper mathematical structure for the attitude of a mechanical system [6], such as the special orthogonal groups $\mathrm{SO}(3)$ for modeling the orientation, and the special Euclidean groups $\mathrm{SE}(2)$ and $\mathrm{SE}(3)$ for modeling the pose [1]. Recent results on the design of dynamic filters on Lie groups can be found in [7]–[9], where Saccon *et al.* [7]–[9] proposed a *second-order-optimal minimum-energy filter*. Unlike the stochastic approach, both uncertainty and noise are considered as unknown deterministic signals, and the optimal filter is obtained by minimizing the square of the estimation error (energy). The filters are based on the results in [10], where a methodology of generating progressive realizable approximations of a minimum-energy functional was proposed. The solution is obtained by differentiating the boundary conditions of the associated optimal control problem. The main theoretical result in [9] is given for a generic Lie group, and the main theorem's conditions are made explicit only for designing an optimal filter on $\mathrm{SO}(3)$. It is called *second-order*-optimal in the sense that it is a truncation of the exact solution that would be an infinite dimensional system. The filter takes the form of a gradient observer coupled with a kind of Riccati differential equation that updates its gain (similarly to the standard Kalman filter).

Their theoretical result is our starting point. We apply the general formulation to (mechanical) systems modeled on $\mathrm{TSE}(2)$, the tangent bundle of the special Euclidean group $\mathrm{SE}(2)$. $\mathrm{TSE}(2)$ is the proper geometric setting for the state of many important systems moving on a plane and allows to jointly estimate pose and linear/angular velocities (see e.g., [6] and [11]).

The main contribution of this article is to derive a second-order-optimal minimum-energy filter for $\mathrm{TSE}(2)$ and to provide all the mathematics for the many technical operations needed to compute it. The measurement equation consists of the position of two global positioning system (GPS)-like antennas and the linear/angular velocities from an inertial measurement unit (IMU)-like device.

The rest of this article is organized as follows. In Section II, we introduce the notation used in this article and some important maps between Lie groups and Lie algebras. The $\mathrm{TSE}(2)$ structure is described in Section III. In Section IV, we formulate the optimal problem, whereas in Section V, we provide our main contribution. Section VI shows some numerical experiments. Finally, Section VII concludes this article together with the future works.

II. Notation

We assume that the reader is familiar with the theory of mechanical control systems, differential geometry, and Lie groups (see

Manuscript received 6 August 2021; revised 14 February 2022; accepted 19 May 2022. Date of publication 3 June 2022; date of current version 30 August 2022. The work of Nicola Sansonetto was supported by Progetto Giovani ricercatori-GNFM-INDAM. This work was supported by the Italian Ministry of Education, University and Research (MIUR), through Project Dipartimenti di Eccellenza 2018-2022. Recommended by Associate Editor Y. L. Gorrec. *(Corresponding author: Damiano Rigo.)*

Damiano Rigo, Nicola Sansonetto, and Riccardo Muradore are with the Department of Computer Science, University of Verona, 37134 Verona, Italy (e-mail: damiano.rigo@univr.it; nicola.sansonetto@gmail.com; riccardo.muradore@univr.it).

Chiara Segala is with the Department of Mathematics, RWTH Aachen University, 52062 Aachen, Germany (e-mail: segala@igpm.rwth-aachen.de).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TAC.2022.3179988.

Digital Object Identifier 10.1109/TAC.2022.3179988

e.g., [6], [12], and [13]). Here, we sum up the notation used throughout this article.

G	A connected Lie group.
g	Element of G.
$\mathfrak{g}$	Lie algebra associated with G.
$[\cdot,\cdot]$	Lie bracket of $\mathfrak{g}$.
$\mathfrak{g}^*$	Dual of the Lie algebra $\mathfrak{g}$.
$L_g : G \to G$	Left translation $L_g h = gh$.
$T_h L_g$	Tangent map of L_g at $h \in G$.
gX	Shorthand for $T_e L_g(X) \in T_g G$.
$\langle \cdot,\cdot \rangle$	Duality paring $\langle \mu, X\rangle = \mu(X)$.
V	Finite-dimensional vector space.
$f : G \to V$	Differentiable map.
$\mathrm{d}f(g)$	Differential of f at g, $\mathrm{d}f(g) : T_g G \to V$ identifying $T_{f(g)}V$ with V.
$\omega : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$	Connection function associated with ∇.
$\omega_X : \mathfrak{g} \to \mathfrak{g}$	$\omega_X(Y) = \omega(X,Y)$.
$T(X,Y) \in \mathfrak{g}$	Torsion function associated with ω.
$T_X : \mathfrak{g} \to \mathfrak{g}$	Partial torsion function $T_X Y = T(X,Y)$.
$\mathrm{Hess} f(g) : T_g G \to L(T_g G, V)$	Hessian operator of a twice differentiable function $f : G \to \mathbb{R}$ (or a map $f : G \to V$).
$(\phi)^W : L(W,U) \to L(W,V)$	Exponential functor $(\cdot)^W$ applied to a linear map $\phi : U \to V$, exponential functor lifts ϕ to $\phi^W : L(W,U) \to L(W,V)$ defined by $\phi^W(\xi) = \phi \circ \xi$.
$\mathbb{I} : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$	Inner product on $\mathfrak{g}$.
$\mathbb{I}^\sharp : \mathfrak{g}^* \to \mathfrak{g}$	$\sharp$-map associated to the inner product $\mathbb{I}$.
$\mathbb{I}^\flat : \mathfrak{g} \to \mathfrak{g}^*$	$\flat$-map associated to the inner product $\mathbb{I}$.
$\mathrm{ad} : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$	Adjoint map on on $\mathfrak{g}$.
$\mathrm{ad}^* : \mathfrak{g}\times\mathfrak{g}^* \to \mathfrak{g}^*$	Dual adjoint map.
$0_{n\times n}$	Null matrix of dimension $n \times n$.
$I_{n\times n}$	Identity matrix of dimension $n \times n$.

III. TSE(2) Structure

In this section, we study the planar rigid body shown in Fig. 1, whose state space is $\bar{G} = \mathrm{TSE}(2)$, the tangent bundle of $G = \mathrm{SE}(2)$ that can be identified with $\mathrm{SE}(2) \times \mathfrak{se}(2)$ via left translation [6] (we will use the two notations interchangeably).

