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journal homepage: www.elsevier.com/locate/jpaaInjectives over Leavitt path algebras of graphs with disjoint cycles [☆]Gene Abrams ^a, Francesca Mantese ^b, Alberto Tonolo ^{c,*}^a Department of Mathematics, University of Colorado, Colorado Springs, CO 80918, USA^b Dipartimento di Informatica, Università degli Studi di Verona, I-37134 Verona, Italy^c Dipartimento di Matematica “Tullio Levi-Civita”, Università degli Studi di Padova, I-35121, Padova, Italy

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ABSTRACT

Let K be any field, and let E be a finite graph with the property that every vertex in E is the base of at most one cycle (i.e., a graph with disjoint cycles). We explicitly construct the injective envelope of each simple left module over the Leavitt path algebra $L_K(E)$. The main idea guiding our construction is that of a “formal power series” extension of modules, thereby developing for all graphs with disjoint cycles the understanding of injective envelopes of simple modules over $L_K(E)$ achieved previously for the simple modules over the Toeplitz algebra.

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1. Introduction

In [4], the three authors presented a complete, explicit description of the injective envelope of every simple left module over $L_K(\mathcal{T})$, the “Toeplitz Leavitt path K -algebra” associated to the “Toeplitz graph” $\mathcal{T} := \bigcirc \bullet \longrightarrow \bullet$, isomorphic to the well-studied Jacobson algebra $K\langle X, Y \mid XY = 1 \rangle$ (see for instance [8,10,12,13]). Clearly \mathcal{T} is an elementary but nontrivial example of a graph that satisfies the following property.

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Definition 1.1. Let E be a finite graph. We say that E has disjoint cycles in case each vertex of E is the base of at most one cycle.

The condition that each vertex in a finite graph E is the base of at most one cycle has been studied in other settings. For example, in [6] the four authors establish that, for any field K , this condition is equivalent to the Leavitt path algebra $L_K(E)$ having finite Gelfand-Kirillov dimension. Subsequently, Ara and Rangaswamy in [7, Theorem 1.1] establish that this condition is also equivalent to the property that all simple modules over $L_K(E)$ are of a specified type (so-called *Chen* modules). More recently, in [11] these graphs have been characterised in terms of the Jordan-Hölder composition series of their talented monoid.

For us, the condition to have disjoint cycles sits at a wonderfully fortunate confluence of the ideas presented in [6] and [7]. On one hand, similar to a process carried out in [6], the condition affords the possibility to consider a well-defined positive integer representing the “length of a chain of cycles”, and in particular affords the possibility of applying an induction argument in such a situation. On the other hand, as provided by [7], having a complete explicit description of all the simple $L_K(E)$ -modules puts us in position to describe the injective envelopes of all simple modules, and consequently to describe an injective cogenerator for $L_K(E)$ -Mod.

We show in the current article that the “formal power series” idea developed in [4, Section 6] in the context of the graph \mathcal{T} can indeed be extended to all graphs with disjoint cycles. In particular, we show that the injective envelopes of all the simple modules over $L_K(E)$ may be realised as a type of formal power series extension. This is the heart of our main result, Theorem 5.2. The proof, rather long and delicate, is obtained first reducing via Morita equivalences to connected graphs which contain no *source vertices* and in which every *source cycle* is a loop, and then on an induction reasoning on the number of cycles. The presence of cycles in the graph gives rise to elements of $L_K(E)$ which can be seen as polynomials with the cycles as indeterminate. It is therefore not surprising to come across formal series when dealing with this class of algebras.

The article is organised as follows. In Section 2 we review some of the basics of Leavitt path algebras, and provide some properties of Leavitt path algebras of graphs having a specified structure. In Section 3 we review the notion of a Chen simple $L_K(E)$ -module, as well as that of a Prüfer $L_K(E)$ -module. In Section 4 we introduce the formal power series construction associated to subsets of $\text{Path}(E)$. With the Sections 3 and 4 material in hand, in Section 5 we state and prove the main result of the article, Theorem 5.2.

We conclude this introductory section by giving a reformulation of the standard Baer Criterion for injectivity, one which is well suited to our situation.

Proposition 1.2. *Let L be an associative ring, and let M be a left L -module. Let I be a fixed left ideal of L . Assume that any morphism $f : X \rightarrow M$ from any left ideal $X \leq I$ extends to a morphism $\hat{f} : I \rightarrow M$, and also assume that any morphism $g : Y \rightarrow M$ from any left ideal $Y \geq I$ extends to a morphism $\hat{g} : L \rightarrow M$. Then M is injective.*

Proof. Let J be any left ideal of L , and let $h : J \rightarrow M$ be an L -module morphism. The restriction $h_0 : J \cap I \rightarrow M$ extends by assumption to $\hat{h}_0 : I \rightarrow M$. Setting

$$h_1(j + i) = h(j) + \hat{h}_0(i)$$

for each $j \in J, i \in I$ yields a morphism $h_1 : J + I \rightarrow M$. That h_1 is well-defined follows from this observation: $j + i = j' + i'$ implies $j - j' = i' - i \in J \cap I$, and hence

$$h(j) - h(j') = h(j - j') = h_0(j - j') = h_0(i' - i) = \hat{h}_0(i' - i) = \hat{h}_0(i') - \hat{h}_0(i),$$

from which $h(j) + \hat{h}_0(i) = h(j') + \hat{h}_0(i')$. Clearly h_1 extends h . Now $h_1 : J + I \rightarrow M$ extends by assumption to $\hat{h}_1 : L \rightarrow M$. By the Baer Criterion we conclude that M is injective. \square

2. Leavitt path algebras

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \rightarrow E^0$, the *range* and the *source* maps. The elements of E^0 are called *vertices*, those of E^1 *edges*. E is *finite* in case both E^0 and E^1 are finite sets. **Unless otherwise indicated, we assume throughout that all graphs are finite.** A *sink* w is a vertex which emits no edges, i.e., $s^{-1}(w) = \emptyset$; a *source* u is a vertex to which no edges arrive, i.e., $r^{-1}(u) = \emptyset$. A *finite path* of length n is a sequence of edges $\rho = e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for $i = 1, 2, \dots, n - 1$. We denote $s(\rho) = s(e_1)$ and $r(\rho) = r(e_n)$. Any vertex is viewed as a (trivial) path of length 0. We denote by $\text{Path}(E)$ the set of all finite paths in E . For each $e \in E^1$, we call e^* the associated *ghost edge*; by definition, $r(e^*) = s(e)$ and $s(e^*) = r(e)$.

Given any field K and (finite) graph E , the *Leavitt path K -algebra* $L_K(E)$ is the free associative K -algebra generated by a set of symbols $\{v : v \in E^0\} \sqcup \{e, e^* : e \in E^1\}$, where $\{v : v \in E^0\}$ are orthogonal idempotents, and for which the following relations are imposed:

- (1) for each $e \in E^1$, $s(e)e = e = er(e)$ and $r(e)e^* = e^* = e^*s(e)$;
- (2) for each pair $e, f \in E^1$, $e^*f = \begin{cases} r(f) & \text{if } e = f, \\ 0 & \text{otherwise;} \end{cases}$ and
- (3) for each non-sink $v \in E^0$, $v = \sum_{e \in s^{-1}(v)} ee^*$.

The elements of $L_K(E)$ are K -linear combinations of paths $\lambda\mu^*$, $\lambda, \mu \in \text{Path}(E)$ [2, Lemma 1.2.12]. A nontrivial path $e_1 e_2 \cdots e_n$ is a *closed path* (resp., a *cycle*¹) if $r(e_n) = s(e_1)$ (resp., and $s(e_i) \neq s(e_j)$ for every $i \neq j$). A cycle of length 1 is called a *loop*. A *source cycle* is a cycle without entrances, i.e., a cycle $c = e_1 \cdots e_n$ such that, for each $e \in E^1$, $r(e) \in c^0 := \{s(e_i) : i = 1, \dots, n\}$ implies $e \in \{e_1, \dots, e_n\}$.

Suppose A is a K -algebra, and suppose that there is a subset $S := \{a_v : v \in E^0\} \sqcup \{a_e, a_{e^*} : e \in E^1\}$ of A for which $\{a_v : v \in E^0\}$ is a set of orthogonal idempotents, and for which the relations analogous to (1), (2), and (3) above are satisfied by the elements of S . Such a set S is called a *Cuntz-Krieger E -family* in A . By the construction of $L_K(E)$, if S is a Cuntz-Krieger E -family in A , then there exists a K -algebra homomorphism $\Phi : L_K(E) \rightarrow A$ for which $\Phi(v) = a_v$, $\Phi(e) = a_e$, and $\Phi(e^*) = a_{e^*}$ for all $v \in E^0$ and $e \in E^1$.

Because E is assumed to be finite, $L_K(E)$ is a unital K -algebra, with multiplicative identity $1_{L_K(E)} = \sum_{v \in E^0} v$. If d is a closed path in E and $p(x) = a_0 + \sum_{i=1}^n a_i x^i \in K[x]$, then $p(d)$ denotes the element

$$p(d) := a_0 1_{L_K(E)} + \sum_{i=1}^n a_i d^i$$

of $L_K(E)$.

A subset $H \subseteq E^0$ is *hereditary* if, whenever $v \in H$ and there exists $p \in \text{Path}(E)$ for which $s(p) = v$ and $r(p) = w$, then $w \in H$. An hereditary set of vertices is *saturated* if for any non-sink $v \in E^0$, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$. In the Fig. 1 the set of vertices $\{t_1, t_2, t_3, w, z\}$ is hereditary, and the set $\{v, t_1, t_2, t_3, w, z\}$ is hereditary and saturated. Let \mathcal{P} be any subset of $\text{Path}(E)$. Let $\ell_1, \ell_2 \in L_K(E)$ denote either a vertex of E or the sum of distinct vertices of E . We denote by

$$\ell_1 \mathcal{P} \ell_2 := \{\ell_1 p \ell_2 : p \in \mathcal{P}\}$$

¹ however, see Definition 3.1 below.

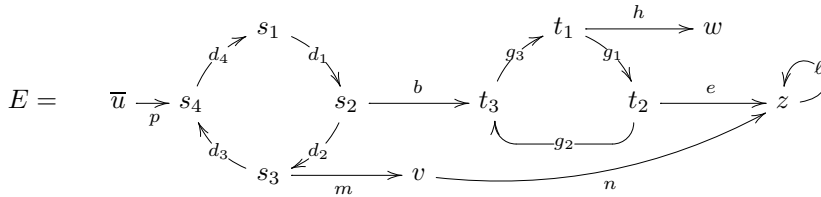


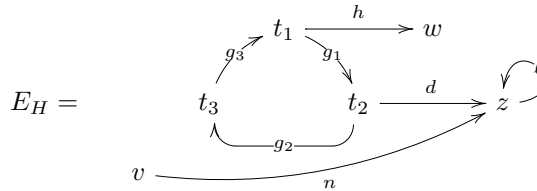
Fig. 1. Reference example of a graph with disjoint cycles.

the set of all paths in \mathcal{P} which start in a vertex summand of ℓ_1 and end in a vertex summand of ℓ_2 . Referring to the Fig. 1,

$$s_2\{g_3h, bg_3, bg_3h, bg_3g_1e\}(w + z) = \{bg_3h, bg_3g_1d\}.$$

Additional general information about Leavitt path algebras may be found in [2].

For a hereditary saturated subset H of E^0 , E_H denotes the restriction of E to H , that is, the graph with $E_H^0 = H$ and $E_H^1 = \{e \in E^1 \mid s(e) \in H\}$. Referring to the Fig. 1, if $H = \{v, t_1, t_2, t_3, w, z\}$, then



Proposition 2.1. *Let K be any field. Let τ be a source loop in the graph E , with $s(\tau) = r(\tau) = t$. Let $H := E^0 \setminus \{t\}$, and let I be the two sided ideal of $L_K(E)$ generated by $\rho := \sum_{u \in H} u$; that is, $I = L_K(E)\rho L_K(E)$. Then:*

- (1) *The ideal I , when viewed as a K -algebra in its own right, is isomorphic to a Leavitt path K -algebra. Further, I is Morita equivalent to $L_K(E_H)$.*
- (2) *There is a lattice isomorphism between the lattice of left $L_K(E)$ -ideals contained in I and the lattice of left $L_K(E_H)$ -submodules of ρI .*
- (3) *Any left $L_K(E)$ -ideal properly containing I is a principal left ideal of $L_K(E)$. It is generated by an element of the form $p(\tau) \in L_K(E)$, where $p(x) \in K[x]$ has $p(0) = 1$.*

Proof. 1.) Since τ is a source cycle, H is clearly hereditary. As τ is a loop based at t , H is necessarily (vacuously) saturated as well. Invoking [2, Theorem 2.5.19] we get that I is isomorphic to a Leavitt path algebra of a (not-necessarily finite) graph ${}_H E$. (The graph ${}_H E$ is a so-called ‘‘hedgehog graph.’’) Note then that $I = L_K({}_H E)$ need not have multiplicative identity. However, I has local units when viewed as a ring (see e.g., [2, Section 1.2]). Since $\rho = \rho^2$, and $I = L_K(E)\rho L_K(E)$, clearly then $I = I\rho I$; as well, $\rho I\rho = \rho L_K(E)\rho$. Hence by [1, Corollary 4.3] we have an equivalence

$$I\text{-Mod} \begin{array}{c} \xrightarrow{\rho I \otimes_I -} \\ \xleftarrow{I \rho \otimes_{\rho L_K(E) \rho} -} \end{array} \rho L_K(E)\rho\text{-Mod}$$

between the category $I\text{-Mod}$ of unitary I -modules (i.e., for each M in $I\text{-Mod}$, $M = IM$ holds) and the category of modules over the full corner ring $\rho L_K(E)\rho$. Since there are no paths containing t that both start and end in H , the ring $\rho L_K(E)\rho$ is isomorphic to the Leavitt path algebra $L_K(E_H)$.

