Minimal silting modules and ring extensions

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Abstract. Ring epimorphisms often induce silting modules and cosilting modules, termed minimal silting or minimal cosilting. The aim of this paper is twofold. Firstly, we determine the minimal tilting and minimal cotilting modules over a tame hereditary algebra. In particular, we show that a large cotilting module is minimal if and only if it has an adic module as a direct summand. Secondly, we discuss the behaviour of minimality under ring extensions. We show that minimal cosilting modules over a commutative noetherian ring extend to minimal cosilting modules along any flat ring epimorphism. Similar results are obtained for commutative rings of small homological dimension.

Keywords. Minimal silting modules. Ring epimorphisms. Ring extensions. Minimal cosilting modules. Tame hereditary algebras.

1 Introduction

Tilting theory and its recent development into silting theory are known to be closely related to localization of rings. For example, every Ore localization $R \hookrightarrow \Sigma^{-1}R$ of a ring R with the property that $\Sigma^{-1}R$ has projective dimension at most one over R gives rise to a tilting module $\Sigma^{-1}R\oplus\Sigma^{-1}R/R$. Of course, such tilting module will often be large, that is, it won't be finitely presented, not even up to equivalence.

More generally, ring epimorphisms with nice homological properties give rise to silting modules. Such modules were introduced in [5] as large analogues of the support τ-tilting modules studied in representation theory and cluster theory. They can be characterized as zero cohomologies of (not necessarily compact) two-term silting complexes.

Building on these connections, it was shown in [6] that the universal localizations of a hereditary ring are parametrized by certain silting modules which are determined by a minimality condition and are called *minimal silting*. A dual version of this result was recently established in [3], leading to the notion of a *minimal cosilting* module. The interest in minimal silting or cosilting modules goes well beyond the hereditary case. For example, minimal cosilting modules also parametrize the flat ring epimorphisms starting in a commutative noetherian ring. We refer to Section 2 for details.

In the present paper, we continue these investigations by analyzing two aspects. The first one concerns an important and widely studied class of hereditary rings: finite dimensional tame hereditary

Version of July 15, 2021

2020 Mathematics Subject Classification: 16E60 16G10 16S85 13B02

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algebras. The large cotilting modules over such algebras were classified in [14]. They are determined by their indecomposable summands, which can be either finite dimensional regular modules, or infinite dimensional pure-injective, thus Prüfer modules, adic modules, or the generic module. A classification of the large tilting modules was established in [9]. Both tilting and cotilting modules are parametrized by pairs (Y, P) where Y is a branch module, that is, a finite dimensional regular module with certain combinatorial features, and P is a subset of the projective line, when the ground field is algebraically closed, or more generally, a subset of the index set $\mathbb{X}$ of the tubular family $\mathbf{t} = \bigcup_{\lambda\in\mathbb{X}} \mathbf{t}_\lambda$ in the Auslander-Reiten quiver.

In Section 3, we determine the minimal tilting and minimal cotilting modules over a tame hereditary algebra. Since the finite dimensional (co)tilting modules are all minimal, we focus on the large ones. We prove that under the parametrization described above, minimal tilting or minimal cotilting modules correspond to the pairs (Y, P) where P is not empty. This result (Theorem 3.14) is achieved by an explicit construction of the universal localization corresponding to the tilting module $T_{(Y,P)}$ when $P \neq \emptyset$. More precisely, we construct the wide subcategory $\mathcal{M}$ of the category of finite dimensional modules that allows to realize $T_{(Y,P)}$ as the tilting module $R_\mathcal{M} \oplus R_\mathcal{M}/R$ arising from the universal localization $R \to R_\mathcal{M}$ of R at $\mathcal{M}$. We also obtain that a large cotilting module is minimal if and only if it has an adic direct summand (Theorem 3.16).

The second aspect we want to address is the behaviour of silting and cosilting modules under ring extensions. A criterion recently established in [12] ensures that every silting module T over a commutative ring R extends to a silting module $T\otimes_R S$ along any ring epimorphism $R \to S$. In Section 4, we give conditions under which minimality is preserved. In particular, we show that all minimal cosilting modules over a commutative noetherian ring extend to minimal cosilting modules along any flat ring epimorphism (Corollary 4.11). Over a commutative hereditary ring, we see that every cosilting module extends to a minimal cosilting module along any ring epimorphism (Corollary 4.12).

The paper is organized as follows. Section 2 contains some preliminaries on ring epimorphisms and a survey on their relation with silting and cosilting modules. In Section 3, we determine the minimal tilting and minimal cotilting modules over a tame hereditary algebra. Section 4 is devoted to extensions of minimal silting (or cosilting) modules along ring epimorphisms. We first study the example of the Kronecker algebra (Example 4.2). Then we turn to commutative rings and provide some useful criteria for preserving minimality. We close the paper with applications to commutative noetherian rings and commutative rings of small homological dimension.

2 Preliminaries

2.1 Notation

Throughout the paper, R will denote a ring, Mod$R$ ($R\,$Mod) the category of all right (left) R-modules, and mod$R$ ($R\,$mod) the category of finitely presented right (left) R-modules.

We fix a commutative ring k such that R is a k-algebra, together with an injective cogenerator W in Mod k, and we denote by $(-)^+ = \mathrm{Hom}_k(-, W)$ the duality functors between Mod$R$ and RMod. For example, one can choose $k = \mathbb{Z}$ and $W = \mathbb{Q}/\mathbb{Z}$. In case R is a finite dimensional algebra over a field k, we will take the usual vector space duality $(-)^+ = D = \mathrm{Hom}_k(-, k)$.