The pose is an element $g \in \mathrm{SE}(2)$. The triple (θ, x, y) is a parameterization of the pose and consists of the position of the center of mass (x, y) of the planar rigid body and the angle θ the body forms with the horizontal axis [6]. A well-known matrix representation of $g \in \mathrm{SE}(2)$ is given by

$$g = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Given the Lie algebra $\mathfrak{se}(2)$, we introduce the isomorphisms $\vee : \mathfrak{se}(2) \to \mathbb{R}^3$ and $\wedge : \mathbb{R}^3 \to \mathfrak{se}(2)$ as

$$\begin{bmatrix} 0 & -\omega & v^x \\ \omega & 0 & v^y \\ 0 & 0 & 0 \end{bmatrix}^\vee = \begin{bmatrix} \omega \\ v^x \\ v^y \end{bmatrix}, \quad \begin{bmatrix} \omega \\ v^x \\ v^y \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\omega & v^x \\ \omega & 0 & v^y \\ 0 & 0 & 0 \end{bmatrix}$$

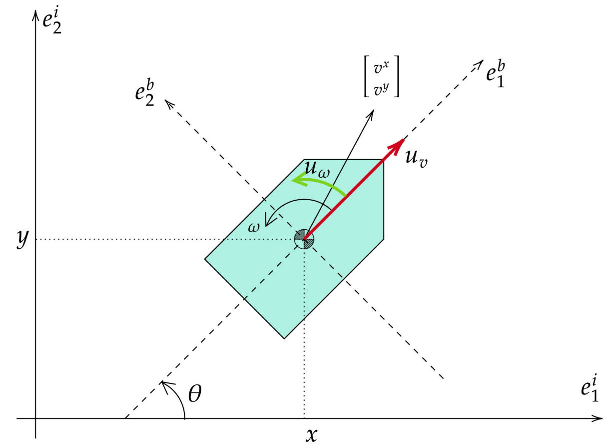

Fig. 1. Planar rigid body.

from the Lie algebra $(\mathbb{R}^3, \star)$ (where ω is the angular velocity and (v^x, v^y) is the linear velocity of the body written in body coordinate) to the matrix Lie algebra $(\mathfrak{se}(2), [\cdot,\cdot])$, where $\star : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the Lie bracket operation defined as

$$\begin{bmatrix} \omega_1 \\ v_1^x \\ v_1^y \end{bmatrix} \star \begin{bmatrix} \omega_2 \\ v_2^x \\ v_2^y \end{bmatrix} = \begin{bmatrix} 0 \\ v_1^y \omega_2 - \omega_1 v_2^y \\ \omega_1 v_2^x - v_1^x \omega_2 \end{bmatrix} \tag{1}$$

and $[\cdot,\cdot]$ is the usual matrix commutator (see e.g., [11]). Given $v = (\omega, v^x, v^y) \in \mathbb{R}^3$, the adjoint operator $\mathrm{ad}_{v^\wedge} \in L(\mathfrak{se}(2); \mathfrak{se}(2))$ admits the matrix representation

$$[\mathrm{ad}_{v^\wedge}] = \begin{bmatrix} 0 & 0 & 0 \\ v^y & 0 & -\omega \\ -v^x & \omega & 0 \end{bmatrix}.$$

The dual adjoint operator $\mathrm{ad}^*_{v^\wedge} \in L(\mathfrak{se}(2)^*; \mathfrak{se}(2)^*)$ satisfies $[\mathrm{ad}^*_{v^\wedge}] = [\mathrm{ad}_{v^\wedge}]^T$. Given the dual basis $\{\hat{e}^1, \hat{e}^2, \hat{e}^3\}$ on $\mathfrak{se}(2)^*$, we denote by $\mathbb{I}$ the constant inertia tensor

$$\mathbb{I} = J\hat{e}^1 \otimes \hat{e}^1 + m\hat{e}^2 \otimes \hat{e}^2 + m\hat{e}^3 \otimes \hat{e}^3 \tag{2}$$

with matrix representation (in the standard basis for $\mathbb{R}^3$, see [6])

$$\mathbb{I} = \mathrm{diag}(J, m, m) \tag{3}$$

where J the inertia along the axis passing to the center of mass and orthogonal to the plane and with m the mass of the rigid body. We denote by $u = (u_\omega/J, u_v/m, 0)^T$ the control inputs that are functions of time such that u_ω and u_v act, respectively, as a torque applied around the center of mass and a force applied along the body first axis, as shown in Fig. 1. The dynamic equations on $\bar{G}$ are

$$\begin{cases} g^{-1}\dot{g} = \Omega & (4) \\ \dot{\Omega} = \mathbb{I}^\sharp \mathrm{ad}^*_\Omega \mathbb{I}^\flat \Omega + u^\wedge & (5) \end{cases}$$

with $(g, \Omega) \in \bar{G}$. We can rewrite the systems (4) and (5) in a componentwise form as

$$\begin{cases} \dot{\theta} = \omega \\ \dot{x} = v^x \cos\theta - v^y \sin\theta \\ \dot{y} = v^x \sin\theta + v^y \cos\theta \\ \dot{\omega} = u_\omega / J \\ \dot{v^x} = \omega v^y + u_v/m \\ \dot{v^y} = -\omega v^x. \end{cases}$$

To take into account unmodeled dynamics in (4) and (5), we consider the unknown error δ and the mapping

$$B : \mathbb{R}^3 \to \mathfrak{se}(2) \times \mathfrak{se}(2), \quad \delta \mapsto (0, B_2\delta) \tag{6}$$

with $B_2 : \mathbb{R}^3 \to \mathfrak{g}$ a linear map.

The full dynamic equations are

$$\begin{cases} g^{-1}\dot{g} = \Omega & (7)\\ \dot{\Omega} = \mathbb{I}^\sharp \mathrm{ad}^*_\Omega \mathbb{I}^\flat \Omega + u^\wedge + B_2\delta & (8)\end{cases}$$

where the model error δ does not affect the reconstruction equation $g^{-1}\dot{g} = \Omega$.

A. Algebraic Aspects of $\mathrm{SE}(2) \times \mathfrak{se}(2)$

In this section, we present some mathematical tools to better face the computations in $\mathrm{SE}(2) \times \mathfrak{se}(2)$. We identify elements of groups and algebras with their matrix representations.

Let $\bar{g} = (g, \Omega)$ be an element of $\bar{G}$, we represent it in matrix form as

$$\bar{g} = \begin{bmatrix} g & 0 & 0 \\ 0 & I & \Omega \\ 0 & 0 & I \end{bmatrix} \tag{9}$$

while group operation is

$$(g, \Omega) \cdot (f, \Psi) = (gf, \Omega + \Psi)$$

that in matrix form reads

$$\begin{bmatrix} g & 0 & 0 \\ 0 & I & \Omega \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & I & \Psi \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} gf & 0 & 0 \\ 0 & I & \Omega + \Psi \\ 0 & 0 & I \end{bmatrix}.$$

The unit element of this group is $e = (I, 0)$ and for each (g, Ω) its inverse is $(g^{-1}, -\Omega)$. Since the group operation and its inverse are smooth, then $\bar{G}$ is a Lie group.