2.) Let J be a left ideal of $L_K(E)$ contained in I . Since I has local units, we have $IJ = J$ and hence J belongs to $I\text{-Mod}$. Conversely, any left I -submodule M of I is a left $L_K(E)$ -ideal contained in I , indeed

$$L_K(E)M = L_K(E)(IM) = (L_K(E)I)M = IM = M.$$

Applying the equivalence described in point 1.), we get the indicated lattice isomorphism between the left $L_K(E)$ -ideals contained in I and the left $L_K(E_H)$ -submodules of $\rho I \otimes_I I \cong \rho I$.

3.) Let J' be any left ideal of $L_K(E)$ properly containing I . We must prove that there exists $p(x) = 1 + \sum_{i=1}^m a_i x^i \in K[x]$ for which

$$J' = L_K(E)p(\tau) = L_K(E)(1_{L_K(E)} + a_1\tau + \dots + a_m\tau^m).$$

By [2, Corollary 2.4.13], $L_K(E)/I \cong L_K(\tau \bigcirc \bullet^t) \cong K[x, x^{-1}]$. Since $0 \neq J'/I$ is then a (left) ideal of the principal ideal domain $L_K(E)/I$, it is generated as an $L_K(E)/I$ -ideal by a nonzero polynomial $p(x)$ of degree $m \geq 0$ with $p(0) = 1$ evaluated in $\tau + I$. We consider the following two elements of $L_K(E)$:

$$p_t(\tau) := t + a_1\tau + \dots + a_m\tau^m \quad \text{and} \\ p(\tau) := 1_{L_K(E)} + a_1\tau + \dots + a_m\tau^m = \rho + p_t(\tau).$$

We show that $J' = L_K(E)p(\tau)$. One can easily check that $J' = L_K(E)p_t(\tau) + I$, and clearly J' contains $p_t(\tau) + \rho = p(\tau)$; since $p_t(\tau) = tp(\tau)$ we have

$$J' \geq L_K(E)p(\tau) \geq L_K(E)p_t(\tau).$$

Since then $J' \leq L_K(E)p(\tau) + I$, to prove that $J' = L_K(E)p(\tau)$ we need to show that I is contained in $L_K(E)p(\tau)$.

To do so, let $s^{-1}(t) = \{\tau, \varepsilon_1, \dots, \varepsilon_n\}$ be the set of edges in E with source $t = s(\tau)$. Any element of I is a sum of the type $\alpha\rho + \beta\rho\gamma s(\tau)$ with $\alpha, \beta, \gamma \in L_K(E)$. Clearly $\alpha\rho = \alpha p(\tau)$ belongs to $L_K(E)p(\tau)$. Since τ is a source loop, $\rho\gamma s(\tau) = \sum_{j=1}^n k_j\rho\gamma_j\varepsilon_j^*(\tau^*)^{\ell_j}$ for suitable $k_j \in K$, $\ell_j \in \mathbb{N}$, and $\gamma_j \in L_K(E)$. We prove by induction on ℓ_j that $\varepsilon_j^*(\tau^*)^{\ell_j}$ belongs to $L_K(E)p(\tau)$ for any $\ell_j \geq 0$; this will then give that $I \leq L_K(E)p(\tau)$. Note that $\varepsilon_j^*\tau = 0$ for all $1 \leq j \leq n$, so $\varepsilon_j^*p(\tau) = \varepsilon_j^*$.

If $\ell_j = 0$ then

$$\varepsilon_j^*(\tau^*)^0 = \varepsilon_j^* = \varepsilon_j^*p(\tau) \in L_K(E)p(\tau).$$

Let $0 < \ell_j \leq m = \deg p(x)$. Then

$$\begin{aligned} \varepsilon_j^*(\tau^*)^{\ell_j}p(\tau) &= \varepsilon_j^*(\tau^*)^{\ell_j}(1_{L_K(E)} + a_1\tau + \dots + a_m\tau^m) \\ &= \varepsilon_j^*((\tau^*)^{\ell_j} + a_1(\tau^*)^{\ell_j-1} + \dots + a_{\ell_j-1}\tau^* + a_{\ell_j}t + \dots + a_m\tau^{m-\ell_j}) \\ &= \varepsilon_j^*(\tau^*)^{\ell_j} + a_1\varepsilon_j^*(\tau^*)^{\ell_j-1} + \dots + a_{\ell_j-1}\varepsilon_j^*(\tau^*) + \varepsilon_j^*(a_{\ell_j}t + \dots + a_m\tau^{m-\ell_j}) \\ &= \varepsilon_j^*(\tau^*)^{\ell_j} + a_1\varepsilon_j^*(\tau^*)^{\ell_j-1} + \dots + a_{\ell_j-1}\varepsilon_j^*(\tau^*) + (\varepsilon_j^*p(\tau))(a_{\ell_j}t + \dots + a_m\tau^{m-\ell_j}) \\ &= \varepsilon_j^*(\tau^*)^{\ell_j} + a_1\varepsilon_j^*(\tau^*)^{\ell_j-1} + \dots + a_{\ell_j-1}\varepsilon_j^*(\tau^*) + \varepsilon_j^*(a_{\ell_j}t + \dots + a_m\tau^{m-\ell_j})p(\tau). \end{aligned}$$

Note that in the last step we used the fact that $\tau t = t\tau = \tau$, and that polynomials in τ commute in $L_K(E)$. Therefore

$$\varepsilon_j^*(\tau^*)^{\ell_j} = \varepsilon_j^*(\tau^*)^{\ell_j}p(\tau) - [a_1\varepsilon_j^*(\tau^*)^{\ell_j-1} + \dots + a_{\ell_j-1}\varepsilon_j^*(\tau^*)] - \varepsilon_j^*(a_{\ell_j}t + \dots + a_m\tau^{m-\ell_j})p(\tau).$$

By the inductive hypothesis, each summand in the bracketed term belongs to $L_K(E)p(\tau)$, and thus so does $\varepsilon_j^*(\tau^*)^{\ell_j}$.

Finally, let $\ell_j > m$. Then

$$\begin{aligned}\varepsilon_j^*(\tau^*)^{\ell_j}p(\tau) &= \varepsilon_j^*(\tau^*)^{\ell_j}(1 + a_1\tau + \cdots + a_m\tau^m) \\ &= \varepsilon_j^*(\tau^*)^{\ell_j} + a_1\varepsilon_j^*(\tau^*)^{\ell_j-1} + \cdots + a_m\varepsilon_j^*(\tau^*)^{\ell_j-m}.\end{aligned}$$

Thus

$$\varepsilon_j^*(\tau^*)^{\ell_j} = \varepsilon_j^*(\tau^*)^{\ell_j}p(\tau) - [a_1\varepsilon_j^*(\tau^*)^{\ell_j-1} + \cdots + a_m\varepsilon_j^*(\tau^*)^{\ell_j-m}],$$

which, again by the inductive hypothesis, belongs to $L_K(E)p(\tau)$. \square

The following result regarding expressions in $L_K(E)$ will be extremely useful in the sequel.

Lemma 2.2. *Let E be a finite graph. Suppose that $\gamma_1, \gamma_2 \in \text{Path}(E)$ have $r(\gamma_1) = r(\gamma_2) = v$ for some $v \in E^0$. Suppose further that neither γ_1 nor γ_2 is of the form $\gamma\delta$, where δ is a closed path having $s(\delta) = r(\delta) = v$. Then in $L_K(E)$,*

$$\gamma_1^*\gamma_2 = \begin{cases} v & \text{if } \gamma_1 = \gamma_2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If γ_1 or γ_2 are vertices, then the result is easy to check. So now assume $\gamma_1 = \alpha_1 \cdots \alpha_h$ and $\gamma_2 = \beta_1 \cdots \beta_k$, where $h, k \geq 1$, and $r(\alpha_h) = r(\beta_k) = v$. Then

$$\gamma_1^*\gamma_2 = \alpha_h^* \cdots \alpha_1^*\beta_1 \cdots \beta_k.$$

Assume $\gamma_1^*\gamma_2 \neq 0$. If $h < k$ we would have $\alpha_1 = \beta_1, \dots, \alpha_h = \beta_h$; since $r(\alpha_h) = r(\beta_h) = r(\beta_k) = v$, we get that the path $\beta_{h+1} \cdots \beta_k$ would be a closed path with source and range vertex v , contradicting the hypothesis on γ_2 . The situation in which $h > k$ follows similarly. Therefore $h = k$ and $\gamma_1 = \gamma_2$, and the result follows. \square

We close this section by noting some graph-theoretic properties related to the condition of having disjoint cycles.

Lemma 2.3. *Let E be a finite graph. Then E has disjoint cycles if and only if all closed paths in E are powers of cycles.²*

Proof. Clearly any graph in which the only closed paths are powers of cycles has disjoint cycles.

Conversely, let $e := e_1 \cdots e_n$ be a closed path in E . We proceed by induction on the length n of e . If $n = 1$, then e is a loop and hence it is a cycle. Let $n > 1$. Assume that e is not a power of a cycle. Consider the sequence of vertices $(s(e_1), \dots, s(e_n))$. Since e is not a cycle, in the sequence there are repetitions. In the set

$$\{(i, j) : i < j \in \{1, \dots, n\}, s(e_i) = s(e_j), j - i \text{ minimal}\}$$

² This lemma has been obtained in one of our fruitful discussions with our friend and colleague Kulumani M. Rangaswamy.

consider the pair (\bar{i}, \bar{j}) with \bar{i} minimal. Then $e_1 \cdots e_{\bar{i}-1} e_{\bar{j}} \cdots e_n$ and $e_{\bar{i}} \cdots e_{\bar{j}-1}$ are respectively a closed path p and a cycle d in E of length $< n$. By the inductive hypothesis we have $p = c^m$ for a suitable cycle c in E and $m \geq 1$. The cycles c and d have in common the vertex $s(e_{\bar{i}}) = r(e_{\bar{i}-1})$: therefore, since distinct cycles are disjoint, $c = d$. By the minimality of \bar{i} , we get $\bar{i} = 1$ and hence

$$c = d = e_1 \cdots e_{\bar{j}-1}, \quad p = c^m = e_{\bar{j}} \cdots e_n.$$

Therefore $e = c^{m+1}$, a power of the cycle c . \square

The set of vertices of a graph E is naturally endowed with a preorder structure. Given $u, v \in E^0$, we write $v \leq u$ if there exists $p \in \text{Path}(E)$ with $s(p) = u$ and $r(p) = v$.

Writing $u \equiv v$ if $u \leq v$ and $v \leq u$, one obtains an equivalence relation \equiv on E^0 . The preorder \leq induces a partial order on the quotient set E^0_{\equiv} . Because E^0 is assumed to be finite, E^0_{\equiv} contains maximal elements. Referring to the Fig. 1, $E^0 = \{\bar{u}, s_1, s_2, s_3, s_4, v, t_1, t_2, t_3, w, z\}$ and $E^0_{\equiv} = \{\bar{u}, [s_1, s_2, s_3, s_4], [v], [t_1, t_2, t_3], [w], [z]\}$; \bar{u} is the only maximal element in E^0_{\equiv} .

Remark 2.4. If the graph E has disjoint cycles, then (using Lemma 2.3) each equivalence class in E^0_{\equiv} is either a single vertex, or the set of vertices of a cycle. In particular, the maximal elements in E^0_{\equiv} are either source vertices or the set of vertices of a source cycle.

Therefore, if E in addition contains no source vertices, then the finiteness of E together with the condition of having disjoint cycles implies that E contains at least one source cycle.

3. Chen simples and Prüfer modules

We reiterate here our standing assumption that E is a finite graph. So the Leavitt path algebra $L_K(E)$ has a multiplicative identity, namely, $1_{L_K(E)} = \sum_{v \in E^0} v$.

In [9] Chen introduced classes of simple left modules over Leavitt path algebras. Here, we concentrate on those simples associated to sinks or to cycles: $L_K(E)w$ (for any sink w) and $V_{[c^\infty]}$ (for any cycle c) in E . We will see that, although these have been viewed as distinct classes in the literature, they in fact share many properties.

A sink w is an idempotent in $L_K(E)$: we consider the left ideal $L_K(E)w$. Since there are no ghost edges ending in w , any element of $L_K(E)w$ is a K -linear combination of paths in $\text{Path}(E)w$, i.e., of (real) paths ending in w . Since $L_K(E) = \bigoplus_{u \in E^0} L_K(E)u$ as left $L_K(E)$ -ideals, we have clearly

$$L_K(E)w \cong L_K(E)/L_K(E)(w - 1).$$

Given a cycle $c = e_1 \cdots e_n$ in E , the “infinite path” obtained by repeating c infinitely many times, denoted c^∞ , can be considered as an “idempotent-like element”: one has $c^n \cdot c^\infty = c^\infty$ for each $n \geq 1$. Extending the product defined between elements of $L_K(E)$ and c , one can in the expected way construct the left $L_K(E)$ -module $L_K(E)c^\infty$, which is usually denoted by $V_{[c^\infty]}$. Since $e^*c^\infty = 0$ for each $e \neq e_1$, and $e_1^*c^\infty = e_2 \cdots e_n c^\infty$, any element in $L_K(E)c^\infty$ is a K -linear combination of paths in $\text{Path}(E)c^\infty$, i.e., of (real) paths ending in $s(c)$ times c^∞ . By [5, Theorem 2.8] we have

$$L_K(E)c^\infty \cong L_K(E)/L_K(E)(c - 1).$$

If one thinks of a sink w as a cycle of length 0, and observes that $w^\infty = w$, the two above constructions of simple modules may be viewed as two manifestations of the same idea.