Let $\mathcal{C} \subset$ Mod$R$ be a class of modules. Denote by Add$\mathcal{C}$ (respectively, add$\mathcal{C}$) the class consisting of all modules isomorphic to direct summands of (finite) direct sums of elements of M. The class consisting of all modules isomorphic to direct summands of products of modules of $\mathcal{C}$ is denoted by Prod$\mathcal{C}$. The class consisting of the right R-modules which are epimorphic images of arbitrary direct sums of elements in $\mathcal{C}$ is denoted by Gen$\mathcal{C}$. Dually, we define Cogen$\mathcal{C}$ as the class of all submodules of arbitrary direct products of elements in $\mathcal{C}$. Moreover, we write

$$\mathcal{C}^{\perp} = \{N_R \mid \mathrm{Ext}^1_R(M,N) = 0 = \mathrm{Hom}_R(M,N) \text{ for each } M \in \mathcal{C}\}.$$

$$\mathcal{C}^{\perp_1} = \{N_R \mid \mathrm{Ext}^1_R(M,N) = 0 \text{ for each } M \in \mathcal{C}\}.$$

$$\mathcal{C}^{\perp_0} = \{N_R \mid \mathrm{Hom}_R(M,N) = 0 \text{ for each } M \in \mathcal{C}\}.$$

and define ${}^{\perp}\mathcal{C}$, ${}^{\perp_1}\mathcal{C}$, ${}^{\perp_0}\mathcal{C}$ dually. If $\mathcal{C}$ contains a unique module M, then we shall denote these subcategories by $M^{\perp}$, $M^{\perp_1}$, and $M^{\perp_0}$ etc.

2.2 Ring epimorphisms

Definition 2.1. A ring homomorphism $\lambda : R \to S$ is a *ring epimorphism* if it is an epimorphism in the category of rings with unit, or equivalently, if the functor given by restriction of scalars $\lambda_* : \mathrm{Mod}S \to \mathrm{Mod}R$ is a full embedding.

A ring epimorphism $\lambda : R \to S$ is said to be

- *homological* if $\mathrm{Tor}_i^R(S,S) = 0$ for $i > 0$, or equivalently, the functor given by restriction of scalars $\lambda_* : D(\mathrm{Mod}S) \to D(\mathrm{Mod}R)$ induces a full embedding of the corresponding derived categories;
- *(right) flat* if S is a flat right R-module;
- *pseudoflat* if $\mathrm{Tor}_1^R(S,S) = 0$.

Two ring epimorphisms $\lambda : R \to S$ and $\lambda^{'} : R \to S^{'}$ are *equivalent* if there is a ring isomorphism $h : S \to S^{'}$ such that $\lambda^{'} = h \cdot \lambda$. We say that λ and λ' lie in the same *epiclass* of R.

Ring epimorphisms are closely related to certain subcategories of Mod$R$.

Definition 2.2. A full subcategory $\mathcal{X}$ of Mod$R$ is called *bireflective* if the inclusion functor $\mathcal{X} \to \mathrm{Mod}R$ admits both a left and right adjoint, or equivalently, $\mathcal{X}$ is closed under products, coproducts, kernels and cokernels.

Theorem 2.3. [16, 11] (1) *The map assigning to a ring epimorphism $\lambda : R \to S$ the essential image $\mathcal{X}$ of the functor λ_* defines a bijection between*

(*i*) *epiclasses of ring epimorphisms $R \to S$;*
(*ii*) *bireflective subcategories $\mathcal{X}$ of* Mod$R$.

(2) *The following statements are equivalent for a ring epimorphism $\lambda : R \to S$.*

(1) *λ is a pseudoflat ring epimorphism;*
(2) *$\mathcal{X}$ is closed under extensions in* Mod$R$*;*
(3) *the functors* Ext^1_R *and* Ext^1_S *agree on S-modules;*
(4) *the functors* Tor_1^R *and* Tor_1^S *agree on S-modules.*

Classical localization of commutative rings at multiplicative sets provides an important class of examples for flat ring epimorphisms. More generally, the notion of universal localization which we recall below yields a large supply of pseudoflat ring epimorphisms. If R is a hereditary ring, then $\lambda : R \to S$ is a homological ring epimorphism if and only if it is pseudoflat, which is equivalent to being a universal localization of R by [19, Theorem 6.1].

Theorem 2.4. [21, Theorem 4.1] *Let R be a ring and Σ be a class of morphisms between finitely generated projective right R-modules. Then there is a pseudoflat ring epimorphism $\lambda : R \to R_\Sigma$, called* universal localization *of R at Σ, such that*

(1) *λ is Σ-inverting: if σ belongs to Σ, then $\sigma \otimes_R R_\Sigma$ is an isomorphism of right R_Σ-modules, and*

(2) *λ is universal Σ-inverting: for any Σ-inverting morphism $\lambda' : R \to S$ there exists a unique ring homomorphism $g : R_\Sigma \to S$ such that $g \circ \lambda = \lambda'$.*

Given a bireflective subcategory $\mathcal{X}$ of Mod$R$ and an R-module M, we will denote by $\psi_M : M \to X_M$ the unit of the adjunction given by the left adjoint of the inclusion functor. The map ψ_M is an $\mathcal{X}$-*reflection*, i.e. $\mathrm{Hom}_R(\psi_M, X)$ is an isomorphism for all X in $\mathcal{X}$. In particular, ψ_M is a left $\mathcal{X}$-approximation which is *left minimal*, i.e. any endomorphism θ of X_M with $\theta \circ \psi_M = \psi_M$ is an isomorphism. This entails that every ring epimorphism $\lambda : R \to S$ is a left minimal R-module homomorphism.