The Lie algebra of the (product) group $\bar{G}$ is the (product) algebra $\bar{\mathfrak{g}} = \mathfrak{se}(2) \times \mathfrak{se}(2)$. Every element $\eta^{\bar{g}} = (\eta^g, \eta^\Omega)$ can be represented as

$$\eta^{\bar{g}} = \begin{bmatrix} \eta^g & 0 & 0 \\ 0 & 0 & \eta^\Omega \\ 0 & 0 & 0 \end{bmatrix}.$$

The left translation on $\bar{G}$ is

$$L_{(g,\Omega)}(f, \Psi) = (gf, \Omega + \Psi).$$

We define the product between an element of $\bar{G}$ and an element of its Lie algebra as

$$(g, \Omega) * (\eta^g, \eta^\Omega) = (g\eta^g, \eta^\Omega)$$

that in matrix form reads

$$\begin{bmatrix} g & 0 & 0 \\ 0 & I & \Omega \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \eta^g & 0 & 0 \\ 0 & 0 & \eta^\Omega \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} g\eta^g & 0 & 0 \\ 0 & 0 & \eta^\Omega \\ 0 & 0 & 0 \end{bmatrix}.$$

The tangent map is

$$T_{(I,0)} L_{(g,\Omega)}(\eta^g, \eta^\Omega) = (g\eta^g, \eta^\Omega).$$

The adjoint representation of the Lie algebra into itself is

$$\mathrm{ad}_{(\eta^g,\eta^\Omega)}(\eta^f, \eta^\Psi) = (\mathrm{ad}_{\eta^g}\eta^f, 0).$$

The matrix form of the $\mathrm{ad}_{(\eta^g,\eta^\Omega)}$ operator is represented by the 6×6 matrix

$$\mathrm{ad}_{(\eta^g,\eta^v)} = \begin{bmatrix} [\mathrm{ad}_{\eta^g}] & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}. \tag{10}$$

Recall that given the control system $\dot{\bar{g}} = f(\bar{g}, u, t)$, the left-trivialized dynamics is defined as $\lambda(\bar{g}, u, t) := \bar{g}^{-1} f(\bar{g}, u, t)$. In our case, we obtain from (4) and (5)

$$\dot{\bar{g}} = f(\bar{g}, u, t) = \left(g\Omega, \mathbb{I}^\sharp \mathrm{ad}^*_\Omega \mathbb{I}^\flat \Omega + u^\wedge\right)$$

and

$$\begin{aligned}\lambda(\bar{g}, u, t) &= (g^{-1}, -\Omega)\left(g\Omega, \mathbb{I}^\sharp \mathrm{ad}^*_\Omega \mathbb{I}^\flat \Omega + u^\wedge\right) \\ &= \left(\Omega, \mathbb{I}^\sharp \mathrm{ad}^*_\Omega \mathbb{I}^\flat \Omega + u^\wedge\right) \in \mathfrak{se}(2) \times \mathfrak{se}(2).\end{aligned}$$

IV. Problem Statement

We now formulate the optimization problem to design the second-order minimum-energy filter. We rewrite the systems (7) and (8) as

$$\dot{\bar{g}}(t) = \bar{g}(t)(\lambda(\bar{g}(t), u(t), t) + B\delta(t)), \quad \bar{g}(t_0) = \bar{g}_0 \tag{11}$$

where $\bar{g}(t) \in \bar{G}$ is the state, $u(t) \in \mathbb{R}^3$ is the input, $\delta(t)$ is the unknown model error, $\lambda : \bar{G} \times \mathbb{R}^2 \times \mathbb{R} \to \bar{\mathfrak{g}}$ the left-trivialized dynamics, $\lambda = \begin{bmatrix} \lambda_g \\ \lambda_\Omega \end{bmatrix} = \begin{bmatrix} \Omega \\ \mathbb{I}^\sharp \mathrm{ad}^*_\Omega \mathbb{I}^\flat \Omega + u^\wedge \end{bmatrix}$, and $B : \mathbb{R}^3 \to \bar{\mathfrak{g}}$ a linear map given by $B\delta = (0_{3\times 3}, B_2\delta)$. From now on, we will indicate with B both this linear operator and its matrix form from $\mathbb{R}^3$ to $\mathbb{R}^6$.

The measurement equation $\mathrm{y} \in \mathbb{R}^7$ is

$$\mathrm{y}(t) = h(g(t), t) + D\varepsilon(t) \tag{12}$$

where $h : \bar{G} \times \mathbb{R} \to \mathbb{R}^7$ is the output map

$$h(\bar{g}(t), t) = \begin{bmatrix} x(t) + \ell \cos\theta(t) \\ y(t) + \ell \sin\theta(t) \\ x(t) - \ell \cos\theta(t) \\ y(t) - \ell \sin\theta(t) \\ \omega(t) \\ v^x(t) \\ v^y(t) \end{bmatrix} \tag{13}$$

which models a GPS-like system (first four rows) and an IMU (last three rows), $\varepsilon \in \mathbb{R}^7$ is the unknown measurement error and

$$D = \mathrm{diag}\{d_2, d_2, d_2, d_2, d_3, d_4, d_4\},\ d_i \in \mathbb{R}^+ \tag{14}$$

is an invertible linear map. The GPS measurements are the positions of two antennas attached to the body at distance ℓ from the center of gravity, as shown in Fig. 2, whereas the IMU provides the angular velocity ω and the linear velocities v^x and v^y.

In the deterministic setting of the minimum-energy filter, the errors δ and ε are assumed to be unknown deterministic functions of time. Given the input u and the measurement output y, the goal is to find the best estimate $\widehat{\bar{g}}(t) = (\widehat{g}(t), \widehat{\Omega}(t))$ of the state $\bar{g}(t) = (g(t), \Omega(t))$ minimizing a cost functional. This cost functional is a measure of energy in the unknown error signals δ and ε and is given by

$$J(\delta, \varepsilon, g_0; t, t_0) := m(g_0, t, t_0) + \int_{t_0}^{t} l(\delta(\tau), \varepsilon(\tau), t, \tau)\mathrm{d}\tau \tag{15}$$

where t_0 is the initial time and t the present time. The incremental cost function $l : \mathbb{R}^3 \times \mathbb{R}^7 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ represents the accumulated error along the trajectory, and is defined by

$$l(\delta, \varepsilon, t, \tau) := 1/2 e^{-\alpha(t-\tau)}(\mathcal{R}(\delta) + \mathcal{Q}(\varepsilon)) \tag{16}$$