Definition 3.1. By a cycle in the graph E we mean either a sink (i.e., a cycle of length 0), or a proper cycle, i.e., a cycle of length $n \geq 1$. We unify notation by setting

$$V_{[w^\infty]} := L_K(E)w$$

for any sink w in E .

Definition 3.2. Let E be a finite graph, and c any cycle in E . By \mathcal{P}_c^E we denote the following subset of $\text{Path}(E)$:

$$\mathcal{P}_c^E := \begin{cases} \{\gamma \in \text{Path}(E) : r(\gamma) = s(c) = c\} = \text{Path}(E)c & \text{if } c \text{ is a sink;} \\ \{\gamma \in \text{Path}(E) : r(\gamma) = s(c), \text{ and } \gamma \neq \gamma'c \ \forall \gamma' \in \text{Path}(E)\} & \text{if } c \text{ is a proper cycle.} \end{cases}$$

Less formally, if c is a sink then \mathcal{P}_c^E is the set of paths in E ending in $s(c) = c$. Otherwise, if c is a proper cycle, \mathcal{P}_c^E is the set of paths in E which end at the source vertex $s(c)$ of the cycle c , but for which the path does not end with a complete traverse of the cycle c .

Referring to the Fig. 1, the set \mathcal{P}_w^E of all paths in E ending in w is infinite: it contains the sink w and all the left truncations of paths of the form $pd_4(d_1d_2d_3d_4)^i d_1bg_3(g_1g_2g_3)^j h$ ($i, j \geq 0$). As well, the sets $\mathcal{P}_{g_1g_2g_3}^E$, and \mathcal{P}_e^E are infinite. The former contains the vertex $t_1 = s(g_1)$, g_2g_3 , and all the left truncations of paths of the form $pd_4(d_1d_2d_3d_4)^i d_1bg_3$ ($i \geq 0$); the latter contains the vertex z and all the left truncations of paths of the form $pd_4(d_1d_2d_3d_4)^i d_1d_2mn$ and $pd_4(d_1d_2d_3d_4)^i d_1bg_3(g_1g_2g_3)^j g_1e$. The set $\mathcal{P}_{d_1d_2d_3d_4}^E$ is finite: it contains $s_1, d_4, pd_4, d_3d_4, d_2d_3d_4$.

Remark 3.3. By Lemma 2.3, if the graph E has disjoint cycles, then for any cycle c the paths in \mathcal{P}_c^E do not end with the traverse of any closed path.

Remark 3.4. It follows by [9, §3.1] and [2, Corollary 1.5.15] that for any cycle c , the set

$$\mathcal{P}_c^E \cdot c^\infty = \{p \cdot c^\infty : p \in \mathcal{P}_c^E\}$$

is a K -base for the simple left $L_K(E)$ -module $V_{[c^\infty]}$.

If c is any cycle and $0 \neq a \in K$, we denote by $\sigma_{c,a}$ the “gauge” automorphism of $L_K(E)$ associated to c and a . This automorphism is defined to be the identity if c is a sink. Otherwise, if $c = e_1e_2 \cdots e_n$ is a proper cycle, $\sigma_{c,a}$ sends u to u for each $u \in E^0$, e to e and e^* to e^* for each $e \in E^1 \setminus \{e_1\}$, and e_1 to ae_1 and e_1^* to $a^{-1}e_1^*$. If $a = 1$, then $\sigma_{c,1}$ is clearly the identity automorphism of $L_K(E)$.

For M in $L_K(E)\text{-Mod}$, we define $M^{\sigma_{c,a}} \in L_K(E)\text{-Mod}$ by setting $M^{\sigma_{c,a}} = M$ as an abelian group, but with the modified left $L_K(E)$ -action

$$\ell \star m := \sigma_{c,a}(\ell)m$$

for each $\ell \in L_K(E)$ and $m \in M$. The automorphism $\sigma_{c,a}$ induces an auto-equivalence of the category $L_K(E)\text{-Mod}$ given by the functor

$$L_K(E)^{\sigma_{c,a}} \otimes_{L_K(E)} - : L_K(E)\text{-Mod} \rightarrow L_K(E)\text{-Mod}$$

which sends any $L_K(E)$ -module M to the “twisted” $L_K(E)$ -module $M^{\sigma_{c,a}} \cong L_K(E)^{\sigma_{c,a}} \otimes M$. The twisted module $V_{[c^\infty]}^{\sigma_{c,a}}$ is a simple left $L_K(E)$ -module for each $0 \neq a \in K$.

A family of simple $L_K(E)$ -modules wider than the one presented in [9] for proper cycles was obtained by Ara and Rangaswamy in [7] as follows. Let K be any field, c be any cycle in E , and $f(x) \in K[x]$ be a *basic irreducible* polynomial in $K[x]$, i.e., a polynomial which is irreducible in $K[x]$, and for which $f(0) = -1$. Denote by K' the field $K[x]/\langle f(x) \rangle$ and by \bar{x} the element $x + \langle f(x) \rangle \in K'$.

Definition 3.5. We denote by $V_{E,[c^\infty]}^f$ the left $L_K(E)$ -module obtained from the twisted left $L_{K'}(E)$ -module $V_{E,[c^\infty]}^{\sigma_{c,\bar{x}}}$ by restricting scalars from K' to K .

Here is a more precise formulation of $V_{E,[c^\infty]}^f$. The K -algebra homomorphism $K \rightarrow K'$ induces a functor \mathcal{U} from $L_{K'}(E)$ -modules to $L_K(E)$ -modules which is the right adjoint of the extension of scalars functor $-\otimes_K K' : L_K(E)\text{-Mod} \rightarrow L_{K'}(E)\text{-Mod}$. Then $V_{E,[c^\infty]}^f$ is defined to be $\mathcal{U}(V_{E,[c^\infty]}^{\sigma_{c,\bar{x}}})$.

In particular, for each $0 \neq a \in K$ we may view the previously defined twisted modules in these more general terms by noting that

$$V_{E,[c^\infty]}^{\sigma_{c,a}} = V_{E,[c^\infty]}^f \quad \text{where } f(x) = a^{-1}x - 1.$$

We will often omit the graph E when it is clear from the context, writing simply $V_{[c^\infty]}^f$. If $c = w$ is a sink, then

$$V_{[w^\infty]}^f = L_K(E)w = K\mathcal{P}_w^E = K\text{Path}(E)w$$

for any basic irreducible polynomial $f(x) \in K[x]$.

Remark 3.6. Since any element of K' is a K -linear combination of elements of the form \bar{x}^h ($0 \leq h < \deg(f(x))$), by Remark 3.4 for any cycle c , the set

$$\mathcal{P}_c^E \cdot \{\bar{x}^i : 0 \leq i < \deg f(x)\} \cdot c^\infty = \{p \cdot \bar{x}^i \cdot c^\infty : p \in \mathcal{P}_c^E, 0 \leq i < \deg f(x)\}$$

is a K -base for the simple left $L_K(E)$ -module $V_{[c^\infty]}^f$.

Definition 3.7. Let c be any sink and $f(x)$ any basic irreducible polynomial in $K[x]$. Following Ara and Rangaswamy [7], we call any simple left $L_K(E)$ -module of the form $V_{[c^\infty]}^f$ a Chen simple module.

Remark 3.8. If c_1 is a cycle of length ≥ 2 obtained by “rotating” the cycle c_2 , then the simple modules $V_{E,[c_1^\infty]}^f$ and $V_{E,[c_2^\infty]}^f$ are isomorphic. If c is a fixed sink, then $V_{E,[c^\infty]}^f = V_{E,[c^\infty]}$, for any basic irreducible polynomial $f(x) \in K[x]$. Aside from these cases, the Chen simple modules are pairwise non-isomorphic. (See [9, Theorem 6.2] and [7, Proposition 3.8].)

Referring to the Fig. 1, a complete list of the simple left $L_K(E)$ -modules up to isomorphisms is given by: $V_{[w^\infty]} = L_K(E)w$, $V_{[(d_1 d_2 d_3 d_4)^\infty]}^f$, $V_{[(g_1 g_2 g_3)^\infty]}^f$, and $V_{[\ell^\infty]}^f$, for any basic irreducible polynomial $f(x) \in K[x]$.

In addition to the Chen simple modules, there is another important (and related) construction of $L_K(E)$ -modules which will be relevant in the sequel. Let E be a finite graph, and c any cycle in E . In [3] the three authors constructed the *Prüfer $L_K(E)$ -module* $U_{E,c-1}$ associated to any proper cycle c in E . The module $U_{E,c-1}$ is an infinite length uniserial artinian $L_K(E)$ -module with all composition factors isomorphic to $V_{[c^\infty]}$. The module $U_{E,c-1}$ is the direct limit

$$U_{E,c-1} := \varinjlim_{i,j \geq 1} \{L_K(E)/L_K(E)(c-1)^i, \psi_{E,i,j}\}$$

of the factor modules $L_K(E)/L_K(E)(c-1)^i$, with respect to the morphisms

$$\begin{aligned} \psi_{E,i,j} : L_K(E)/L_K(E)(c-1)^i &\rightarrow L_K(E)/L_K(E)(c-1)^j, \text{ defined by setting} \\ 1 + L_K(E)(c-1)^i &\mapsto \frac{(c-1)^j}{(c-1)^i} + L_K(E)(c-1)^j \quad \forall 1 \leq i < j. \end{aligned}$$

Denoting by

$$\psi_{E,i} : L_K(E)/L_K(E)(c-1)^i \rightarrow U_{E,c-1}$$

for each $i \geq 1$ the induced monomorphism, the Prüfer module $U_{E,c-1}$ is generated as an $L_K(E)$ -module by the elements

$$\alpha_{c,i} := \psi_{E,i}(1 + L_K(E)(c-1)^i), \quad i \geq 1.$$

If $c = w$ is a sink, then $\alpha_{w,i} = (-1)^{i+1}\alpha_{w,1}$. The Prüfer module corresponding to a sink then becomes

$$U_{E,w-1} = V_{[w^\infty]} = L_K(E)w = K\mathcal{P}_w^E.$$

Clearly, for any cycle c , $L_K(E)\alpha_{c,1}$ is isomorphic to the simple module $V_{[c^\infty]}$, and hence it is the socle of the module $U_{E,c-1}$ (it is equal to $V_{[c^\infty]}$ if c is a sink). In particular $V_{[c^\infty]}$ is essential in $U_{E,c-1}$. In general we omit reference to the graph E and write simply U_{c-1} .

We now provide some additional information about the Prüfer $L_K(E)$ -module U_{c-1} for any cycle c of E . Let $m \geq 1$. Setting $\alpha_{c,0} = 0$, we have

$$(c-1)\alpha_{c,m} = \begin{cases} 0 & \text{if } c \text{ is a sink,} \\ \alpha_{c,m-1} & \text{if } c \text{ is a proper cycle.} \end{cases}$$

If $u \in E^0 \setminus \{s(c)\}$, then easily one shows that $u = (-1)^m u(c-1)^m$ for each $m \geq 1$, so we get $u\alpha_{c,m} = 0$. (As a result, $s(c)\alpha_{c,m} = \alpha_{c,m}$.)

If c has length 0 and $e \in E^1$, or c is a proper cycle and $e \in E^1 \setminus \{e_1\}$, it is also easily shown that $e^* = (-1)^m e^*(c-1)^m$ for each $m \geq 1$. So $e^*\alpha_{c,m} = 0$ for all $m \geq 1$ in this situation as well.

Finally, if c is a proper cycle $c = e_1 e_2 \cdots e_n$, then for each $m \geq 1$, multiplying each side of the equation $(c-1)\alpha_{c,m} = \alpha_{c,m-1}$ by e_1^* and rearranging terms, we get that $e_1^*\alpha_{c,m} = e_2 \cdots e_n \alpha_{c,m} - e_1^*\alpha_{c,m-1}$. (We interpret $e_2 \cdots e_n$ as $t = s(c)$ if $n = 1$.) Continuing in this way, we iteratively get

$$e_1^*\alpha_{c,m} = e_2 \cdots e_n \left(\sum_{\ell=0}^{m-1} (-1)^\ell \alpha_{c,m-\ell} \right).$$

Therefore, by the three previous paragraphs, any element of U_{c-1} can be written in the form

$$\sum_{j=1}^m \sum_{\gamma \in F} k_{j,\gamma} \alpha_{c,j}$$

for suitable $k_{j,\gamma} \in K$ and finite subset $F \subseteq \mathcal{P}_c^E$. Less formally, we have established that each element of U_{c-1} can be written as an $L_K(E)$ -linear combination of the $\alpha_{c,m}$ in such a way that the $L_K(E)$ -coefficients do not involve any ghost edges. This observation will play a key role later on.

Analogous to the construction of the simple module $V_{[c^\infty]}^f$ associated to any proper cycle c and any basic irreducible $f(x) \in K[x]$, we have

Definition 3.9. Let E be any finite graph, and K any field. Let c be any cycle, $f(x)$ be any basic irreducible polynomial in $K[x]$, and $\bar{x} := x + \langle f(x) \rangle \in K' = K[x]/\langle f(x) \rangle$. We define

$$U_{f(c)} := \mathcal{U}(U_{c-1}^{\sigma_{c,\bar{x}}}),$$

the left $L_K(E)$ -module obtained from the twisted $L_{K'}(E)$ -module $U_{E,c-1}^{\sigma_{c,\bar{x}}}$ via the functor \mathcal{U} which restricts scalars to K .