2.3 Silting theory

Given a morphism $\sigma : P \to Q$ between projective modules, we define the subcategory

$$\mathcal{D}_\sigma = \{X \in \mathrm{Mod}R \mid \mathrm{Hom}_R(\sigma, X) \text{ is surjective}\}.$$

Definition 2.5. [5, 7] We say that an R-module T

- admits a *presilting presentation* if there is a projective presentation $P \xrightarrow{\sigma} Q \to T \to 0$ such that $\mathrm{Hom}_{D(\mathrm{Mod}R)}(\sigma, \sigma^{(I)}[1]) = 0$ for all sets I or, equivalently, Gen$T \subseteq D_\sigma$;
- is a *silting* module if it admits a projective presentation $P \xrightarrow{\sigma} Q \to T \to 0$ with Gen$T = \mathcal{D}_\sigma$, in which case we say that T is silting *with respect to* σ;
- is a *tilting* module if it is silting with respect to an injective map σ, or equivalently, Gen$T = T^{\perp_1}$. This amounts to the following conditions:

  $(T1)$ proj.dim(T)≤ 1,

  $(T2)$ $\mathrm{Ext}^1_{\mathrm{R}}(T, T^{(\kappa)}) = 0$ for any cardinal κ,

  $(T3)$ there is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ with T_0, $T_1 \in$ Add$T$.

Note that every silting module T satisfies Add$T$ = Gen$T \cap {}^{\perp_1}$(Gen$T$). Moreover, T gives rise to a torsion pair with torsion class Gen$T$ and torsion-free class $T^{\perp_0}$. The class Gen$T$ is called a *silting class*, or a *tilting class* in case T is a tilting module. Silting modules having the same silting class are said to be *equivalent*. We say that a silting module is *large* if it not equivalent to a finitely presented silting module. *Cosilting* or *cotilting* modules and classes are defined dually, and equivalence of cosilting or cotilting modules is defined correspondingly.

If T is a silting module with respect to σ, then by [23] there is a triangle

$$R \xrightarrow{\phi} \sigma_0 \to \sigma_1 \to R[1] \tag{1.1}$$

in the derived category $D(\mathrm{Mod}R)$, where σ_0 and σ_1 lie in Addσ and ϕ is a left Addσ-approximation of R. Applying the cohomology functor $\mathrm{H}^0(-)$ to this triangle, we obtain an exact sequence

$$R \xrightarrow{f} T_0 \to T_1 \to 0 \tag{1.2}$$

in Mod$R$, where $T_0, T_1 \in$ Add$T$ and f is a left Add$T$-approximation of R.

Definition 2.6. [7] Let T be a silting module with respect to σ. If the map ϕ in the triangle (1.1) can be chosen left minimal, then T is said to be a *minimal silting module*.

For example, all finite dimensional silting (that is, support τ-tilting) modules over a finite dimensional algebra are minimal, cf. [7, Remark 1.6]. Minimal silting modules are closely related with pseudoflat ring epimorphisms. Indeed, there is a map assigning a pseudoflat ring epimorphism to every minimal silting module [7, Corollary 2.4]. Conversely, every ring epimorphism $\lambda : R \to S$ for which the right R-module S_R admits a presilting presentation induces a silting R-module of the form $T = S \oplus \mathrm{Coker}\lambda$, see [7, Proposition 1.3].

Definition 2.7. We say that a silting module T *arises from a ring epimorphism* if there is a ring epimorphism $\lambda : R \to S$ such that $S \oplus \mathrm{Coker}\lambda$ is a silting R-module equivalent to T.

Proposition 2.8. *The map assigning to a ring epimorphism $\lambda : R \to S$ the right R-module $S \oplus \mathrm{Coker}\lambda$ yields an injection from (i) to (ii), where*

(i) epiclasses of ring epimorphisms $\lambda : R \to S$ such that S_R admits a presilting presentation,

(ii) equivalence classes of silting right R-modules arising from ring epimorphisms.

If R is right perfect, then this map is a bijection, and all modules in (ii) are minimal silting modules.

Proof. If a silting module $T = S \oplus \mathrm{Coker}\lambda$ arises from a ring epimorphism $\lambda : R \to S$, then the map λ, viewed as an R-module homomorphism, is a minimal left $\mathrm{Add}T$-approximation of R and is thus uniquely determined up to isomorphism. This shows that the equivalence class of T determines the bireflective subcategory $\mathcal{X} = \{X \in \mathrm{Mod}R \mid \mathrm{Hom}_R(\lambda, X) \text{ is bijective}\}$ and therefore the epiclass of λ, proving the injectivity of the assignment.

Moreover, every silting module $T = S \oplus \mathrm{Coker}\lambda$ as in (ii) yields an exact sequence of the form (1.2) where f is left minimal. Now, if R is right perfect, we infer from [7, Remark 1.6] that T is left minimal, and its presilting presentation entails the existence of a presilting presentation for the direct summand S_R (compare the minimal projective presentations of T_R and S_R). Thus the assignment is also surjective. □

The bijection for right perfect rings in Proposition 2.8 has a dual version, which holds over arbitrary rings thanks to the existence of minimal injective copresentations. Let us first introduce the necessary terminology. If C is a cosilting left R-module, then by [24] there is an exact sequence

$$0 \to C_1 \to C_0 \stackrel{g}{\to} R^+ \tag{2.1}$$

where C_0, C_1 are in $\mathrm{Prod}C$, and g is a right $\mathrm{Prod}C$-approximation of ${}_RR^+$.