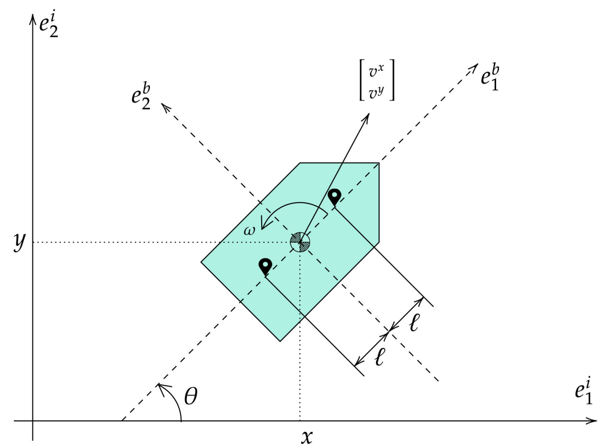

Fig. 2. Planar rigid body with two antennas.

where $\alpha \geq 0$ is the forgetting factor, and

$$\mathcal{R} : \mathbb{R}^3 \to \mathbb{R}, \quad \mathcal{Q} : \mathbb{R}^7 \to \mathbb{R}$$

are two quadratic forms with matrix representations

$$R = \mathrm{diag}\left\{r_3, r_4, r_4\right\}, r_i \in \mathbb{R}^+ \tag{17}$$

$$Q = \mathrm{diag}\left\{q_2, q_2, q_2, q_2, q_3, q_4, q_4\right\}, q_i \in \mathbb{R}^+. \tag{18}$$

For an easier computation, we choose $R = B^T B$ and $Q = D^T D$. Since we require that the matrix R in (17) is invertible, we assume that the operator B_2 in (6) is invertible, and for simplicity to be

$$B_2 = \mathrm{diag}\{b_3, b_4, b_4\}, \quad b_i > 0.$$

The function $m : G \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ can be considered as an initial cost, since it depends on the initial state g_0, and takes the form

$$m(g_0, t, t_0) := 1/2 e^{-\alpha(t-t_0)} m_0(g_0) \tag{19}$$

with $m_0 : G \to \mathbb{R}$ a bounded smooth function with a unique global minimum on G, which encodes the *a priori* information about the initial state at t_0.

For the derivation of the exact filter, it would be necessary to solve an infinite dimension Hamilton–Jacobi–Bellman equation, but since it is impossible to solve it exactly, the terms of order higher than two of the associated value function will be neglected; thus, the filter that we consider corresponds to a second-order approximation of the value function. For more details, we refer the reader to [9] and the references therein.

V. TSE(2) Optimal Filter

We are now ready to design the second-order-optimal filter for the dynamics (7) and (8), which minimize the cost functional (15)–(19) using the measurement equation (13). We consider the Cartan–Schouten (0) connection form $\omega^{(0)} = \frac{1}{2}\mathrm{ad}$. This choice is justified by the fact that this connection has null torsion (see [12]) and works better than the others (see [9]). We use the shorthand notation $h_t(\bar{g})$ and $\lambda_t(\bar{g}, u)$ for $h(\bar{g}(t), t)$ and $\lambda(\bar{g}(t), u(t), t)$, respectively, and we will denote with $\widehat{\cdot}$ the estimated variables.

Proposition 1: The second-order-optimal filter for the dynamic systems (7) and (8) with measurement equation (13) and with respect to the cost functional (15)–(19) is given by

$$\begin{cases} \widehat{g}^{-1}\dot{\widehat{g}} = \widehat{\Omega} + (K_{11} r^g + K_{12} r^\Omega)^\wedge & (20) \\ \dot{\widehat{\Omega}} = \mathbb{I}^\sharp \mathrm{ad}^*_{\widehat{\Omega}} \mathbb{I}^\flat \widehat{\Omega} + u^\wedge + (K_{21} r^g + K_{22} r^\Omega)^\wedge & (21) \end{cases}$$

where the residual r_t is

$$r_t = \begin{bmatrix} r^g \\ r^\Omega \end{bmatrix} = \begin{bmatrix} -(\tilde{\mathrm{y}}_1 - \tilde{\mathrm{y}}_3)\ell \sin\widehat{\theta} + (\tilde{\mathrm{y}}_2 - \tilde{\mathrm{y}}_4)\ell \cos\widehat{\theta} \\ (\tilde{\mathrm{y}}_1 + \tilde{\mathrm{y}}_3)\cos\widehat{\theta} + (\tilde{\mathrm{y}}_2 + \tilde{\mathrm{y}}_4)\sin\widehat{\theta} \\ -(\tilde{\mathrm{y}}_1 + \tilde{\mathrm{y}}_3)\sin\widehat{\theta} + (\tilde{\mathrm{y}}_2 + \tilde{\mathrm{y}}_4)\cos\widehat{\theta} \\ \tilde{\mathrm{y}}_5 \\ \tilde{\mathrm{y}}_6 \\ \tilde{\mathrm{y}}_7 \end{bmatrix}^T \tag{22}$$

and

$$\mathbb{R}^7 \ni \tilde{\mathrm{y}} = \left[\mathrm{diag}\left\{\frac{q_2}{d_2^2}, \frac{q_2}{d_2^2}, \frac{q_2}{d_2^2}, \frac{q_2}{d_2^2}, \frac{q_3}{d_3^2}, \frac{q_4}{d_4^2}, \frac{q_4}{d_4^2}\right\}\right](\mathrm{y} - \widehat{\mathrm{y}}). \tag{23}$$

The second-order-optimal gain $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$ is the solution of the perturbed matrix Riccati differential equation

$$\begin{aligned} \dot{K} = & -\alpha K + AK + KA^T - KEK + BR^{-1}B^T \\ & - W(K, r_t)K - KW(K, r_t)^T \end{aligned} \tag{24}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -\widehat{v}^y & 0 & \widehat{\omega} & 0 & 1 & 0 \\ \widehat{v}^x & -\widehat{\omega} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widehat{v}^y & 0 & \widehat{\omega} \\ 0 & 0 & 0 & -\widehat{v}^x & -\widehat{\omega} & 0 \end{bmatrix} \tag{25}$$

$$E = \begin{bmatrix} E_1 & 0_{3\times 3} \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix}, \quad E_1 = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & 2 & 0 \\ a_{3,1} & 0 & 2 \end{bmatrix} \tag{26}$$