Observe that if $c = w$ is a sink, then Definition 3.9 reduces to

$$U_{f(w)} = U_{w-1} = V_{[w^\infty]} = L_K(E)w = K\mathcal{P}_w^E.$$

Remark 3.10. Since for each $0 \neq a \in K$, any automorphism of $L_K(E)$ induces an autoequivalence of $L_K(E)\text{-Mod}$, it is clear that for any proper cycle $c = e_1 \cdots e_n$ the twisted $L_K(E)$ -module $U_{E,c-1}^{\sigma_{c,a}}$ has the same submodule structure as $U_{E,c-1}$. Precisely, $U_{E,c-1}^{\sigma_{c,a}}$ is an infinite length uniserial artinian left $L_K(E)$ -module with composition factors isomorphic to $V_{[c^\infty]}^{\sigma_{c,a}}$. Moreover, setting

$$\alpha_{c,m}^a := 1_{L_K(E)} \otimes_{L_K(E)} \alpha_{c,m} \quad \text{in} \quad L_K(E)^{\sigma_{c,a}} \otimes_{L_K(E)} U_{E,c-1} \cong U_{E,c-1}^{\sigma_{c,a}}$$

we have

$$\begin{aligned} (a^{-1}c - 1)\alpha_{c,m}^a &= ((a^{-1}c - 1) \star 1_{L_K(E)}) \otimes_{L_K(E)} \alpha_{c,m} \\ &= \sigma_{c,a}(a^{-1}c - 1) \otimes_{L_K(E)} \alpha_{c,m} \\ &= (c - 1) \otimes_{L_K(E)} \alpha_{c,m} \\ &= 1_{L_K(E)} \otimes_{L_K(E)} (c - 1)\alpha_{c,m} \\ &= \alpha_{c,m-1}^a. \end{aligned}$$

If $f(x) \in K[x]$ is basic irreducible and $K' = K[x]/\langle f(x) \rangle$, taking a equal to the invertible element $\bar{x} = x + \langle f(x) \rangle$ of K' we get that

$$(\bar{x}^{-1}c - 1)\alpha_{c,m}^{\bar{x}} = \alpha_{c,m-1}^{\bar{x}}$$

in the twisted left $L_{K'}(E)$ -module $U_{E,c-1}^{\sigma_{c,\bar{x}}}$. Multiplying by \bar{x} on both sides one gets $\bar{x}\alpha_{c,m}^{\bar{x}} = c\alpha_{c,m}^{\bar{x}} - \bar{x}\alpha_{c,m-1}^{\bar{x}}$, which iteratively gives

$$\bar{x}\alpha_{c,m}^{\bar{x}} = c\alpha_{c,m}^{\bar{x}} - c\alpha_{c,m-1}^{\bar{x}} + \cdots + (-1)^{m-1}c\alpha_{c,1}^{\bar{x}}.$$

In such a way, multiplication of $\alpha_{c,m}^{\bar{x}}$ by \bar{x} yields a linear combination of $\alpha_{c,j}^{\bar{x}}$'s having coefficients in $L_K(E)$. Moreover for each edge $e \neq e_1 \in E^1$ we have

$$e^* \bar{x}^h \alpha_{c,m}^{\bar{x}} = \bar{x}^h e^* \alpha_{c,m}^{\bar{x}} = \bar{x}^h (1_{L_K(E)} \otimes_{L_K(E)} e^* \alpha_{c,m}) = 0,$$

while when $e = e_1$ we have

$$\begin{aligned} e_1^* \bar{x}^h \alpha_{c,m}^{\bar{x}} &= \bar{x}^h e_1^* \alpha_{c,m}^{\bar{x}} = \bar{x}^h e_1^* (1_{L_K(E)} \otimes_{L_K(E)} \alpha_{c,m}) \\ &= \bar{x}^h (1_{L_K(E)} \otimes_{L_K(E)} e_1^* \alpha_{c,m}) \\ &= \bar{x}^h (1_{L_K(E)} \otimes_{L_K(E)} e_2 \cdots e_n \sum_{\ell=0}^{m-1} (-1)^\ell \alpha_{c,m-\ell}) \end{aligned}$$

$$= \sum_{\ell=0}^{m-1} (-1)^\ell e_2 \cdots e_n \bar{x}^\ell \alpha_{c,m-\ell}^{\bar{x}}.$$

The discussion over the previous paragraphs has established the following.

Proposition 3.11. *For any sink w , any proper cycle c and any basic irreducible $f(x) \in K[x]$ we have the following equalities of K -vector spaces*

$$U_{w-1} = K\mathcal{P}_w^E \alpha_{w,1} = K\mathcal{P}_w^E \quad \text{and} \quad U_{f(c)} = \bigoplus_{j \geq 1} \sum_{h=0}^{\deg f(x)-1} K\mathcal{P}_c^E \bar{x}^h \alpha_{c,j}^{\bar{x}}.$$

Specifically, any element of $U_{f(c)}$ is a finite sum of elements of the form $\{k\gamma\bar{x}^h \alpha_{c,j}^{\bar{x}} \mid k \in K, \gamma \in \mathcal{P}_c^E, h \geq 0, j \geq 1\}$.

As mentioned previously, the cases of simple $L_K(E)$ -modules corresponding to sinks and those corresponding to proper cycles have historically been treated separately in the literature. We believe that there is enough commonality to these two types of simples that merits treating them as two cases of the same construction. Indeed, this is the case in all of our subsequent results. Thus, as we already said in Definition 3.1, when we use the word *cycle* we will mean either a sink, or a proper cycle. Admittedly, however, there is a small price of additional notation to pay in order to achieve this single approach.

Definition 3.12. We set

$$\mathbb{I}_c := \begin{cases} \mathbb{N}_{\geq 1} & \text{if } c \text{ is a proper cycle} \\ \{1\} & \text{if } c = w \text{ is a sink} \end{cases}$$

$$\deg_c f(x) := \begin{cases} \deg f(x) & \text{if } c \text{ is a proper cycle,} \\ 1 & \text{if } c = w \text{ is a sink.} \end{cases}$$

Moreover, if $c = w$ is a sink, then $\alpha_{c,1}^{\bar{x}} := w$.

Lemma 3.13. *Let c be any cycle in the graph E , and let $f(x) \in K[x]$ be basic irreducible. Assume the graph E has disjoint cycles. Then in $U_{f(c)}$ the set*

$$\{\gamma\bar{x}^h \alpha_{c,j}^{\bar{x}} \mid \gamma \in \mathcal{P}_c^E, 0 \leq h < \deg_c f(x), j \in \mathbb{I}_c\}$$

is K -linearly independent.

Proof. Assume

$$\sum_{j \in F_1} \sum_{h=0}^{\deg_c f(x)-1} \sum_{\gamma \in F_2} k_{j,\gamma,h} \gamma \bar{x}^h \alpha_{c,j}^{\bar{x}} = 0$$

for suitable finite subsets F_1 of \mathbb{I}_c and F_2 of \mathcal{P}_c^E . Since the cycles are disjoint, we can apply Lemma 2.3 to get that the elements in \mathcal{P}_c^E do not end with any closed path. Therefore, multiplying on the left by any $\bar{\gamma}^*$ (with $\bar{\gamma} \in \mathcal{P}_c^E$), by Lemma 2.2 we get

$$\sum_{j \in F_1} \left(\sum_{h=0}^{\deg_c f(x)-1} k_{j,\bar{\gamma},h} \bar{x}^h \right) \alpha_{c,j}^{\bar{x}} = 0.$$

Since the set $\{\alpha_{c,j}^{\bar{x}} \mid j \in F_1\}$ is linearly independent over the field extension $K' = K[x]/\langle f(x) \rangle$ of K , we have that $\sum_{h=0}^{\deg_c f(x)-1} k_{j,\bar{\gamma},h} \bar{x}^h = 0$ in K' for each $1 \leq j \leq m$. Since $\{\bar{x}^h \mid 0 \leq h < \deg_c f(x)\}$ is K -linearly independent, we get $k_{j,\bar{\gamma},h} = 0$ for any $j \in F_1$ and $0 \leq h < \deg_c f(x)$. Because $\bar{\gamma}$ was arbitrary, we are done. \square

As expected, the wider class of modules $U_{f(c)}$ satisfy properties similar to the specific modules U_{c-1} .

Proposition 3.14. *Let E be a finite graph, K any field, c a proper cycle in E , and $f(x) = xg(x) - 1$ a basic irreducible polynomial in $K[x]$. The $L_K(E)$ -module $U_{f(c)}$ is an infinite length uniserial artinian module with all composition factors isomorphic to $V_{[c^\infty]}^f$.*

Proof. As sets, we have $U_{f(c)} = U_{g(\bar{x})c-1}$. It is sufficient to prove that the lattice of $L_K(E)$ -submodules of $U_{f(c)}$ is equal to the lattice of $L_{K'}(E)$ -submodules of $U_{g(\bar{x})c-1}$. Clearly any $L_{K'}(E)$ -submodule of $U_{g(\bar{x})c-1}$ is also a $L_K(E)$ -submodule of $U_{f(c)}$. Consider now an $L_K(E)$ -submodule M of $U_{f(c)}$. Since, as observed in Remark 3.10,

$$\bar{x}\alpha_{c,j}^{\bar{x}} = c\alpha_{c,j}^{\bar{x}} - c\alpha_{c,j-1}^{\bar{x}} + \dots + (-1)^{j-1}c\alpha_{c,1}^{\bar{x}},$$

we have that $\bar{x}m$ belongs to M for each $m \in M$. Then M is also a $L_{K'}(E)$ -submodule of $U_{g(\bar{x})c-1}$. \square

We end the section with an observation about the left-module action of $L_K(E)$ on $U_{f(c)}$.

Remark 3.15. Let c be either a sink in E , or $c = e_1e_2 \dots e_n$ a cycle in E . Let $e \in E^1$ and $\gamma \in \mathcal{P}_c^E$ for which $r(e) = s(\gamma)$. In general there are two possible interpretations of an expression of the form $e\gamma$. One is simply the concatenation of the paths e and γ . The other is as $e \cdot \gamma$, where we view $e \in L_K(E)$, $\gamma \in U_{f(c)}$, and \cdot denotes the left $L_K(E)$ -action on $U_{f(c)}$.

If for example $e = e_1$ and $\gamma = e_2 \dots e_n$ then $\gamma \in \mathcal{P}_c^E$, but the concatenation $e\gamma$ is not in \mathcal{P}_c^E . So in this situation the concatenation $e\gamma$ is not an element of the form indicated in Proposition 3.11, and the two interpretations of $e\gamma$ differ.

However, consideration of the lengths of paths shows that the situation described in the previous paragraph is the only configuration one needs to consider in order to avoid the situation in which the concatenation of an edge e with a path $\gamma \in \mathcal{P}_c^E$ having $r(e) = s(\gamma)$ yields a path which is not in \mathcal{P}_c^E . Consequently, suppose for example that $\gamma \in \mathcal{P}_c^E$ for which $|\gamma| \geq |c|$, and let $e \in E^1$ with $r(e) = s(\gamma)$. Then the concatenation path $e\gamma$ is necessarily in \mathcal{P}_c^E . Thus for an element y of $U_{f(c)}$ of the form $y = k\gamma\bar{x}^h\alpha_{c,j}$ with $k \in K$, $|\gamma| \geq |c|$ and $r(e) = s(\gamma)$, the element $e \cdot y$ of $U_{f(c)}$ is equal to the element $k(e\gamma)\alpha_{c,j}$ of $U_{f(c)}$.

A similar observation holds for e^* where $e \in E^1$. Specifically, suppose $\gamma \in \mathcal{P}_c^E$ has $\gamma = e\gamma'$ for some $\gamma' \in \text{Path}(E)$. Then necessarily $\gamma' \in \mathcal{P}_c^E$. So the two possible interpretations of the expressions $e^*\gamma$ (as either the element γ' of \mathcal{P}_c^E , or as the element $e^* \cdot \gamma$ of $U_{f(c)}$) coincide. We note that the only elements of the form $k\gamma\bar{x}^h\alpha_{c,j}$ of $U_{f(c)}$ for which $e^*k\gamma\bar{x}^h\alpha_{c,j} \neq 0$ and $\gamma \neq e\gamma'$ must have $\gamma = v = s(c)$. In this case, the iterative version of the expression for $e_1^*\bar{x}^h\alpha_{c,m}$ given in Remark 3.10 yields a sum of elements of the indicated form, where the lengths of the paths appearing are each equal to $|c| - 1$.

4. Formal power series built from E

In this section we introduce the key construction by which we will produce the injective envelopes of Chen simple modules.

For any cycle c , the set $\mathcal{P}_c^E \subseteq \text{Path}(E)$ is a K -linearly independent set in $L_K(E)$ by [2, Corollary 1.5.15].

Definition 4.1. We denote by $K[[\mathcal{P}_c^E]]$ the K -vector space of all mappings from \mathcal{P}_c^E to K . Any element \mathfrak{p} in $K[[\mathcal{P}_c^E]]$ can be represented as a “ \mathcal{P}_c^E -formal series”

$$\mathfrak{p} = \sum_{\mu \in \mathcal{P}_c^E} \mathfrak{p}(\mu) \cdot \mu.$$

If $\mathfrak{p}(\mu) \neq 0$ only for a finite number of $\mu \in \mathcal{P}_c^E$, then \mathfrak{p} belongs to the K vector space $K\mathcal{P}_c^E$ generated by \mathcal{P}_c^E . Otherwise \mathfrak{p} is called a *proper* \mathcal{P}_c^E -formal series.