Definition 2.9. [3] (1) A cosilting left R-module C is a *minimal cosilting module* if the exact sequence (2.1) can be chosen such that the subcategory $\mathrm{Cogen}C \cap {}^{\perp_0}C_1$ is bireflective, and $\mathrm{Hom}_R(C_0, C_1) = 0$.
(2) We say that a module C admits a *precosilting copresentation* if there is an injective copresentation $0 \to C \to Q_0 \stackrel{\omega}{\to} Q_1$ such that $\mathrm{Hom}_{D(A)}(\omega^I, \omega[1]) = 0$ for all sets I.

Theorem 2.10. [3, Theorem 4.17] *The map assigning to a ring epimorphism $\lambda : R \to S$ the left R-module $S^+ \oplus \mathrm{Ker}\lambda^+$, yields a bijection between*

(i) epiclasses of ring epimorphisms $\lambda : R \to S$ such that ${}_RS^+$ has a precosilting copresentation,

(ii) equivalence classes of minimal cosilting left R-modules.

Also minimal cosilting modules are intimately related with pseudoflat ring epimorphisms.

Remark 2.11. [3, Example 4.15],[4, Proposition 4.5] The ring epimorphisms satisfying condition (i) above are all pseudoflat, and the converse holds true if S_R has weak dimension at most one. If R is commutative noetherian, then (i) consists precisely of the epiclasses of flat ring epimorphisms.

In the hereditary case we obtain the following result.

Theorem 2.12. [6, Theorem 5.8 and Corollary 5.17], [3, Corollary 4.22]. *If R is hereditary, there are bijections between*

(i) epiclasses of homological ring epimorphisms $R \to S$,

(ii) equivalence classes of minimal silting right R-modules,

(iii) equivalence classes of minimal cosilting left R-modules,

(iv) subcategories of mod$R$ *which are wide, i.e. closed under kernels, cokernels, and extensions.*

The bijections $(i) \to (ii), (iii)$ map a homological ring epimorphism $\lambda : R \to S$ to the silting right R-module $S \oplus \mathrm{Coker}\lambda$ and to the cosilting left R-module $S^+ \oplus \mathrm{Ker}\lambda^+$, and restrict to bijections between injective homological ring epimorphisms, tilting right modules and cotilting left modules. The assignment $(ii) \to (iii)$ is given by $T \mapsto T^+$. The bijection $(iv) \to (i)$ maps a wide subcategory $\mathcal{M}$ to the universal localization $R \to R_{\mathcal{M}}$ at (projective resolutions of) the modules in $\mathcal{M}$.

3 Minimal tilting modules over tame hereditary algebras

In this section, let R be a finite dimensional tame hereditary algebra over a field k, which we assume to be indecomposable. We want to determine the minimal tilting modules over R. Since every finite dimensional tilting module is obviously minimal, we will focus on the large tilting modules.

3.1 Preliminaries on tame hereditary algebras

It is well known that the finite dimensional indecomposable (right) R-modules are depicted by the Auslander-Reiten quiver of mod$R$, which consists of a preprojective and a preinjective component, denoted by $\mathbf{p}$ and $\mathbf{q}$, respectively, and a family of orthogonal tubes $\mathbf{t} = \bigcup_{\lambda \in \mathbb{X}} \mathbf{t}_\lambda$ containing the regular modules. For details we refer e.g. to [10].

Given a quasi-simple (or simple regular) module S, that is, a module at the mouth of a tube $\mathbf{t}_\lambda$, we denote by $S[m]$ the module of regular length m on the ray

$$S = S[1] \subset S[2] \subset \cdots \subset S[m] \subset S[m+1] \subset \cdots$$

and let $S[\infty] = \varinjlim_{m \to \infty} S[m]$ be the corresponding *Prüfer* module. The *adic* module $S[-\infty]$ corresponding to S is defined as the inverse limit along the coray ending at S. We denote by G the *generic* module. It is the unique indecomposable infinite dimensional module which has finite length over its endomorphism ring.

3.2 Over the Kronecker algebra

We start by reviewing the case when R is the Kronecker algebra, i.e. the path algebra of the quiver $\bullet \rightrightarrows \bullet$. Denote by P_i (respectively Q_i), with $i \in \mathbb{N}$, the (finite dimensional) indecomposable preprojective (respectively, preinjective) right R-modules, indexed such that $\dim_k \mathrm{Hom}_R(P_i, P_{i+1}) = 2$ (respectively, $\dim_k \mathrm{Hom}_R(Q_{i+1}, Q_i) = 2$). Recall that P_1 is simple projective and embeds in all Kronecker modules but the modules in $\mathrm{Add}Q_1$, and Q_1 is simple injective with a surjection from all Kronecker modules but the modules in $\mathrm{Add}P_1$.

By [19, Theorem 6.1], every homological ring epimorphism $R \to S$ is equivalent to a universal localization at a set of (projective resolutions of) finitely presented modules. Here is a complete list of the epiclasses of R, together with the corresponding bireflective subcategories $\mathcal{X}$ of $\mathrm{Mod}R$:

- $R \to 0$ and $id_R : R \to R$,
- the universal localization at P_1 with $\mathcal{X} = \mathrm{Add}Q_1$,
- the universal localization at $P_{i+1}, i \geq 1$, with $\mathcal{X} = \mathrm{Add}P_i$,
- the universal localization at $Q_i, i \geq 1$, with $\mathcal{X} = \mathrm{Add}Q_{i+1}$,
- the universal localization at a non-empty set $\mathcal{U}$ of simple regular modules, with $\mathcal{X} = \mathcal{U}^{\perp}$.

Notice that the epimorphisms in this list are either surjective with an idempotent kernel, or injective, and the only non-injective ones are $R \to 0$ and the universal localizations at the projective modules P_1 and P_2.