with

$$\begin{aligned} a_{1,1} &= (\tilde{\mathrm{y}}_1 - \tilde{\mathrm{y}}_3)\ell \cos\widehat{\theta} + (\tilde{\mathrm{y}}_2 - \tilde{\mathrm{y}}_4)\ell \sin\widehat{\theta} + 2\ell^2 \\ a_{1,2} &= -\frac{1}{2}(\tilde{\mathrm{y}}_2 + \tilde{\mathrm{y}}_4)\cos\widehat{\theta} + \frac{1}{2}(\tilde{\mathrm{y}}_1 + \tilde{\mathrm{y}}_3)\sin\widehat{\theta} \\ a_{1,3} &= +\frac{1}{2}(\tilde{\mathrm{y}}_2 + \tilde{\mathrm{y}}_4)\sin\widehat{\theta} + \frac{1}{2}(\tilde{\mathrm{y}}_1 + \tilde{\mathrm{y}}_3)\cos\widehat{\theta} \\ a_{2,1} &= +\frac{1}{2}(\tilde{\mathrm{y}}_2 + \tilde{\mathrm{y}}_4)\cos\widehat{\theta} - \frac{1}{2}(\tilde{\mathrm{y}}_1 + \tilde{\mathrm{y}}_3)\sin\widehat{\theta} \\ a_{3,1} &= -\frac{1}{2}(\tilde{\mathrm{y}}_2 + \tilde{\mathrm{y}}_4)\sin\widehat{\theta} - \frac{1}{2}(\tilde{\mathrm{y}}_1 + \tilde{\mathrm{y}}_3)\cos\widehat{\theta} \end{aligned}$$

$$BR^{-1}B^T = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & B_2 R^{-1} B_2^T \end{bmatrix}$$

$$W(K, r_t) = \begin{bmatrix} \frac{1}{2}\mathrm{ad}_{(K_{11}r^g + K_{12}r^\Omega)^\wedge} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}. \tag{27}$$

Proof: The general formulas of the residual r_t and the operators A, E, and W for a generic Lie group G are presented in [9]. Here, we derive them for the TSE(2) case. In what follows, we will use $\eta = [\eta^g, \eta^\Omega] = [\eta^\theta, \eta^x, \eta^y, \eta^\omega, \eta^{v^x}, \eta^{v^y}]^T \in \mathbb{R}^6$ to indicate the vector form of an element of the Lie algebra $\bar{\mathfrak{g}}$ and $\widehat{\bar{g}}\eta = [\widehat{g}\eta^g, \eta^\Omega] = T_e L_{\widehat{\bar{g}}}(\eta) = [\widehat{\theta},' \widehat{x},' \widehat{y},' \widehat{\omega},' \widehat{v}^{x\prime}, \widehat{v}^{y\prime}]^T \in \mathbb{R}^6$ for the vector form of an element of the tangent space $T_{\widehat{\bar{g}}}\bar{G}$.

A. Computation of r_t

The residual $r_t(\widehat{g}) \in \bar{\mathfrak{g}}^*$ represents the weighted errors that depend on the Lie group structure, the measurement equation, and the matrices Q and D; its formula is given by

$$r_t(\widehat{g}) = T_e L^*_{\widehat{g}}[((D^{-1})^* \circ Q \circ D^{-1}(\mathrm{y} - h_t(\widehat{g}))) \circ \mathrm{d}h_t(\widehat{g})].$$

Given $T_e L_{\widehat{g}}(\eta) \in \mathbb{R}^6$, the differential of h in $\widehat{g}$ applied to $T_e L_{\widehat{g}}(\eta)$ is

$$\mathrm{d}h(\widehat{g})(T_e L_{\widehat{g}}(\eta)) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \begin{bmatrix} \widehat{x}(s) + \ell\cos\widehat{\theta}(s) \\ \widehat{y}(s) + \ell\sin\widehat{\theta}(s) \\ \widehat{x}(s) - \ell\cos\widehat{\theta}(s) \\ \widehat{y}(s) - \ell\sin\widehat{\theta}(s) \\ \widehat{\omega}(s) \\ \widehat{v}^x(s) \\ \widehat{v}^y(s) \end{bmatrix} = \begin{bmatrix} \widehat{x}' - \ell\widehat{\theta}'\sin\widehat{\theta} \\ \widehat{y}' + \ell\widehat{\theta}'\cos\widehat{\theta} \\ \widehat{x}' + \ell\widehat{\theta}'\sin\widehat{\theta} \\ \widehat{y}' - \ell\widehat{\theta}'\cos\widehat{\theta} \\ \widehat{\omega}' \\ \widehat{v}^{x\prime} \\ \widehat{v}^{y\prime} \end{bmatrix} \quad (28)$$

and thus, we can write the operator $\mathrm{d}h_t(\widehat{g})$ as

$$\mathrm{d}h_t(\widehat{g}) = \begin{bmatrix} -\ell\sin\widehat{\theta} & 1 & 0 & 0_{1\times 3} \\ +\ell\cos\widehat{\theta} & 0 & 1 & 0_{1\times 3} \\ +\ell\sin\widehat{\theta} & 1 & 0 & 0_{1\times 3} \\ -\ell\cos\widehat{\theta} & 0 & 1 & 0_{1\times 3} \\ 0_{3\times 1} & 0_{3\times 1} & 0_{3\times 1} & I_{3\times 3} \end{bmatrix}. \quad (29)$$

From the definition of the matrices (14) and (18), it follows that:

$$(D^{-1})^* \circ Q \circ D^{-1}(\mathrm{y} - h_t(\widehat{g})) = \left[\mathrm{diag}\left\{\frac{q_2}{d_2^2}, \frac{q_2}{d_2^2}, \frac{q_2}{d_2^2}, \frac{q_2}{d_2^2}, \frac{q_3}{d_3^2}, \frac{q_4}{d_4^2}, \frac{q_4}{d_4^2}\right\} (\mathrm{y} - \widehat{\mathrm{y}})\right]^T \quad (30)$$

composing (29) and (30), and evaluating $T_e L^*_{\widehat{g}}$, we obtain (22).