Definition 4.2. Let E be any finite graph, c a cycle in E , K any field, and $f(x)$ a basic irreducible polynomial in $K[x]$. We define the K -vector space $\widehat{U}_{f(c)}$ by setting

$$\widehat{U}_{f(c)} := \bigoplus_{j \in \mathbb{I}_c} \sum_{h=0}^{\deg_c f(x)-1} K[[\mathcal{P}_c^E]] \bar{x}^h \alpha_{c,j}^{\bar{x}}.$$

Clearly $\widehat{U}_{f(c)}$ is an enlarged version of the Prüfer module

$$U_{f(c)} = \bigoplus_{j \in \mathbb{I}_c} \sum_{h=0}^{\deg_c f(x)-1} K\mathcal{P}_c^E \bar{x}^h \alpha_{c,j}^{\bar{x}}.$$

Proposition 4.3. *The K -vector space $\widehat{U}_{f(c)}$ is a left $L_K(E)$ -module.*

Proof. The set of paths \mathcal{P}_c^E is a disjoint union

$$\{\mu \in \mathcal{P}_c^E : |\mu| \leq |c|\} \sqcup \{\mu \in \mathcal{P}_c^E : |\mu| = |c| + 1\} \sqcup \{\mu \in \mathcal{P}_c^E : |\mu| \geq |c| + 2\},$$

which we denote respectively by $\mathcal{P}_{\leq |c|}^E$, $\mathcal{P}_{=|c|+1}^E$, and $\mathcal{P}_{\geq |c|+2}^E$. This disjoint union induces a K -vector space decomposition $\widehat{U}_{f(c)} = A \oplus B \oplus C$, where

$$A := \bigoplus_{j \in \mathbb{I}_c} \sum_{h=0}^{\deg_c f(x)-1} K\mathcal{P}_{\leq |c|}^E \bar{x}^h \alpha_{c,j}^{\bar{x}}, \quad B := \bigoplus_{j \in \mathbb{I}_c} \sum_{h=0}^{\deg_c f(x)-1} K\mathcal{P}_{=|c|+1}^E \bar{x}^h \alpha_{c,j}^{\bar{x}},$$

$$\text{and } C := \bigoplus_{j \in \mathbb{I}_c} \sum_{h=0}^{\deg_c f(x)-1} K\mathcal{P}_{\geq |c|+2}^E \bar{x}^h \alpha_{c,j}^{\bar{x}}.$$

We note that $A \oplus B \subseteq U_{f(c)}$. Additionally, for $\mu \in \mathcal{P}_{=|c|+1}^E \sqcup \mathcal{P}_{\geq |c|+2}^E := \mathcal{P}_{\geq |c|+1}^E$ and $e \in E_1$ we have either $e\mu = 0$ or, having $r(e) = s(\mu)$, $e\mu \in \mathcal{P}_c^E$, more specifically, $e\mu \in \mathcal{P}_{\geq |c|+2}^E$.

So each element of $\widehat{U}_{f(c)}$ is a finite sum of elements of the form

$$\sum_{\mu_{h,j} \in \mathcal{P}_c^E} \mathfrak{p}(\mu_{h,j}) \mu_{h,j} \bar{x}^h \alpha_{c,j}^{\bar{x}} \in A \oplus B \oplus C, \quad \text{written as}$$

$$\sum_{\mu_{h,j} \in \mathcal{P}_{\leq |c|}^E} \mathfrak{p}(\mu_{h,j}) \mu_{h,j} \bar{x}^h \alpha_{c,j}^{\bar{x}} + \sum_{\mu_{h,j} \in \mathcal{P}_{=|c|+1}^E} \mathfrak{p}(\mu_{h,j}) \mu_{h,j} \bar{x}^h \alpha_{c,j}^{\bar{x}} + \sum_{\mu_{h,j} \in \mathcal{P}_{\geq |c|+2}^E} \mathfrak{p}(\mu_{h,j}) \mu_{h,j} \bar{x}^h \alpha_{c,j}^{\bar{x}}$$

where $0 \leq h < \deg f(x)$, $j \in F_{h,c}$ for some finite subset $F_{h,c}$ of \mathbb{I}_c , and $\mathfrak{p}(\mu_{h,j}) \in K$ for all $\mu_{h,j} \in \mathcal{P}_c^E$.

For $v \in E^0$ we define $P_v : \widehat{U}_{f(c)} \rightarrow \widehat{U}_{f(c)}$ by setting

$$P_v \left(\sum_{\mu_{h,j} \in \mathcal{P}_c^E} \mathfrak{p}(\mu_{h,j}) \mu_{h,j} \bar{x}^h \alpha_{c,j}^{\bar{x}} \right) = \sum_{\substack{\mu_{h,j} \in \mathcal{P}_c^E \\ s(\mu_{h,j})=v}} \mathfrak{p}(\mu_{h,j}) \mu_{h,j} \bar{x}^h \alpha_{c,j}^{\bar{x}}.$$

Let \cdot denote the left $L_K(E)$ -action on $U_{f(c)}$, and let ι denote the inclusion map from $U_{f(c)}$ to $\widehat{U}_{f(c)}$.

For $e \in E^1$ we define $S_e : \widehat{U}_{f(c)} \rightarrow \widehat{U}_{f(c)}$ as follows. We define two maps: first, $S_{e,A \oplus B} : A \oplus B \rightarrow \widehat{U}_{f(c)}$ by setting

$$S_{e,A \oplus B}(a + b) = \iota(e \cdot (a + b)),$$

and second, $S_{e,B \oplus C} : B \oplus C \rightarrow \widehat{U}_{f(c)}$ by setting

$$S_{e,B \oplus C} \left(\sum_{\mu_{h,j} \in \mathcal{P}_{\geq |c|+1}^E} \mathfrak{p}(\mu_{h,j}) \mu_{h,j} \bar{x}^h \alpha_{c,j}^{\bar{x}} \right) = \sum_{\substack{\mu_{h,j} \in \mathcal{P}_{\geq |c|+1}^E \\ r(e)=s(\mu_{h,j})}} \mathfrak{p}(\mu_{h,j}) e \mu_{h,j} \bar{x}^h \alpha_{c,j}^{\bar{x}}.$$

By Remark 3.15, $S_{e,A \oplus B}$ and $S_{e,B \oplus C}$ coincide on B . So taken together these two maps yield $S_e : \widehat{U}_{f(c)} \rightarrow \widehat{U}_{f(c)}$. We note that $S_e(A) \subseteq A \oplus B$ and $S_e(B \oplus C) \subseteq C$.

For $f \in E^1$ we define $S_{f^*} : \widehat{U}_{f(c)} \rightarrow \widehat{U}_{f(c)}$ as follows. Let $\mathcal{P}_c^E(f)$ denote $\{\mu \in \mathcal{P}_c^E \mid \mu = f\mu'\}$ for some $\mu' \in \text{Path}(E)$; note that in this case necessarily $\mu' \in \mathcal{P}_c^E$. We define two maps: first, $S_{f^*,A \oplus B} : A \oplus B \rightarrow \widehat{U}_{f(c)}$ by setting

$$S_{f^*,A \oplus B}(a + b) = \iota(f^* \cdot (a + b)),$$

and second, $S_{f^*,B \oplus C} : B \oplus C \rightarrow \widehat{U}_{f(c)}$ by setting

$$S_{f^*,B \oplus C} \left(\sum_{\mu_{h,j} \in \mathcal{P}_{\geq |c|+1}^E} \mathfrak{p}(\mu_{h,j}) \mu_{h,j} \bar{x}^h \alpha_{c,j}^{\bar{x}} \right) = \sum_{\substack{\mu_{h,j} \in \mathcal{P}_{\geq |c|+1}^E(f) \\ \mu_{h,j}=f\mu'_{h,j}}} \mathfrak{p}(\mu_{h,j}) \mu'_{h,j} \bar{x}^h \alpha_{c,j}^{\bar{x}}.$$

Again using Remark 3.15, $S_{f^*,A \oplus B}$ and $S_{f^*,B \oplus C}$ coincide on B . So taken together these two maps yield $S_{f^*} : \widehat{U}_{f(c)} \rightarrow \widehat{U}_{f(c)}$. We note that $S_{f^*}(A \oplus B) \subseteq A$.

Claim: The subset $\{P_v, S_e, S_{e^*} \mid v \in E^0, e \in E^1\}$ forms a Cuntz Krieger E -family in $\text{End}_K(\widehat{U}_{f(c)})$.

Proof of Claim: We suppress the composition operator \circ in the following equations; δ denotes the Kronecker delta. By definition it must be checked that:

- (1) $P_v P_w = \delta_{v,w} P_v$ for all $v, w \in E^0$;
- (2) $P_{s(e)} S_e = S_e = S_e P_{r(e)}$ for all $e \in E^1$;
- (3) $P_{r(e)} S_{e^*} = S_{e^*} = S_{e^*} P_{s(e)}$ for all $e \in E^1$;
- (4) $S_{f^*} S_e = \delta_{e,f} P_{r(e)}$ for all $e, f \in E^1$; and
- (5) $\sum_{e \in s^{-1}(v)} S_e S_{e^*} = P_v$ for all non-sink $v \in E^0$.

Statements (1), (2), and (3) are straightforward to check. We give now the key details of the verification of statement (4). We use the decomposition of $\widehat{U}_{f(c)}$ given at the start of the proof of the proposition.

Proof of (4). Let $a \in A$. Recall that $S_e(a) \in A \oplus B$. Then

$$\begin{aligned} S_{f^*}S_e(a) &= S_{f^*}(e \cdot a) = S_{f^*, A \oplus B}(e \cdot a) = f^* \cdot (e \cdot a) \\ &= (f^*e) \cdot a = \delta_{e,f}P_r(e) \cdot a = \delta_{e,f}P_{r(e)}(a), \end{aligned}$$

so that $S_{f^*}S_e = \delta_{e,f}P_{r(e)}$ on A .

Now let $y \in B \oplus C$. Then y is a finite sum of terms of the form $\sum_{\mu_{h,j} \in \mathcal{P}_{\geq |c|+1}^E} \mathfrak{p}(\mu_{h,j})\mu_{h,j}\bar{x}^h\alpha_{c,j}^{\bar{x}}$, where $0 \leq h < \deg f(x)$, $j \in F_{h,c}$ where $F_{h,c}$ is some finite subset of \mathbb{I}_c , and $\mathfrak{p}(\mu_{h,j}) \in K$ for all $\mu_{h,j} \in \mathcal{P}_{\geq |c|+1}^E$. Recall that $S_e(y) \in C$. Then

$$\begin{aligned} S_e(y) &= S_{e, B \oplus C}(y) = S_{e, B \oplus C} \left(\sum_{\mu_{h,j} \in \mathcal{P}_{\geq |c|+1}^E} \mathfrak{p}(\mu_{h,j})\mu_{h,j}\bar{x}^h\alpha_{c,j}^{\bar{x}} \right) \\ &= \sum_{\substack{\mu_{h,j} \in \mathcal{P}_{\geq |c|+1}^E \\ r(e)=s(\mu_{h,j})}} \mathfrak{p}(\mu_{h,j})e\mu_{h,j}\bar{x}^h\alpha_{c,j}^{\bar{x}}. \end{aligned}$$

If $f \neq e$, then $\{e\mu_{h,j} \mid r(e) = s(\mu_{h,j})\} \cap \mathcal{P}_{\geq |c|+1}^E(f) = \emptyset$ by definition, so that

$$S_{f^*}(S_e(y)) = S_{f^*, B \oplus C} \left(\sum_{\substack{\mu_{h,j} \in \mathcal{P}_{\geq |c|+1}^E \\ r(e)=s(\mu_{h,j})}} \mathfrak{p}(\mu_{h,j})e\mu_{h,j}\bar{x}^h\alpha_{c,j}^{\bar{x}} \right) = 0.$$

On the other hand, since $S_e(y) \in C$, we have

$$\begin{aligned} S_{e^*}(S_e(y)) &= S_{e^*, B \oplus C} \left(\sum_{\substack{\mu_{h,j} \in \mathcal{P}_{\geq |c|+1}^E \\ r(e)=s(\mu_{h,j})}} \mathfrak{p}(\mu_{h,j})e\mu_{h,j}\bar{x}^h\alpha_{c,j}^{\bar{x}} \right) \\ &= \sum_{\substack{\mu_{h,j} \in \mathcal{P}_{\geq |c|+1}^E \\ r(e)=s(\mu_{h,j})}} \mathfrak{p}(\mu_{h,j})\mu_{h,j}\bar{x}^h\alpha_{c,j}^{\bar{x}} = P_{r(e)}(y). \end{aligned}$$

Thus we have that $S_{f^*}S_e = \delta_{e,f}P_{r(e)}$ on $B \oplus C$ as well. We conclude that $S_{f^*}S_e = \delta_{e,f}P_{r(e)}$ on all of $\widehat{U}_{f(c)}$, as desired.

Proof of (5). This follows in a manner similar to the proof of (4), and is left to the reader. The key observation here is that for each non-sink $v \in E^0$ we have $\{\mu \in \mathcal{P}_c^E \mid s(\mu) = v\} = \bigsqcup_{e \in s^{-1}(v)} \mathcal{P}_c^E(e)$. (Less formally: every path in E having source vertex v must have as its first edge one of the finitely many elements of $s^{-1}(v)$.)

With the Claim established, we invoke the universal property of $L_K(E)$ to conclude there exists a K -algebra homomorphism

$$\Phi : L_K(E) \rightarrow \text{End}_K(\widehat{U}_{f(c)})$$

for which $\Phi(v) = P_v$, $\Phi(e) = S_e$, and $\Phi(e^*) = S_{e^*}$ for all $v \in E^0, e \in E^1$.