The following is a complete list of silting right R-modules, up to equivalence:

- $0, P_1, Q_1$, the only silting modules that are not tilting,
- $P_i \oplus P_{i+1}, i \geq 1$,
- $Q_{i+1} \oplus Q_i, i \geq 1$,
- $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$, where $\mathcal{U}$ is a non-empty set of simple regular modules,
- the Lukas tilting module $\mathbf{L}$ with $\mathrm{Gen}\mathbf{L} = {}^{\perp_0}\mathbf{p}$, the unique non-minimal silting module.

For details we refer to [9], [6, Example 5.19].

3.3 Classification of tilting modules

Let us now return to an arbitrary tame hereditary algebra R. The large tilting R-modules have been classified in [9]. In contrast to the Kronecker case, they can have finite dimensional summands. This is due to the existence of (at most three) non-homogeneous tubes. Notice, however, that the finite dimensional part of a large tilting module can be described explicitly. In order to explain this, we need to recall some terminology.

Definition 3.1. (1) An R-module Y is said to be *exceptional* if $\mathrm{Ext}^1_R(Y, Y) = 0$.

(2) Given a tube $\mathbf{t}_\lambda$ of rank $r > 1$ and a module $X = U[m] \in \mathbf{t}_\lambda$ of regular length $m < r$, we consider the full subquiver $\mathcal{W}_X$ of $\mathbf{t}_\lambda$ which is isomorphic to the Auslander-Reiten-quiver $\Theta(m)$ of the linearly oriented quiver of type $\mathbb{A}_m$ with X corresponding to the projective-injective vertex of $\Theta(m)$. The set $\mathcal{W}_X$ is called a *wing* of $\mathbf{t}_\lambda$ of size m, which is *rooted in the vertex* $X = U[m]$.

(3) A finite dimensional regular multiplicity-free exceptional R-module Y is a *branch module* if it satisfies the following condition: For each quasi-simple module S and $m \in \mathbb{N}$ such that $S[m]$ is a direct summand of Y, there exist precisely m direct summands of Y that belong to $\mathcal{W}_{S[m]}$.

In other words, a branch module is a regular multiplicity-free exceptional module whose indecomposable summands are arranged in disjoint wings, and the number of summands from each wing equals the size of that wing.

Now it is shown in [9] that the large tilting modules over R are parametrized by pairs (Y, P) where Y is a branch module, and P is subset of $\mathbb{X}$. More precisely, every such pair (Y, P) determines two sets of quasi-simple modules:

- the set $\mathcal{V} = \mathcal{V}_{(Y,P)}$ given by all quasi-simple modules in $\bigcup_{\lambda \in P} \mathbf{t}_\lambda$ and all regular composition factors of Y,
- the set $\mathcal{U} = \mathcal{U}_{(Y,P)}$ given by all quasi-simple modules in $\bigcup_{\lambda \in P} \mathbf{t}_\lambda$ that are not regular composition factors of $\tau^- Y$.

With these sets one can construct a tilting module

$$T_{(Y,P)} = Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{V}}) \oplus \bigoplus_{S \in \mathcal{U}} S[\infty],$$

and it turns out that the modules $T_{(Y,P)}$ form a complete irredundant list of all large tilting modules, up to equivalence. For details we refer to [9, 1].

Our aim is to show that a large tilting module $T_{(Y,P)}$ is minimal if and only if the set $P \subset \mathbb{X}$ is non-empty. The only-if part of this statement is already contained in a result from [9], which we briefly recall for the reader's convenience.

Proposition 3.2. (cf. [9, Corollary 5.10]) *If $T_{(Y,P)}$ is a minimal tilting module, then the set P is not empty.*

Proof. We just outline the argument and refer to [9] for details. Assume $T = T_{(Y,P)}$ is equivalent to a tilting module of the form $S \oplus S/R$ for some ring epimorphism $R \to S$. If $P = \emptyset$, then $T = Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{V}})$ where $\mathcal{V}$ is the set of regular composition factors of Y. Now one considers the torsion and torsion-free part of T with respect to the torsion pair $(\mathrm{Gen}\mathbf{t}, \mathbf{t}^{\perp_0})$ generated by $\mathbf{t}$. It turns out that Y is torsion and $\mathbf{L} \otimes_R R_{\mathcal{V}}$ is torsion-free. Moreover, $\mathrm{Add}Y$ contains all torsion modules in $\mathrm{Add}T$, and $\mathrm{Add}(\mathbf{L} \otimes_R R_{\mathcal{V}})$ contains all torsion-free modules in $\mathrm{Add}T$. Next, one shows that S/R is torsion and therefore lies in $\mathrm{Add}Y$. One then deduces that $(S/R)^{\perp_1} = Y^{\perp_1}$. Notice that $(S/R)^{\perp_1} = \mathrm{Gen}T$, hence our tilting class $\mathrm{Gen}T$ coincides with $Y^{\perp_1}$. On the other hand, by a well-known result due to Bongartz, the finite dimensional exceptional module Y can be completed to a finite dimensional tilting module with tilting class $Y^{\perp_1}$. But this contradicts the assumption that T is large. □

3.4 Minimal tilting modules

Let us fix a large tilting module $T = T_{(Y,P)}$ with $P \neq \emptyset$. We want to prove that T is minimal. To this end, we will use Theorem 2.12 and show that $T_{(Y,P)}$ arises from universal localization at a wide subcategory $\mathcal{M}$ of $\mathrm{mod}R$.

We start out with an easy, but useful observation.

Remark 3.3. (cf. [9, Example 4.4]) If S is a quasi-simple module, then $S[\infty]^{\perp_1} = \bigcap_{n \geq 1} S[n]^{\perp_1}$.

More generally, let $\mathcal{E}$ be a class of modules, and suppose that a module X lies in $\mathcal{E}^{\perp_1}$. Then X lies in $E^{\perp_1}$ for every module E which is filtered by modules from $\mathcal{E}$, or which is a submodule of some module in $\mathcal{E}$.