B. Computation of A

The operator A represents the linear part of the Riccati equation, and its formula $A(t) : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$ is given by

$$A(t) = \mathrm{d}_1\lambda_t(\widehat{g}, u) \circ T_e L_{\widehat{g}} - \mathrm{ad}_{\lambda_t(\widehat{g},u)} - T_{\lambda_t(\widehat{g},u)}$$

where d_1 is the differential with respect to the first component. Given $T_e L_{\widehat{g}}(\eta) \in \mathbb{R}^6$, we have

$$\mathrm{d}_1\lambda(\widehat{g}, u)(T_e L_{\widehat{g}}(\eta)) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \begin{bmatrix} \lambda_g(s) \\ \lambda_\Omega(s) \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \begin{bmatrix} \widehat{\omega}(s) \\ \widehat{v}^x(s) \\ \widehat{v}^y(s) \\ u_\omega/J \\ \widehat{\omega}\widehat{v}^y + u_v/m \\ -\widehat{\omega}\widehat{v}^x \end{bmatrix} = \begin{bmatrix} \widehat{\omega}' \\ \widehat{v}^{x\prime} \\ \widehat{v}^{y\prime} \\ 0 \\ \widehat{\omega}'\widehat{v}^y + \widehat{\omega}\widehat{v}^{y\prime} \\ -\widehat{\omega}'\widehat{v}^x - \widehat{\omega}\widehat{v}^{x\prime} \end{bmatrix} \quad (31)$$

and thus

$$\mathrm{d}_1\lambda(\widehat{g}, u) \circ T_e L_{\widehat{g}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widehat{v}^y & 0 & \widehat{\omega} \\ 0 & 0 & 0 & -\widehat{v}^x & -\widehat{\omega} & 0 \end{bmatrix}. \quad (32)$$

The adjoint matrix representation (10) implies

$$\mathrm{ad}_{\lambda_t(\widehat{g},u)} = \begin{bmatrix} 0 & 0 & 0 & 0_{1\times 3} \\ \widehat{v}^y & 0 & -\widehat{\omega} & 0_{1\times 3} \\ -\widehat{v}^x & \widehat{\omega} & 0 & 0_{1\times 3} \\ 0_{3\times 1} & 0_{3\times 1} & 0_{3\times 1} & 0_{3\times 3} \end{bmatrix}. \quad (33)$$

Considering the Cartan–Schouten (0)-connection form $\omega^{(0)} = \frac{1}{2}\mathrm{ad}$, the torsion function T vanishes (see [12]); thus, in matrix form, it is given by

$$T_{\lambda_t(\widehat{g},u)} = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}. \quad (34)$$

Using (32), (33), and (34), we obtain the matrix (25).

C. Computation of E

The function $E(t) : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}^*$ takes the form

$$E(t) = -T_e L^*_{\widehat{g}} \circ \Big[\left((D^{-1})^* \circ Q \circ D^{-1}(\mathrm{y} - h_t(\widehat{g}))\right)^{T_{\widehat{g}}\bar{G}} \circ \mathrm{Hess}h_t(\widehat{g}) - (\mathrm{d}h_t(\widehat{g}))^* \circ (D^{-1})^* \circ Q \circ D^{-1} \circ \mathrm{d}h_t(\widehat{g})\Big] \circ T_e L_{\widehat{g}}$$

and constitutes the second-order term of the Riccati equation that does not depend on the connection function. Now we can find the compositions

$$(\mathrm{d}h_t(\widehat{g}))^* \circ (D^{-1})^* \circ Q \circ D^{-1} \circ \mathrm{d}h_t(\widehat{g}) = \mathrm{diag}\left\{2\ell^2\frac{q_2}{d_2^2}, 2\frac{q_2}{d_2^2}, 2\frac{q_2}{d_2^2}, \frac{q_3}{d_3^2}, \frac{q_4}{d_4^2}, \frac{q_4}{d_4^2}\right\} \quad (35)$$

and

$$(D^{-1})^* \circ Q \circ D^{-1}(\mathrm{y} - h_t(\widehat{g})) = \left[\mathrm{diag}\left\{\frac{q_2}{d_2^2}, \frac{q_2}{d_2^2}, \frac{q_2}{d_2^2}, \frac{q_2}{d_2^2}, \frac{q_3}{d_3^2}, \frac{q_4}{d_4^2}, \frac{q_4}{d_4^2}\right\} (\mathrm{y} - \widehat{\mathrm{y}})\right]^T. \quad (36)$$

Given the twice differentiable function $h_t : \bar{G} \to \mathbb{R}^7$ and $T_e L_{\widehat{g}}(\eta_1) = (\widehat{g}\eta^{g_1}, \eta^{\Omega_1})$, $T_e L_{\widehat{g}}(\eta_2) = (\widehat{g}\eta^{g_2}, \eta^{\Omega_2}) \in T_{\bar{g}}\bar{G}$, the Hessian operator $\mathrm{Hess}h_t(\bar{g}) : T_{\bar{g}}\bar{G} \to T_{\bar{g}}\bar{G}^*$ is defined by

$$\begin{aligned} &\mathrm{Hess}h(\widehat{g})(T_e L_{\widehat{g}}(\eta_1))(T_e L_{\widehat{g}}(\eta_2)) \\ &= \mathrm{d}(\mathrm{d}h(\widehat{g})(T_e L_{\widehat{g}}(\eta_2)))(T_e L_{\widehat{g}}(\eta_1)) \\ &\quad - \mathrm{d}h(\widehat{g})(\nabla_{T_e L_{\widehat{g}}(\eta_1)}(T_e L_{\widehat{g}}(\eta_2))). \end{aligned} \quad (37)$$

We have

$$\mathrm{d}(\mathrm{d}h_t(\widehat{g})(T_e L_{\widehat{g}}(\eta_2)))(T_e L_{\widehat{g}}(\eta_1)) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \begin{bmatrix} \widehat{x}'_2 - \widehat{\theta}'_2\ell\sin\widehat{\theta}(s) \\ \widehat{y}'_2 + \widehat{\theta}'_2\ell\cos\widehat{\theta}(s) \\ \widehat{x}'_2 + \widehat{\theta}'_2\ell\sin\widehat{\theta}(s) \\ \widehat{y}'_2 - \widehat{\theta}'_2\ell\cos\widehat{\theta}(s) \\ 0_{3\times 1} \end{bmatrix} = \begin{bmatrix} -\widehat{\theta}'_1\widehat{\theta}'_2\ell\cos\widehat{\theta} \\ -\widehat{\theta}'_1\widehat{\theta}'_2\ell\sin\widehat{\theta} \\ +\widehat{\theta}'_1\widehat{\theta}'_2\ell\cos\widehat{\theta} \\ +\widehat{\theta}'_1\widehat{\theta}'_2\ell\sin\widehat{\theta} \\ 0_{3\times 1} \end{bmatrix}. \quad (38)$$