Since every path in E has source vertex in E^0 , it is clear that $1_{\text{End}_K(\widehat{U}_{f(c)})} = \sum_{v \in E^0} P_v$. Thus $\Phi(1_{L_K(E)}) = \Phi(\sum_{v \in E^0} v) = 1_{\text{End}_K(\widehat{U}_{f(c)})}$.

With the above discussion and computations in hand, by a standard ring-theory argument we conclude that $\widehat{U}_{f(c)}$ admits a left $L_K(E)$ -module action (which we appropriately denote by \cdot) by setting

$$r \cdot u = \Phi(r)(u)$$

for all $r \in L_K(E)$ and all $u \in \widehat{U}_{f(c)}$. Finally, by again using Remark 3.15, we have that this $L_K(E)$ -action on $\widehat{U}_{f(c)}$ extends the $L_K(E)$ -action on $U_{f(c)}$, i.e., that $U_{f(c)}$ is an $L_K(E)$ -submodule of $\widehat{U}_{f(c)}$. \square

Referring to the Fig. 1, the modules \widehat{U}_{w-1} , $\widehat{U}_{f(g_1g_2g_3)}$, and $\widehat{U}_{f(\ell)}$ contain proper formal series (and thus properly contain U_{w-1} , $U_{f(g_1g_2g_3)}$, and $U_{f(\ell)}$, respectively). On the other hand, $\widehat{U}_{f(d_1d_2d_3d_4)} = U_{f(d_1d_2d_3d_4)}$.

Modules of the form $\widehat{U}_{f(c)}$ will play a central role in our analysis, as these will be shown to be the injective envelopes of the Chen simple modules of the form $V_{[c^\infty]}^f$. We first check that $\widehat{U}_{f(c)}$ is an essential extension of $V_{[c^\infty]}^f$.

Proposition 4.4. *Let E be a finite graph with disjoint cycles. For any cycle c , the simple left $L_K(E)$ -module $V_{[c^\infty]}^f$ is essential in $\widehat{U}_{f(c)}$.*

Proof. We establish that $U_{f(c)}$ is essential in $\widehat{U}_{f(c)}$; the result will then follow directly from Proposition 3.14, which in particular implies that $V_{[c^\infty]}^f$ is essential in $U_{f(c)}$.

By definition, any element in $\widehat{U}_{f(c)}$ is of the form

$$\sum_{j \in F_1} \sum_{h=0}^{\deg_c f(x)-1} \left(\sum_{\gamma \in \mathcal{P}_c^E} k_{j,\gamma,h} \gamma \right) \bar{x}^h \alpha_{c,j}^{\bar{x}}$$

for a suitable (finite) subset F_1 of \mathbb{I}_c . Assume this element is not equal to zero. Then there exist $j_0 \in F_1$, $\gamma_0 \in \mathcal{P}_c^E$, and $0 \leq h_0 < \deg_c f(x)$ such that $k_{j_0,\gamma_0,h_0} \neq 0$. By Remark 3.3, the paths in \mathcal{P}_c^E do not end in any closed path. So by Lemma 2.2 we have

$$\gamma_0^* \sum_{j \in F_1} \sum_{h=0}^{\deg_c f(x)-1} \sum_{\gamma \in \mathcal{P}_c^E} k_{j,\gamma} \bar{x}^h \alpha_{c,j}^{\bar{x}} = \sum_{j \in F_1} \sum_{h=0}^{\deg_c f(x)-1} k_{j,\gamma_0,h} \bar{x}^h \alpha_{c,j}^{\bar{x}}$$

The latter is a nonzero element in $U_{f(c)}$ since by Lemma 3.13 the set $\{\bar{x}^h \alpha_{c,j}^{\bar{x}} \mid j \in F_1, 0 \leq h < \deg_c f(x)\}$ is K -linearly independent, and $k_{j_0,\gamma_0,h_0} \neq 0$. \square

5. The main result

The aim of this paper is to explicitly construct the injective envelope of all simple modules over any Leavitt path algebra $L_K(E)$ associated to a finite graph E with disjoint cycles. A first important step has already been established. Using a number of powerful tools (e.g., pure injectivity, and the Bézout property of Leavitt path algebras), the three authors proved the following.

Theorem 5.1. [3, Theorem 6.4] *Let E be any finite graph, and let c be a proper cycle in E . Then the Prüfer module $U_{E,c-1}$ is injective if and only if c is a maximal cycle (i.e., there are no cycles in E other than cyclic shifts of c which connect to $s(c)$).*

Referring to the Fig. 1, $U_{E,c-1}$ is injective if and only if $c = d_1d_2d_3d_4$.

With Theorem 5.1 and the results established in the previous sections in hand, we are now in position to state the main theorem of the article.

Theorem 5.2. *Let K be any field, and let E be any finite graph with disjoint cycles. Let c be a cycle in E , and let $f(x) \in K[x]$ be a basic irreducible polynomial in $K[x]$. Then the injective envelope of $V_{[c^\infty]}^f$ is the left $L_K(E)$ -module $\widehat{U}_{f(c)}$.*

In particular, if $c = w$ is a sink, then the injective envelope of $V_{[w^\infty]} = L_K(E)w$ is the left $L_K(E)$ -module $\widehat{U}_{w-1} = K[[E]]w$.

Observe that for a proper cycle c , \mathcal{P}_c^E is finite if and only if c is a maximal cycle. In such a situation we get $U_{f(c)} = \widehat{U}_{f(c)}$. Thus Theorem 5.2 may be viewed as a significant extension of Theorem 5.1 for graphs with disjoint cycles, as it provides the injective envelopes of simples associated to all sinks, and to all cycles (not only the maximal ones) together with appropriate irreducible polynomials of $K[x]$.

Remark 5.3. As mentioned in the Introduction, by [7, Theorem 1.1], when E has disjoint cycles then the Chen simple modules represent all the simple $L_K(E)$ -modules. So, once the proof of Theorem 5.2 is completed, we will get as a consequence an explicit description of an injective cogenerator for $L_K(E)$ -Mod, namely, the direct product of the injective envelopes of all the simple modules.

We will present the proof of Theorem 5.2 below. Here is an overview of how we will proceed. In Proposition 4.4 we have already checked that $V_{[c^\infty]}^f$ is an essential submodule of $\widehat{U}_{f(c)}$. In Proposition 5.4 we reduce the task of proving the injectivity of $\widehat{U}_{f(c)}$ for any basic irreducible $f(x) \in K[x]$ to the specific polynomial $f(x) = x - 1$. To do so, we will use that $\widehat{U}_{f(c)}$ is obtained from the left $L_{K'}(E)$ -module $\widehat{U}_{\bar{x}^{-1}c-1}$ by applying the functor \mathcal{U} which restricts the scalars from K' to K .

Then, to prove the injectivity of \widehat{U}_{c-1} , for any cycle c in E , we reduce first to connected graphs, then reduce to graphs which contain no source vertices, and finally reduce to graphs in which every source cycle is a loop.

After these four reductions, the core of the proof is based on an induction argument on the number of cycles in the graph E . It is here that the main ideas come to light.

Proposition 5.4. *Let c be a proper cycle in E and $f(x) \in K[x]$ a basic irreducible polynomial. Denote by K' the field $K[x]/\langle f(x) \rangle$. If \widehat{U}_{c-1} is an injective left $L_{K'}(E)$ -module, then $\widehat{U}_{f(c)}$ is an injective left $L_K(E)$ -module.*

Proof. Given $0 \neq a \in K$, we first consider the case $f(x) = a^{-1}x - 1$. In this situation, since $\deg(f(x)) = 1$ we get $K = K'$. Then $\widehat{U}_{f(c)}$ is the $L_K(E)$ -module $\widehat{U}_{a^{-1}c-1}$, which is the twisted version of the $L_K(E)$ -module \widehat{U}_{c-1} . Since twisting is an autoequivalence of $L_K(E)$ -Mod, $\widehat{U}_{a^{-1}c-1}$ is injective.

If $f(x)$ is a basic irreducible polynomial of degree > 1 , writing $f(x) = xg(x) - 1$ we get from the previous paragraph that $\widehat{U}_{g(\bar{x})c-1}$ is an injective left $L_{K'}(E)$ -module. Now consider the left $L_K(E)$ -module $\widehat{U}_{f(c)} = \mathcal{U}(\widehat{U}_{g(\bar{x})c-1})$. By [14, Lemma 12.29.1], since \mathcal{U} is right adjoint to

$$- \otimes_K K' : L_K(E)\text{-Mod} \rightarrow L_{K'}(E)\text{-Mod},$$

and the latter is exact, \mathcal{U} transforms injectives into injectives. \square

The next three lemmas will permit us to reduce our study to connected graphs having no source vertices and in which the source cycles are loops.

Lemma 5.5. *Let E be a finite graph and $E_i, i = 1, 2, \dots, m$, its connected components. The Leavitt path algebra associated to E decomposes as the direct sum of two-sided ideals, each of which is the Leavitt path algebra associated to a connected component of E :*

$$L_K(E) = \bigoplus_{i=1}^m L_K(E_i).$$

For any cycle c in E_1 , we have

$$\widehat{U}_{E,c-1} = \widehat{U}_{E_1,c-1}.$$

Moreover, let J be an ideal of $L_K(E)$ and ρ_1 be the sum of all vertices in E_1 . We have $J = \rho_1 J \oplus (1 - \rho_1)J$ and $\varphi((1 - \rho_1)J) = 0$ for each morphism $\varphi : J \rightarrow \widehat{U}_{E,c-1} = \widehat{U}_{E_1,c-1}$.

Proof. For the decomposition $L_K(E) = \bigoplus_{i=1}^m L_K(E_i)$ see [2, Proposition 1.2.14]. Clearly $\mathcal{P}_c^E = \mathcal{P}_c^{E_1}$. Therefore we get $\widehat{U}_{E,c-1} = \widehat{U}_{E_1,c-1}$. Finally, for any

$$\varphi : J \rightarrow \widehat{U}_{E,c-1} = \widehat{U}_{E_1,c-1}$$

we have

$$\varphi((1 - \rho_1)J) = (1 - \rho_1)\varphi(J) \subseteq (1 - \rho_1)\widehat{U}_{E_1,c-1}.$$

Since $(1 - \rho_1)\mathcal{P}_c^{E_1} = 0$ we get $\varphi((1 - \rho_1)J) = 0$. \square

The reduction to graphs having no source vertices and in which the source cycles are loops will be achieved using relatively standard equivalence functors between categories of modules over Leavitt path algebras associated to general graphs, and categories of modules over Leavitt path algebras associated to graphs having no source vertices and all source cycles being loops. Of course such equivalences will preserve various homological properties, including injectivity.

However, we need more. The modules $\widehat{U}_{E,c-1}$ are built in a specific way from the data corresponding to the graph E and cycle c . So if, for example, F is a subgraph of E (or F is otherwise built from E) which contains the cycle c , and if $T : L_K(F)\text{-Mod} \rightarrow L_K(E)\text{-Mod}$ is an equivalence of categories, and if $\widehat{U}_{F,c-1}$ has been shown to be injective as an $L_K(F)$ -module, then certainly $T(\widehat{U}_{F,c-1})$ is an injective $L_K(E)$ -module. But it is not at all immediate from purely categorical considerations that

$$T(\widehat{U}_{F,c-1}) \cong \widehat{U}_{E,c-1}.$$

Fortunately, the displayed isomorphism does indeed hold for each of the equivalence functors that we will utilise.

Lemma 5.6. *Let E be a finite graph with disjoint cycles, and \bar{u} a source vertex in E (see Fig. 2). Let H denote the hereditary subset $E^0 \setminus \{\bar{u}\}$ of E^0 . Setting $\varepsilon = \sum_{u \in E^0, u \neq \bar{u}} u$, there is a Morita equivalence*

$$L_K(E)\text{-Mod} \begin{array}{c} \xrightarrow{\varepsilon L_K(E) \otimes_{L_K(E)} -} \\ \xleftarrow{L_K(E)\varepsilon \otimes_{L_K(E)\varepsilon} -} \end{array} L_K(E_H)\text{-Mod} = \varepsilon L_K(E)\varepsilon\text{-Mod}.$$

Moreover, for any cycle c of E , necessarily c is in E_H , and we have

$$\widehat{U}_{E,c-1} \cong L_K(E)\varepsilon \otimes_{L_K(E)\varepsilon} \widehat{U}_{E_H,c-1}.$$

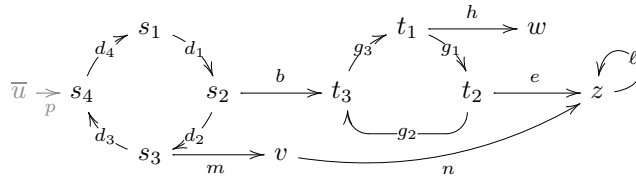


Fig. 2. Eliminating source vertices.