This follows immediately from the fact that any class of the form ${}^{\perp_1}X$ is closed under filtrations by [18, 3.1.2], and when X has injective dimension at most one (as in our case), then it is also closed under submodules.

Next, we compute the tilting class of T. This amounts to computing the subcategory

$$\mathcal{S} = {}^{\perp_1}\mathrm{Gen}T \cap \mathrm{mod}R,$$

that is, the largest subcategory $\mathcal{S}$ of mod$R$ with the property that $\mathrm{Gen}T = \mathcal{S}^{\perp_1}$. We are going to see that the indecomposable non-preprojective modules in $\mathcal{S}$ either lie on a ray starting in $\mathcal{U} = \mathcal{U}_{(Y,P)}$, or lie "below" an indecomposable summand of Y. Here is the precise statement.

Lemma 3.4. (1) *A finitely generated indecomposable R-module M belongs to $\mathcal{S}$ if and only if one of the following statements holds true:*

- *M is preprojective,*
- *$M \cong S[n]$ where $n \geq 1$ and S is in $\mathcal{U}$,*
- *there is a module $S[h] \in \mathrm{Add}Y$ such that $M \cong S[i]$ for some $1 \leq i \leq h$.*

(2) $\mathrm{Gen}T = \bigcap_{S\in\mathcal{U}} S[\infty]^{\perp_1} \cap Y^{\perp_1}$.

Proof. (1) We know from [9, Theorem 2.7] that $\mathcal{S} = \mathrm{add}(\mathbf{p} \cup \mathbf{t}^{'})$, where $\mathbf{t}^{'}$ is a set of regular modules. Take a quasi-simple S whose ray $\{S[n] \,|\, n \geq 1\}$ contains some modules from $\mathbf{t}^{'}$. If the whole ray is contained in $\mathbf{t}^{'}$, then by [9, Theorem 4.5], the Prüfer module $S[\infty] \in \mathrm{Add}T$, and $S \in \mathcal{U}$. If $\mathbf{t}'$ contains some, but not all modules from the ray, then $\mathbf{t}^{'} \cap \{S[n] \mid n \geq 1\} = \{S[i] \mid i \leq h\}$ with $S[h] \in \mathrm{Add}Y$ by [9, Lemma 3.3]. Hence $\mathbf{t}^{'}$ and $\mathcal{S}$ have the stated shape.

(2) $\mathrm{Gen}T = T^{\perp_1} \subseteq \bigcap_{S\in\mathcal{U}} S[\infty]^{\perp_1} \cap Y^{\perp_1}$ since $S[\infty]$, $S \in \mathcal{U}$, and Y are summands of T.

Now take $X \in \bigcap_{S\in\mathcal{U}} S[\infty]^{\perp_1} \cap Y^{\perp_1}$. By Remark 3.3, it follows that X lies in $\bigcap_{n\geq 1} S[n]^{\perp_1}$ for all $S \in \mathcal{U}$. Moreover, if $S[h] \in \mathrm{Add}Y$, then X lies in $S[h]^{\perp_1}$ and also in $S[i]^{\perp_1}$ for each $1 \leq i \leq h$, due to the inclusion $S[i] \hookrightarrow S[h]$. This shows that $X \in \mathbf{t}^{'\perp_1} = \mathcal{S}^{\perp_1} = \mathrm{Gen}T$. □

We want to find a wide subcategory $\mathcal{M} \subset \mathrm{mod}R$ which corresponds to T under the bijection in Theorem 2.12. From [8, Corollary 4.13] and [20, Theorem 2.6], we know that the tilting module $R_{\mathcal{M}} \oplus R_{\mathcal{M}}/R$ given by a wide subcategory $\mathcal{M}$ has tilting class $\mathrm{Gen}(R_{\mathcal{M}}) = \mathcal{M}^{\perp_1}$. Hence $\mathcal{M}$ must satisfy $\mathrm{Gen}T = \mathcal{M}^{\perp_1}$, and in particular it must be contained in $\mathcal{S}$. Furthermore, the class $\mathcal{M}^{\perp_1}$ must be contained in $Y^{\perp_1}$ and in each $S[\infty]^{\perp_1}$ with $S \in \mathcal{U}$.

To construct such $\mathcal{M}$, we will therefore pick modules from $\mathcal{S}$ which filter the Prüfer modules $S[\infty]$ with $S \in \mathcal{U}$. This will ensure the inclusion $\mathcal{M}^{\perp_1} \subseteq \bigcap_{S\in\mathcal{U}} S[\infty]^{\perp_1}$ by Remark 3.3.

Moreover, for each ray $\{S[n] \,|\, n \geq 1\}$ containing a module in $\mathrm{Add}Y$, we will include in $\mathcal{M}$ the module $S[h] \in \mathrm{Add}Y$ of maximal regular length on that ray. This will entail the inclusion $\mathcal{M}^{\perp_1} \subseteq Y^{\perp_1}$, again by Remark 3.3. In fact, a closer look shows that it suffices to take care of the rays starting in quasi-simple modules that are not in $\mathcal{U}$, because for $S \in \mathcal{U}$ we already have $\mathcal{M}^{\perp_1} \subseteq \bigcap_{n\geq 1} S[n]^{\perp_1}$.