Since we are working with the Cartan–Schouten (0)-connection, we end up with

$$\begin{aligned} \nabla_{(\widehat{g}\eta^{g_1}, \eta^{\Omega_1})}(\widehat{g}\eta^{g_2}, \eta^{\Omega_2}) &= \frac{1}{2}\widehat{g}\,\mathrm{ad}_{(\eta^{g_1}, \eta^{\Omega_1})}(\eta^{g_2}, \eta^{\Omega_2}) \\ &= \frac{1}{2}\left(\widehat{g}\,\mathrm{ad}_{\eta^{g_1}}\eta^{g_2}, 0\right) \end{aligned} \quad (39)$$

and so

$$\mathrm{d}h_t(\widehat{g})(\nabla_{(\widehat{g}\eta^{g_1},\eta^{\Omega_1})}(\widehat{g}\eta^{g_2},\eta^{\Omega_2})) = \frac{1}{2}\begin{bmatrix} +\widehat{\theta}_2'\tilde{y}_1' - \widehat{\theta}_1'\tilde{y}_2' \\ -\widehat{\theta}_2'\tilde{x}_1' + \widehat{\theta}_1'\tilde{x}_2' \\ +\widehat{\theta}_2'\tilde{y}_1' - \widehat{\theta}_1'\tilde{y}_2' \\ -\widehat{\theta}_2'\tilde{x}_1' + \widehat{\theta}_1'\tilde{x}_2' \\ 0_{3\times 1} \end{bmatrix}. \quad (40)$$

The Hessian evaluated in $(\widehat{g}\eta^{g_1},\eta^{\Omega_1})$ and $(\widehat{g}\eta^{g_2},\eta^{\Omega_2})$ takes the form

$$\mathrm{Hess}h_t(\widehat{g})(\widehat{g}\eta^{g_1},\eta^{\Omega_1})(\widehat{g}\eta^{g_2},\eta^{\Omega_2}) =$$

$$= \begin{bmatrix} -\widehat{\theta}_1'\widehat{\theta}_2'\ell\cos\widehat{\theta} - \frac{1}{2}\widehat{\theta}_2'\tilde{y}_1' + \frac{1}{2}\widehat{\theta}_1'\tilde{y}_2' \\ -\widehat{\theta}_1'\widehat{\theta}_2'\ell\sin\widehat{\theta} + \frac{1}{2}\widehat{\theta}_2'\tilde{x}_1' - \frac{1}{2}\widehat{\theta}_1'\tilde{x}_2' \\ +\widehat{\theta}_1'\widehat{\theta}_2'\ell\cos\widehat{\theta} - \frac{1}{2}\widehat{\theta}_2'\tilde{y}_1' + \frac{1}{2}\widehat{\theta}_1'\tilde{y}_2' \\ +\widehat{\theta}_1'\widehat{\theta}_2'\ell\sin\widehat{\theta} + \frac{1}{2}\widehat{\theta}_2'\tilde{x}_1' - \frac{1}{2}\widehat{\theta}_1'\tilde{x}_2' \\ 0_{3\times 1} \end{bmatrix}. \quad (41)$$

From (36) and (41), it follows that:

$$\left((D^{-1})^* \circ Q \circ D^{-1}(\mathrm{y} - h_t(\widehat{g}))\right)^{T_{\widehat{g}}G} \circ \mathrm{Hess}h_t(\widehat{g})$$

$$= \begin{bmatrix} a_{1,1} & -\frac{1}{2}(\tilde{\mathrm{y}}_2+\tilde{\mathrm{y}}_4) & \frac{1}{2}(\tilde{\mathrm{y}}_1+\tilde{\mathrm{y}}_3) & 0_{1\times 3} \\ \frac{1}{2}(\tilde{\mathrm{y}}_2+\tilde{\mathrm{y}}_4) & 0 & 0 & 0_{1\times 3} \\ -\frac{1}{2}(\tilde{\mathrm{y}}_1+\tilde{\mathrm{y}}_3) & 0 & 0 & 0_{1\times 3} \\ 0_{3\times 1} & 0_{3\times 1} & 0_{3\times 1} & 0_{3\times 3} \end{bmatrix} \quad (42)$$

where $a_{1,1} = -\tilde{\mathrm{y}}_1\ell\cos\widehat{\theta} - \tilde{\mathrm{y}}_2\ell\sin\widehat{\theta} + \tilde{\mathrm{y}}_3\ell\cos\widehat{\theta} + \tilde{\mathrm{y}}_4\ell\sin\widehat{\theta}$. In conclusion, combining (35) and (42) with $T_eL^*_{\widehat{g}}$ and $T_eL_{\widehat{g}}$, the result follows.

D. Computation of W

The operator W represents a quadratic term that depends on the choice of the connection function and that cannot be incorporated in the standard quadratic term $K \circ E \circ K$. From the adjoint matrix form (10) and recalling that we consider the Cartan–Schouten (0)-connection (see e.g., [7], [8]), characterized by the connection function $\omega^{(0)} = \frac{1}{2}\mathrm{ad}$, we have

$$W(K, r_t) = \frac{1}{2}\mathrm{ad}_{((K_{11}r^g + K_{12}r^\Omega)^\wedge,\ (K_{21}r^g+K_{22}r^\Omega)^\wedge)}$$

$$= \begin{bmatrix} \frac{1}{2}\mathrm{ad}_{(K_{11}r^g+K_{12}r^\Omega)^\wedge} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}.$$

■

VI. Numerical Experiments

In this section, we explain how the second-order-optimal minimum-energy filter on $\bar{G} = \mathrm{TSE}(2)$ works. We set $J = 6.125\ \mathrm{kg\,m}^2$ and $m = 125$ kg. We consider the control inputs $u_v = 50\sin(\frac{1}{50}t)$ and $u_\omega = \sin(t)$ with total simulation time $T = 50\,\mathrm{s}$. The input u is used to generate the trajectory g. The initial conditions are $\theta(0) = \pi/6$ rad, $x(0) = 1\,\mathrm{m}$, $y(0) = 2\,\mathrm{m}$, $\omega(0) = 0\,\mathrm{rad/s}$, $v^x(0) = 0\,\mathrm{m/s}$, and $v^y(0) = 0\,\mathrm{m/s}$ for the system, and $\theta(0) = \pi/8$ rad, $x(0) = 1.1\,\mathrm{m}$, $y(0) = 2.1\,\mathrm{m}$, $\omega(0) = 0.1\,\mathrm{rad/s}$, $v^x(0) = -0.1\,\mathrm{m/s}$, and $v^y(0) = -0.1\,\mathrm{m/s}$ for the filter. The matrix forms of the operators B_2, R, and B related to the model uncertainty (11), (24) are

$$B_2 = \mathrm{diag}\{0.1, 0.1, 0.1\}, \quad B = \begin{bmatrix} 0_{3\times 3} \\ B_2 \end{bmatrix}, \quad R = B^T B.$$

The measurements in (12) are obtained by adding a Gaussian white noise with zero mean and standard deviation depending on D to the velocities and to the "antennas" components $x_1 = x + \ell\cos\theta$, $y_1 = x + \ell\sin\theta$, $x_2 = x - \ell\cos\theta$, and $y_2 = x - \ell\sin\theta$, where we chose $\ell = 0.2\,\mathrm{m}$. For the matrices D and Q in (14) and (18), we set $d_2 = 0.5$, $d_3 = 0.1$, $d_4 = 0.1$, and $q_i = d_i^2$, $i = 2,\ldots,4$. To solve all the differential equations we used a 4th order Runge–Kutta method with step $h = 1$ms. The nominal (θ_n, x_n, y_n, ω_n, v_n^x, v_n^y), measured (θ, x, y, ω, v^x, v^y), and filtered trajectories ($\widehat{\theta}$, $\widehat{x}$, $\widehat{y}$, $\widehat{\omega}$, $\widehat{v}^x$, $\widehat{v}^y$) for the poses and velocities and their errors

$$\delta\theta \triangleq \theta_n - \widehat{\theta}, \quad \delta x \triangleq x_n - \widehat{x}, \quad \ldots$$

are shown in Figs. 3 and 4. The filter performs very well even if the noises are relevant, and it provides accurate pose estimations despite the fact that it does not measure them directly.