Proof. That the indicated functors give a Morita equivalence has been proved in [7, Lemma 4.3]. Let c be a cycle in E . So

$$\mathcal{P}_c^E = \varepsilon \mathcal{P}_c^E \cup \bar{u} \mathcal{P}_c^E = \mathcal{P}_c^{E_H} \cup \bar{u} \mathcal{P}_c^E.$$

Any element γ in $\bar{u} \mathcal{P}_c^E$ has the form $\gamma = \gamma_1 \varepsilon \gamma_2 \cdots \gamma_\ell$, where $s(\gamma_1) = \bar{u}$ and $r(\gamma_\ell) \in H$. The map

$$K \mathcal{P}_c^E \alpha_{c,j} \rightarrow L_K(E) \varepsilon \otimes_{\varepsilon L_K(E) \varepsilon} K \varepsilon \mathcal{P}_c^E \alpha_{c,j} = L_K(E) \varepsilon \otimes_{\varepsilon L_K(E) \varepsilon} K \mathcal{P}_c^{E_H} \alpha_{c,j}$$

defined by $\gamma \alpha_{c,j} \mapsto \varepsilon \otimes \gamma \alpha_{c,j}$ for each $\gamma \in \varepsilon \mathcal{P}_c^E$, and $\gamma \alpha_{c,j} \mapsto \gamma_1 \otimes \gamma_2 \cdots \gamma_\ell \alpha_{c,j}$ for each $\gamma = \gamma_1 \varepsilon \gamma_2 \cdots \gamma_\ell \in \bar{u} \mathcal{P}_c^E$, induces an isomorphism

$$\begin{array}{ccc} \widehat{U}_{E,c-1} & \xrightarrow{\quad\quad\quad} & L_K(E) \varepsilon \otimes_{\varepsilon L_K(E) \varepsilon} \widehat{U}_{E_H,c-1} \\ \parallel & & \parallel \\ \bigoplus_{j \in \mathbb{I}_c} K[[\mathcal{P}_c^E]] \alpha_{c,j} & & L_K(E) \varepsilon \otimes_{\varepsilon L_K(E) \varepsilon} \left(\bigoplus_{j \in \mathbb{I}_c} K[[\mathcal{P}_c^{E_H}]] \alpha_{c,j} \right) \end{array}$$

(where the direct sums are as K -vector spaces), thus establishing the result. \square

Assume now that E contains a source cycle $d = d_1 \cdots d_r$ which is not a loop. By [7, Lemma 4.4] a finite graph $F_{E,d}$ can be constructed from E in which the cycle d (and all its vertices) is replaced by a loop in such a way that $L_K(E)$ and $L_K(F_{E,d})$ are Morita equivalent. More precisely, let d^0 denote the set of vertices $\{s(d_1), \dots, s(d_r)\}$, and define $F_{E,d}^0 = \{\bar{v}\} \cup (E^0 \setminus d^0)$ where \bar{v} is a new vertex. Then to define $F_{E,d}^1$, we first set $s_{F_{E,d}^1}^{-1}(u) = s_E^{-1}(u)$ for each $u \in E^0 \setminus d^0$; for each edge f with $s(f) \in d^0$ and $r(f) \in E^0 \setminus d^0$, define an edge $\varphi(f)$ with $s_{F_{E,d}^1}(\varphi(f)) = \bar{v}$ and $r_{F_{E,d}^1}(\varphi(f)) = r_E(f)$. Finally, we define a loop d' at \bar{v} so that $s_{F_{E,d}^1}(d') = \bar{v} = r_{F_{E,d}^1}(d')$. Observe that there are no edges connecting any pair of vertices in d^0 other than the edges d_1, \dots, d_r .

Lemma 5.7. *Let $d = d_1 \cdots d_r$ be a source cycle in the graph E with $r \geq 2$. Let $F_{E,d}$ be the graph described in the previous paragraph, in which d is replaced by the loop d' . Then there are inverse equivalence functors*

$$L_K(E)\text{-Mod} \begin{array}{c} \xrightarrow{G_1} \\ \xleftarrow{G_2} \end{array} L_K(F_{E,d})\text{-Mod}.$$

Moreover, $\widehat{U}_{E,c-1} \cong G_2 \left(\widehat{U}_{F_{E,d},c-1} \right)$ for any cycle c in E .

Proof. We describe how the functors G_1 and G_2 work. (The details are given in [7, Lemma 4.4].) Consider the map $\theta : L_K(F_{E,d}) \rightarrow L_K(E)$ defined on vertices by:

$$\theta(u) = u \text{ for } u \in F_{E,d}^0 \setminus \{\bar{v}\};$$

$$\theta(\bar{v}) = s_E(d);$$

and defined on edges by:

$$\theta(e) = e \text{ for } e \text{ having } s_{F_{E,d}}(e) \in F_{E,d}^0 \setminus \{\bar{v}\};$$

$$\theta(\varphi(f)) = d_1 \cdots d_{i-1}f \text{ for } f \text{ having } s_E(f) = s(d_i) \text{ and } r_E(f) \in E^0 \setminus d^0;$$

$$\theta(d') = d = d_1 \cdots d_r.$$

By [7, Lemma 4.4] the map θ extends to a well-defined K -algebra isomorphism

$$\theta : L_K(F_{E,d}) \rightarrow \omega L_K(E)\omega,$$

where $\omega := s(d_1) + \sum_{u \in E^0 \setminus d^0} u$. We denote by $\hat{\theta}$ the induced natural isomorphism of categories

$$\hat{\theta} : L_K(F_{E,d})\text{-Mod} \rightarrow \omega L_K(E)\omega\text{-Mod}.$$

The ring $\omega L_K(E)\omega$ is easily seen to be a full corner of $L_K(E)$ and so is Morita equivalent to $L_K(E)$. The functors G_1 and G_2 which realise the Morita equivalence between $L_K(E)\text{-Mod}$ and $L_K(F_{E,d})\text{-Mod}$ are given by the following compositions

$$L_K(E)\text{-Mod} \begin{array}{c} \xrightarrow{\omega L_K(E) \otimes_{L_K(E)} -} \\ \xleftarrow{L_K(E) \omega \otimes_{\omega L_K(E) \omega} -} \end{array} \omega L_K(E)\omega\text{-Mod} \begin{array}{c} \xrightarrow{\hat{\theta}^{-1}} \\ \xleftarrow{\hat{\theta}} \end{array} L_K(F_{E,d})\text{-Mod} .$$

Now let c be a cycle in E (possibly equal to d). Then

$$\mathcal{P}_c^E = \omega \mathcal{P}_c^E \cup (1 - \omega) \mathcal{P}_c^E.$$

The map

$$U_{E,c-1} = \bigoplus_{j \in \mathbb{I}_c} K \mathcal{P}_c^E \alpha_{c,j} \rightarrow L_K(E)\omega \otimes_{\omega L_K(E)\omega} \left(\bigoplus_{j \in \mathbb{I}_c} K \omega \mathcal{P}_c^E \alpha_{c,j} \right)$$

defined by

$$\rho \alpha_{c,j} \mapsto \omega \otimes \rho \alpha_{c,j} \text{ for each } \rho \in \omega \mathcal{P}_c^E$$

and by

$$s(d_i) p \alpha_{c,j} \mapsto d_{i-1}^* \cdots d_1^* \otimes d_1 \cdots d_{i-1} s(d_i) p \alpha_{c,j} \text{ for each } s(d_i) p \in (1 - \omega) \mathcal{P}_c^E$$

extends to an isomorphism

$$\hat{U}_{E,c-1} = \bigoplus_{j \in \mathbb{I}_c} K [[\mathcal{P}_c^E]] \alpha_{c,j} \rightarrow L_K(E)\omega \otimes_{\omega L_K(E)\omega} \left(\bigoplus_{j \in \mathbb{I}_c} K [[\omega \mathcal{P}_c^E]] \alpha_{c,j} \right).$$

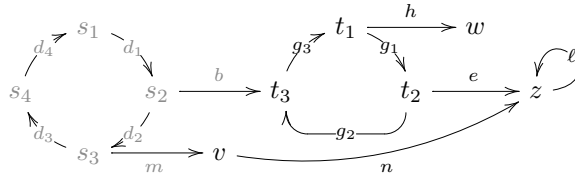
Since $\theta(\mathcal{P}_c^{F_{E,d}}) = \omega \mathcal{P}_c^E$, we have $\hat{\theta}(\hat{U}_{F_{E,d},c-1}) \cong \bigoplus_{j \in \mathbb{I}_c} K [[\omega \mathcal{P}_c^E]] \alpha_{c,j}$, and hence

$$G_2 \left(\hat{U}_{F_{E,d},c-1} \right) = L_K(E)\omega \otimes_{\omega L_K(E)\omega} \hat{\theta}(\hat{U}_{F_{E,d},c-1}) \cong \hat{U}_{E,c-1},$$

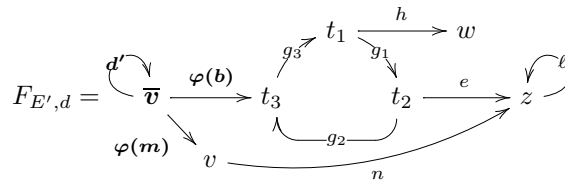
as desired. \square

In the following example we clarify in a concrete situation how the isomorphisms described in Lemma 5.7 work.

Example 5.8. Let E' be the graph



and



the graph in which the cycle $d_1d_2d_3d_4$ is substituted by the loop d' . Denote by ω the idempotent $s_1 + \sum_{u \in E'^0 \setminus d^0} u$ of $L_K(E')$, and by c either one of the cycles $g_1g_2g_3$ and ℓ , or the sink w . In the isomorphism

$$\widehat{U}_{E',c-1} \cong L_K(E')\omega \otimes_{\omega L_K(E')\omega} \widehat{\theta}(\widehat{U}_{F_{E',d},c-1}),$$

for any $\bar{j} \in \mathbb{I}_c$ the element

$$\sum_{i=0}^{\infty} d_4 d^i d_1 b \gamma_i \alpha_{c,\bar{j}}$$

of $\widehat{U}_{E',c-1}$ corresponds to the element

$$d_3^* d_2^* d_1^* \otimes \sum_{i=0}^{\infty} \theta((d')^{i+1} \varphi(b) \gamma_i) \alpha_{c,\bar{j}} = d_3^* d_2^* d_1^* \otimes \sum_{i=0}^{\infty} d^{i+1} d_1 b \gamma_i \alpha_{c,\bar{j}}$$

of $L_K(E')\omega \otimes_{\omega L_K(E')\omega} \left(\bigoplus_{j \geq 1} K[[\omega \mathcal{P}_c^{E'}]] \alpha_{c,j}\right)$.

We are now ready to give the

Proof of Theorem 5.2. Recall that we consider any sink $w \in E^0$ as a cycle of length 0.

The essentiality of $V_{[c^\infty]}^f$ in $\widehat{U}_{f(c)}$ has been shown in Proposition 4.4. So we need only demonstrate that $\widehat{U}_{f(c)}$ is injective for each cycle c and each basic irreducible polynomial $f(x) \in K[x]$.

Given any cycle c , by Lemma 5.5 we can establish the injectivity of $\widehat{U}_{f(c)}$ in the Leavitt path algebra associated to the connected component containing c . By Lemmas 5.6 and 5.7, we can assume that the connected component containing c has no source vertices, and that all its source cycles are source loops. Moreover by Proposition 5.4, it suffices to show the result in the case $f(x) = x - 1$, and hence we may assume $\widehat{U}_{f(c)} = \widehat{U}_{c-1}$.

We proceed by induction on the number N of cycles in the connected component of E containing the cycle c . We continue to call this connected component E .

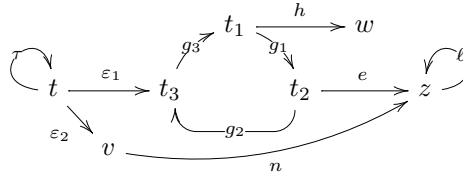


Fig. 3. Graph with disjoint cycles, without source vertices, whose source cycle is a loop.

If $N = 0$, then by the reduction assumptions the graph E has one vertex and no edges. So $L_K(E) \cong K$, and therefore all left $L_K(E)$ -modules are injective.

If $N = 1$, either E consists of one vertex u and one loop at u or, invoking Remark 2.4 and the assumption that $N = 1$ the graph E contains a source loop τ and at least two vertices. In the first case by Theorem 5.1 we conclude that $U_{c-1} = \widehat{U}_{c-1}$ is injective. In the second case, since τ is a maximal cycle, by Theorem 5.1 the Prüfer module $U_{E,\tau-1} = \widehat{U}_{E,\tau-1}$ is injective.

Now let $N > 1$. If τ is a maximal cycle, then again by Theorem 5.1 the Prüfer module $U_{E,\tau-1} = \widehat{U}_{E,\tau-1}$ is injective. So we have to prove that $\widehat{U}_{E,c-1}$ is injective for any non-maximal cycle c . Denote $s(\tau)$ by t . Denote by I the two sided ideal of $L_K(E)$ generated by the hereditary set of vertices $H = E^0 \setminus \{t\}$, and by $\rho \in I$ the idempotent $\sum_{u \in H} u = 1_{L_K(E)} - t$. Moreover, let $s^{-1}(t) = \{\varepsilon_1, \dots, \varepsilon_n\}$ (see, e.g., Fig. 3).

By Proposition 2.1, I is isomorphic to a Leavitt path algebra, and there is an equivalence between the category $I\text{-Mod}$ of (unitary) I -modules and the category $\rho L_K(E)\rho\text{-Mod} = \rho I\rho\text{-Mod}$, induced by the functors

$$\rho I \otimes_I - : I\text{-Mod} \rightarrow \rho L_K(E)\rho\text{-Mod} \quad \text{and}$$

$$I\rho \otimes_{\rho L_K(E)\rho} - : \rho L_K(E)\rho\text{-Mod} \rightarrow I\text{-Mod}.$$

Moreover, the K -algebra $\rho L_K(E)\rho$ is isomorphic to the Leavitt path K -algebra $L_K(E_H)$.

Now let c be any cycle in E , $c \neq \tau$. Then c lives also in E_H . Since each of the connected components of the graph E_H (there might be more than one such component) has fewer cycles than does E , we can apply the inductive hypothesis to conclude that $\widehat{U}_{E_H,c-1}$ is an injective $L_K(E_H)$ -module.

We prove that $\widehat{U}_{E,c-1}$ is injective in $L_K(E)\text{-Mod}$. To do so we use the version of Baer’s Criterion presented in Lemma 1.2, with respect to the ideal I .