So, given a wing $\mathcal{W} = \mathcal{W}_{U[m]}$ inside a tube $\mathbf{t}_\lambda$, where the vertex $U[m]$ belongs to $\mathrm{Add}Y$, we consider the following two sets

$$\mathcal{X}_{\mathcal{W}} = \{S[h] \,|\, S[h] \in \mathrm{Add}Y \text{ of maximal regular length on a ray starting in } S \in \mathcal{W}\},$$

and

$$\widetilde{\mathcal{X}}_{\mathcal{W}} = \{S[h] \,|\, S[h] \in \mathrm{Add}Y \text{ of maximal regular length on a ray starting in } S \in \mathcal{W} \setminus \{U\}\}.$$

We will now follow this strategy and construct the category $\mathcal{M}$.

Construction 3.5. Let $Q := \{\lambda \in \mathbb{X} | \ \mathbf{t}_\lambda \cap \mathrm{Add}Y \neq \emptyset\}$, and fix $\lambda \in Q$. Let $\{S_1, \cdots, S_r\}$ be a complete irredundant set of the quasi-simple modules in $\mathbf{t}_\lambda$. By [9, Proposition 3.7], the summands of Y belonging to $\mathbf{t}_\lambda$ are arranged in disjoint wings $\mathcal{W}_1, \ldots, \mathcal{W}_l$ in $\mathbf{t}_\lambda$ whose vertices $S_{n_1}[m_1], \ldots, S_{n_l}[m_l]$ belong to $\mathrm{Add}Y$. In other words,

$$\mathbf{t}_\lambda \cap \mathrm{Add}Y \subseteq \bigcup_{j=1}^{l} \mathcal{W}_j.$$

Observe that $S_{n_1}, \ldots, S_{n_l} \in \mathcal{U}$ whenever $\lambda \in Q \cap P$. We now define the set

$$\mathcal{X}(\lambda) = \bigcup_{j=1}^{l} \mathcal{X}_j$$

where for each $1 \leq j \leq l$ the set $\mathcal{X}_j$ is given as follows:

$$\begin{cases} \mathcal{X}_j = \mathcal{X}_{\mathcal{W}_j} & \text{if } \lambda \in Q \setminus P, \\ \mathcal{X}_j = \widetilde{\mathcal{X}}_{\mathcal{W}_j} & \text{if } \lambda \in Q \cap P. \end{cases}$$

When $\lambda \in Q \cap P$, we also need to filter the Prüfer modules $\{S[\infty] | \ S \in \mathcal{U}\}$, so we take

$$\mathcal{R}(\lambda) = \bigcup_{j=1}^{l} \mathcal{R}_j$$

where

$$\begin{aligned} \mathcal{R}_j &= \{S_{n_j}[k] \mid \ m_j < k \leq n_{j+1} - n_j\} \cup \{S_i | \ n_j + m_j + 1 \leq i < n_{j+1}\} \quad \text{for } 1 \leq j < l, \\ \mathcal{R}_l &= \{S_{n_l}[k] \mid \ m_l < k \leq r + n_1 - n_l\} \cup \{S_i | \ n_l + m_l + 1 \leq i \leq r\} \cup \{S_i | \ 1 \leq i < n_1\}. \end{aligned}$$

Now we define

$$\mathcal{Q}(\lambda) = \begin{cases} \mathcal{X}(\lambda) & \text{if } \lambda \in Q \setminus P \\ \mathcal{X}(\lambda) \cup \mathcal{R}(\lambda) & \text{if } \lambda \in Q \cap P \end{cases}$$

and let $\mathcal{M} = {}^{\perp}(\mathcal{Q}^{\perp}) \cap \mathrm{mod}R$ be the wide closure of the set

$$\mathcal{Q} = \bigcup_{\lambda \in Q} \mathcal{Q}(\lambda) \cup \bigcup_{\mu \in P \setminus Q} \mathbf{t}_\mu.$$

Example 3.6. In Figure 1 we illustrate the definition of $\mathcal{R}_\lambda$ by considering the case of $l = 2$ wings $\mathcal{W}_1$ and $\mathcal{W}_2$ rooted in the vertex $S_1[3]$ and $S_7[4]$, respectively, inside a tube $\mathbf{t}_\lambda$ of rank $r = 12$. Then $\mathcal{R}_1 = \{S_1[4], S_1[5], S_1[6], S_5, S_6\}$, and $\mathcal{R}_2 = \{S_7[5], S_7[6], S_{12}\}$.

Now suppose that $S_1[2], S_1[3], S_2$ are the indecomposable summands of Y lying in the wing $\mathcal{W}_1$. Then $\mathcal{X}_1 = \{S_2\}$ if $\lambda \in Q \cap P$, while $\mathcal{X}_1 = \{S_1[3], S_2\}$ if $\lambda \in Q \setminus P$.

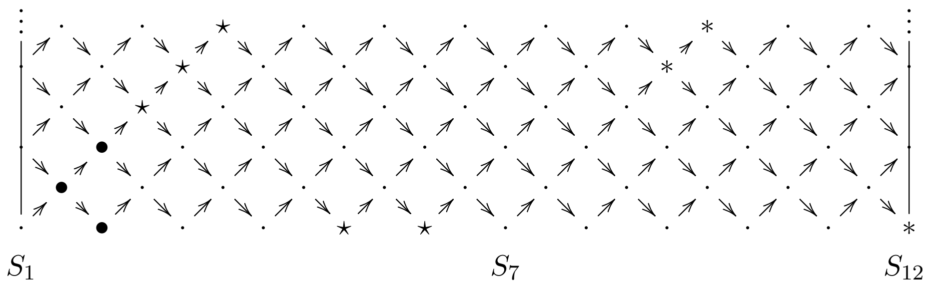

Figure 1: $\bullet$ = direct summands of Y in the wing $\mathcal{W}_1$; $\star$ and $*$ = modules in $\mathcal{R}_1$ and $\mathcal{R}_2$

Example 3.7. Let $\mathbf{t}_\lambda$ be a tube of rank $r > 1$ with quasi-simple modules $S = S_1, \dots, S_r$.