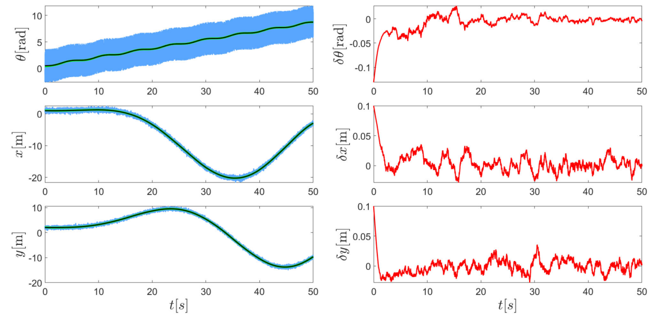

Fig. 3. Left-hand side: Nominal (black) and filtered (green) poses with measurements (blue). Right-hand side: Pose errors (red).

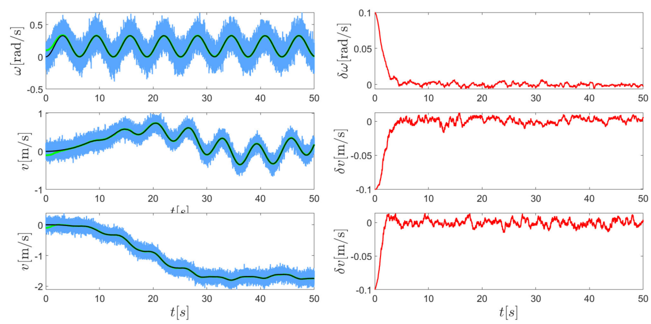

Fig. 4. Left-hand side: Nominal (black) and filtered (green) velocities with measurements (blue). Right-hand side: Velocity errors (red).

VII. Conclusion

In this article, we proposed a second-order-optimal minimum-energy filters on the Lie group $\mathrm{TSE}(2) = \mathrm{SE}(2) \times \mathfrak{se}(2)$. This filter exploits the theory of Lie groups and provides good estimations for both pose and velocity. The measurement equation models GPS-like and IMU-like devices, but the geometric structure lets to get good estimations even if one of them should be missing, for example, this filter should be replaced with another one considering only the IMU instrument anytime the GPS measurements are not available due to trees, high buildings and tunnels [14]. In the present setup, we assumed addictive noises for both the state equation and the measurement equation. In the future, we plan to consider also multiplicative noise, and, as anticipated in the introduction, we will take into account nonholonomic constraints (see, e.g., [15] and [16]). A comparison between the present filter and the extended Kalman filter can be found in [17].

REFERENCES

[1] R. Mahony, V. Kumar, and P. Corke, "Multirotor aerial vehicles: Modeling, estimation, and control of quadrotor," *IEEE Robot. Automat. Mag.*, vol. 19, no. 3, pp. 20–32, Sep. 2012.

[2] S. Thrun, W. Burgard, and D. Fox, *Probabilistic Robotics. Intelligent Robotics and Autonomous Agents Series*. Cambridge, MA, USA: MIT Press, 2005.

[3] R. E. Kalman, "A new approach to linear filtering and prediction problems," *J. Basic Eng.*, vol. 82, no. 1, pp. 35–45, 1960.

[4] S. J. Julier and J. K. Uhlmann, "Reduced sigma point filters for the propagation of means and covariances through nonlinear transformations," in *Proc. IEEE Amer. Control Conf.*, 2002, pp. 887–892.

[5] M. S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking," *IEEE Trans. Signal Process.*, vol. 50, no. 2, pp. 174–188, Feb. 2002.

[6] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems*. Berlin, Germany: Springer, 2019.

[7] A. Saccon, A. P. Aguiar, and J. Hauser, "Lie group projection operator approach: Optimal control on TSO(3)," in *Proc. 50th IEEE Conf. Decis. Control Eur. Control Conf.*, 2011, pp. 6973–6978.

[8] A. Saccon, J. Hauser, and A. P. Aguiar, "Optimal control on lie groups: The projection operator approach," *IEEE Trans. Autom. Control*, vol. 58, no. 9, pp. 2230–2245, Sep. 2013.

[9] A. Saccon, J. Trumpf, R. Mahony, and A. P. Aguiar, "Second-order-optimal minimum-energy filters on lie groups," *IEEE Trans. Autom. Control*, vol. 61, no. 10, pp. 2906–2919, Oct. 2016.

[10] R. Mortensen, "Maximum-likelihood recursive nonlinear filtering," *J. Optim. Theory Appl.*, vol. 2, no. 6, pp. 386–394, 1968.

[11] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry*. Berlin/Heidelberg, Germany: Springer, 1999, vol. 17.

[12] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*. Hoboken, NJ, USA: Wiley, 1963.

[13] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*. Berlin, Germany: Springer, 2013.

[14] C. Spagnol, R. Muradore, M. Assom, A. Beghi, and R. Frezza, "Trajectory reconstruction by integration of GPS and a swarm of MEMS accelerometers: Model and analysis of observability," in *Proc. 7th Int. IEEE Conf. Intell. Transp. Syst.*, 2004, pp. 64–69.

[15] J. M. Osborne and D. V. Zenkov, "Steering the Chaplygin sleigh by a moving mass," in *Proc. 44th IEEE Conf. Decis. Control*, 2005, pp. 1114–1118.

[16] N. Sansonetto and M. Zoppello, "On the trajectory generation of the hydrodynamic Chaplygin sleigh," *IEEE Control Syst. Lett.*, vol. 4, no. 4, pp. 922–927, Oct. 2020.

[17] D. Rigo, N. Sansonetto, and R. Muradore, "A comparison between the extended Kalman filter and a minimum-energy filter in the TSE(2) case," in *Proc. 60th IEEE Conf. Decis. Control*, 2021, pp. 6175–6180.