So first, let J be a left ideal of $L_K(E)$ contained in I . Since I has local units, we have $IJ = J$ and hence J belongs to $I\text{-Mod}$. Consider an arbitrary $L_K(E)$ -homomorphism

$$\varphi : J \rightarrow \widehat{U}_{E,c-1}.$$

Since $IJ = J$, φ factors through

$$\overline{\varphi} : J \rightarrow I\widehat{U}_{E,c-1}.$$

We show that it is possible to extend φ to I . Clearly $I\widehat{U}_{E,c-1}$ is also a left I -module and $\overline{\varphi}$ is an I -linear map. Applying the functor $\rho I \otimes_I -$ we get the $\rho L_K(E)\rho$ -homomorphism

$$\rho I \otimes_I \overline{\varphi} : \rho I \otimes_I J \rightarrow \rho I \otimes_I I\widehat{U}_{E,c-1}.$$

Claim: $\rho I \otimes_I I\widehat{U}_{E,c-1} \cong \widehat{U}_{E_H,c-1}$ as left $\rho L_K(E)\rho \cong L_K(E_H)$ -modules. **Proof.** Any element in $\rho I \otimes_I I\widehat{U}_{E,c-1}$ is of the form $\rho\iota_1 \otimes \iota_2 z$ where $\iota_1, \iota_2 \in I$ and $z \in \widehat{U}_{E,c-1}$. Denoting by ι the product $\iota_1 \iota_2$ we have

$$\rho\iota_1 \otimes \iota_2 z = \rho^2 \iota_1 \otimes \iota_2 z = \rho \otimes \rho\iota_1 \iota_2 z = \rho \otimes \rho\iota z,$$

and ρz has the form

$$\sum_{j \in F_1} \sum_{\gamma \in \mathcal{P}_c^{EH}} k_{j,\gamma} \rho \iota \gamma \alpha_{c,j} + \sum_{j \in F_2} \sum_{\gamma \in \mathcal{P}_c^{EH}} \sum_{y=1}^n \sum_{x=0}^{\infty} k_{j,\gamma,x,y} \rho \iota \tau^x \varepsilon_y \gamma \alpha_{c,j}.$$

We establish now the key point, which is that if $\iota \in I$ and $z \in I\widehat{U}_{E,c-1}$ then $\rho z \in \widehat{U}_{EH,c-1}$. Note that for each $\mu \in \mathcal{P}_c^E$ we have

$$\rho \mu = \begin{cases} \mu & \text{if } \mu \text{ is a path in } \mathcal{P}_c^{EH}, \\ 0 & \text{if } s(\mu) = t. \end{cases}$$

Separating the paths in \mathcal{P}_c^E starting from the vertex t and those starting from a vertex in H , any element z of $\widehat{U}_{E,c-1}$ has the form (see, e.g., Fig. 3)

$$z = \sum_{j \in F_1} \sum_{\gamma \in \mathcal{P}_c^{EH}} k_{j,\gamma} \gamma \alpha_{c,j} + \sum_{j \in F_2} \sum_{\gamma \in \mathcal{P}_c^{EH}} \sum_{y=1}^n \sum_{x=0}^{\infty} k_{j,\gamma,x,y} \tau^x \varepsilon_y \gamma \alpha_{c,j}$$

for suitable finite subsets F_1 and F_2 of \mathbb{I}_c and $k_{j,\gamma}, k_{j,\gamma,x,y} \in K$. Then

$$\rho z = \sum_{j \in F_1} \sum_{\gamma \in \mathcal{P}_c^{EH}} k_{j,\gamma} \rho \iota \gamma \alpha_{c,j} + \sum_{j \in F_2} \sum_{\gamma \in \mathcal{P}_c^{EH}} \sum_{y=1}^n \sum_{x=0}^{\infty} k_{j,\gamma,x,y} \rho \iota \tau^x \varepsilon_y \gamma \alpha_{c,j}.$$

The left hand double summand is an element of $\widehat{U}_{EH,c-1}$. In the right hand quadruple summand consider each $\rho \iota \tau^x \varepsilon_y \gamma \alpha_{c,j}$. The term $\rho \iota$ is a linear combination of monomials in $L_K(E)$ starting from vertices in H , and γ is an element of \mathcal{P}_c^{EH} . In the linear combination of monomials $\rho \iota$, let $\rho \iota'_{x,y} \varepsilon_y^*(\tau^*)^x$ be the sub-linear combination of those monomials ending with $\varepsilon_y^*(\tau^*)^x$. Since ι is fixed, we have $\iota'_{x,y} \neq 0$ for a finite number of $(x, y) \in \mathbb{N} \times \{1, \dots, n\}$, and, in particular, $\iota'_{x,y} = 0$ for x greater than a suitable $\ell \in \mathbb{N}$. Therefore

$$\sum_{j \in F_2} \sum_{\gamma \in \mathcal{P}_c^{EH}} \sum_{y=1}^n \sum_{x=0}^{\infty} k_{j,\gamma,x,y} \rho \iota \tau^x \varepsilon_y \gamma \alpha_{c,j} = \sum_{j \in F_2} \sum_{\gamma \in \mathcal{P}_c^{EH}} \sum_{y=1}^n \sum_{x=0}^{\ell} k_{j,\gamma,x,y} \rho \iota'_{x,y} \rho \gamma \alpha_{c,j},$$

which is an element of $\widehat{U}_{EH,c-1}$. Thus $\rho z \in \widehat{U}_{EH,c-1}$ as desired.

Now consider the map $(\rho \iota_1, \iota_2 z) \mapsto \rho \iota_1 \iota_2 z$ where $\iota_1, \iota_2 \in I$ and $z \in \widehat{U}_{E,c-1}$. Clearly this is an I -balanced linear map, which with the previously established key point yields a left $\rho L_K(E)\rho$ -module map $\psi : \rho I \otimes_I I\widehat{U}_{E,c-1} \rightarrow \widehat{U}_{EH,c-1}$. But by the induction hypothesis, since E_H has fewer cycles than does E , $\widehat{U}_{EH,c-1}$ is an injective module over the ring $\rho I \rho = \rho L_K(E)\rho \cong L_K(E_H)$.

Since $J \leq I$ and $IJ = J$, applying the exact functor $\rho I \otimes_I -$ we get the solid part of the following commutative diagram of left $\rho L_K(E)\rho$ -modules:

$$\begin{array}{ccc} 0 & \longrightarrow & \rho I \otimes_I J & \longrightarrow & \rho I \otimes_I I \\ & & \rho I \otimes_{\widehat{\varphi}} \downarrow & & \swarrow \psi \\ & & \rho I \otimes_I I\widehat{U}_{E,c-1} \cong \widehat{U}_{EH,c-1} & & \end{array}$$

The existence of the dashed arrow ψ follows by the injectivity of $\widehat{U}_{EH,c-1}$ in $\rho L_K(E)\rho\text{-Mod}$ established above. Applying the equivalence functor $I\rho \otimes_{\rho L_K(E)\rho} -$, we get the following commutative diagram of I -modules:

$$\begin{array}{ccc}
 I\rho \otimes_{\rho L_K(E)\rho} (\rho I \otimes_I J) \cong J & \xrightarrow{\quad} & I\rho \otimes_{\rho L_K(E)\rho} (\rho I \otimes_I I) \cong I \\
 \downarrow I\rho \otimes (\rho I \otimes \bar{\varphi}) \cong \bar{\varphi} & & \swarrow \bar{\psi} \\
 I\rho \otimes_{\rho L_K(E)\rho} (\rho I \otimes_I \widehat{IU}_{E,c-1}) \cong \widehat{IU}_{E,c-1} & &
 \end{array}$$

Moreover, the I -linear map $\bar{\psi}$ is in fact $L_K(E)$ -linear. Indeed, since I has local units, for each $\iota \in I$ there exists $\zeta \in I$ such that $\zeta \iota = \iota$. Then for each $\lambda \in L_K(E)$, since $\lambda \zeta$ belongs to I , we have

$$\bar{\psi}(\lambda \iota) = \bar{\psi}((\lambda \zeta) \iota) = (\lambda \zeta) \bar{\psi}(\iota) = \lambda \cdot \bar{\psi}(\zeta \iota) = \lambda \cdot \bar{\psi}(\iota).$$

Composing $\bar{\psi}$ with the inclusion of $\widehat{IU}_{E,c-1}$ inside $\widehat{U}_{E,c-1}$ we get the desired extension of φ . This completes the first step of the application of Baer’s Criterion Lemma 1.2.

To complete the second step required to apply Lemma 1.2, we have to prove that any morphism of left $L_K(E)$ -modules from a left ideal J' containing I extends to $L_K(E)$. As before, we denote $s^{-1}(t)$ by $\{\tau, \varepsilon_1, \dots, \varepsilon_n\}$.

We first assume $J' = I$. Let

$$\chi : I \rightarrow \widehat{U}_{E,c-1}$$

be a morphism of $L_K(E)$ -modules. We have to extend χ to a morphism $\hat{\chi} : L_K(E) = I + L_K(E)t \rightarrow \widehat{U}_{E,c-1}$. The elements of the intersection $I \cap L_K(E)t$ have the following form:

$$\sum_{i,j} \ell_{i,j}(\varepsilon_i)^*(\tau^*)^j \quad \text{with } \ell_{i,j} \in L_K(E).$$

In particular the restriction of χ to $I \cap L_K(E)t$ is determined by $\chi((\varepsilon_i)^*(\tau^*)^j)$, $i \in \{1, \dots, n\}$, $j \in \mathbb{N}$. Observe that

$$z_{i,j} := \chi((\varepsilon_i)^*(\tau^*)^j) = \chi(\rho(\varepsilon_i)^*(\tau^*)^j) = \rho\chi((\varepsilon_i)^*(\tau^*)^j) \in \rho\widehat{U}_{E,c-1}.$$

Setting

$$t \mapsto \sum_{i=0}^{\infty} \sum_{j=1}^n t \tau^i \varepsilon_j z_{i,j} = \sum_{i=0}^{\infty} \sum_{j=1}^n \tau^i \varepsilon_j z_{i,j},$$

we define a map $\bar{\chi} : L_K(E)t \rightarrow \widehat{U}_{E,c-1}$ whose restriction to $I \cap L_K(E)t$ coincides with χ . Specifically,

$$\begin{aligned}
 \bar{\chi}((\varepsilon_{i_0})^*(\tau^*)^{j_0}) &= \bar{\chi}((\varepsilon_{i_0})^*(\tau^*)^{j_0}t) = ((\varepsilon_{i_0})^*(\tau^*)^{j_0})\bar{\chi}(t) \\
 &= ((\varepsilon_{i_0})^*(\tau^*)^{j_0}) \sum_{i=0}^{\infty} \sum_{j=1}^n \tau^i \varepsilon_j z_{i,j} = z_{i_0,j_0}.
 \end{aligned}$$

Thus we can define

$$\hat{\chi} : I + L_K(E)t = L_K(E) \rightarrow \widehat{U}_{E,c-1}$$

by setting $\hat{\chi}(\iota + \lambda t) := \chi(\iota) + \bar{\chi}(\lambda t)$. This completes the verification of the second step, in the specific case $J' = I$.

Assume now J' is a left ideal of $L_K(E)$ properly containing I . By Proposition 2.1, J' is equal to $L_K(E)p(\tau)$ for a suitable polynomial $p(x) \in K[x]$ with $p(0) = 1$. Consider a map $\theta : J' = L_K(E)p(\tau) \rightarrow \widehat{U}_{E,c-1}$.

Denote by $P(x)$ the formal series in $K[[x]]$ such that $p(x)P(x) = 1$, and write $P(x) = \sum_{i=0}^{\infty} h_i x^i$. As observed previously, any element z of $\widehat{U}_{E,c-1}$ has the form

$$z = \sum_{j \in F_1} \sum_{\gamma \in \rho \mathcal{P}_c^E} k_{j,\gamma} \gamma \alpha_{c,j} + \sum_{j \in F_2} \sum_{\gamma \in \rho \mathcal{P}_c^E} \sum_{y=1}^n \sum_{x=0}^{\infty} k_{j,\gamma,x,y} \tau^x \varepsilon_y \gamma \alpha_{c,j}.$$

Therefore

$$\begin{aligned} P(\tau)z &= \left(\sum_{i=0}^{\infty} h_i \tau^i \right) z = \\ &= \sum_{i=0}^{\infty} \sum_{j \in F_1} \sum_{\gamma \in \rho \mathcal{P}_c^E} k_i k_{j,\gamma} \tau^i \gamma \alpha_{c,j} + \sum_{i=0}^{\infty} \sum_{j \in F_2} \sum_{\gamma \in \rho \mathcal{P}_c^E} \sum_{y=1}^n \sum_{x=0}^{\infty} k_i k_{j,\gamma,x,y} \tau^{x+i} \varepsilon_y \gamma \alpha_{c,j} \end{aligned}$$

is again an element of $\widehat{U}_{E,c-1}$.

Setting $\pi(1) = P(\tau)\theta(p(\tau))$ we get a map $\pi : L_K(E) \rightarrow \widehat{U}_{E,c-1}$ which extends θ . This completes the verification of the second step, in the case $I \subsetneq J'$.

Therefore, by Lemma 1.2, we conclude that $\widehat{U}_{E,c-1}$ is an injective $L_K(E)$ -module, thereby completing the proof of Theorem 5.2. \square

CRediT authorship contribution statement

Gene Abrams: Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Validation, Writing – original draft, Writing – review & editing. **Francesca Mantese:** Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Validation, Writing – original draft, Writing – review & editing. **Alberto Tonolo:** Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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