(1) (cf. [9, Example 4.4]) Take $T = T_{(Y,P)}$ where $Y = S \oplus \cdots \oplus S[r-1]$ and $P = \{\lambda\}$. Then $\mathcal{V} = \{S_1, \dots, S_r\}$ and $\mathcal{U} = \{S\}$. Moreover, $P = Q$ and $\mathcal{Q} = \mathcal{R}(\lambda) = \{S[r]\}$.
So, the tilting module

$$T = S \oplus \cdots \oplus S[r-1] \oplus R_{\mathcal{V}} \oplus S[\infty]$$

is equivalent to $T_{\mathcal{M}} = R_{\mathcal{M}} \oplus R_{\mathcal{M}}/R$ for $\mathcal{M} = {}^{\perp}(S[r]^{\perp}) \cap \operatorname{mod} R$.

(2) Let now $r = 3$. Take $T = T_{(Y,P)}$ where $Y = S_1$ and $P = \{\lambda\}$.
Then $\mathcal{V} = \{S_1, S_2, S_3\}$ and $\mathcal{U} = \{S_1, S_3\}$. Moreover, $P = Q$ and $\mathcal{Q} = \mathcal{R}(\lambda) = \{S_1[2], S_1[3], S_3\}$.
So, the tilting module

$$T = S_1 \oplus R_{\mathcal{V}} \oplus S_1[\infty] \oplus S_3[\infty]$$

is equivalent to $T_{\mathcal{M}} = R_{\mathcal{M}} \oplus R_{\mathcal{M}}/R$ for $\mathcal{M} = {}^{\perp}(\{S_1[2], S_1[3], S_3\}^{\perp}) \cap \operatorname{mod} R$.

(3) Let $r = 4$, and let $\mathbf{t}_\mu$ be a tube of rank $r_\mu = 3$ with quasi-simples $\{U_1, U_2, U_3\}$. Take $T_{(Y,P)}$ where $Y = S_1[3] \oplus S_2[2] \oplus S_2 \oplus U_1[2] \oplus U_2$ and $P = \{\lambda\}$.
Then

$$Q = \{\lambda, \mu\},\ \mathcal{X}(\lambda) = \{S_2[2]\}, \mathcal{X}(\mu) = \{U_1[2], U_2\} \text{ and } \mathcal{R}(\lambda) = \{S_1[4]\}.$$

Moreover, $\mathcal{Q} = \mathcal{X}(\lambda) \cup \mathcal{X}(\mu) \cup \mathcal{R}(\lambda)$. So, the tilting module $T_{(Y,P)}$ is equivalent to $T_{\mathcal{M}} = R_{\mathcal{M}} \oplus R_{\mathcal{M}}/R$ for $\mathcal{M} = {}^{\perp}(\mathcal{Q}^{\perp}) \cap \operatorname{mod} R$.

We collect some easy observations.

Remark 3.8. (1) Since $\mathcal{Q} \subseteq \bigcup_{\lambda \in P \cup Q} t_\lambda$, its wide closure $\mathcal{M}$ is contained in $\operatorname{add}(\bigcup_{\lambda \in P \cup Q} t_\lambda)$.
(2) By construction, we have $\bigcup_{j=1}^{l} \mathcal{W}_j \subseteq \mathcal{R}(\lambda)^{\perp} \cap \mathbf{t}_\lambda$. Moreover, $\operatorname{Hom}_R(X, Y) \neq 0$ for any $X \in \mathcal{R}(\lambda)$ and $Y \in \mathbf{t}_\lambda \setminus \bigcup_{j=1}^{l} \mathcal{W}_j$. It follows that $\mathcal{R}(\lambda)^{\perp} \cap \mathbf{t}_\lambda = \bigcup_{j=1}^{l} \mathcal{W}_j$.

The indecomposable modules in $\mathcal{M}$ are therefore of the form $M = S[i] \in \mathbf{t}_\lambda$ with $\lambda \in P \cup Q$. If $\lambda \in P \setminus Q$, all modules in $\mathbf{t}_\lambda$ do occur. Let us turn to the remaining cases.

Proposition 3.9. *Any indecomposable module in $\mathcal{M}$ either lies on a ray starting in $\mathcal{U}$, or there is some $S[h] \in \bigcup_{\lambda \in Q} \mathcal{X}(\lambda)$ such that $M \cong S[i]$ with $1 \leq i \leq h$.*

In particular, $\mathcal{M}$ is contained in $\mathcal{S}$.

Proof. Let $M \in \mathcal{M}$ be indecomposable. Then w.l.o.g. $M = S[i] \in \mathbf{t}_\lambda$ for some $\lambda \in P \cup Q$. If M does not lie on a ray starting in $\mathcal{U}$, we have one of the following cases: (1) $\lambda \in Q \cap P$ and $S \notin \mathcal{U}$, or (2) $\lambda \in Q \backslash P$. We will show that in both cases there is $h \geq i$ such that $S[h] \in \mathcal{X}(\lambda)$.

(1) Assume that $\lambda \in Q \cap P$ and $S \notin \mathcal{U}$. By assumption $S[i] \in {}^\perp(\mathcal{Q}^\perp) \subseteq {}^\perp(\mathcal{Q}^\perp \cap \mathbf{t}_\lambda)$ and $\mathcal{Q}^\perp = \bigcap_{\mu \in Q} \mathcal{Q}(\mu)^\perp \cap (\bigcup_{\mu \in P \backslash Q} \mathbf{t}_\mu)^\perp$. Now, we have $\mathbf{t}_\lambda \subseteq \mathcal{Q}_\mu^\perp$, when $\mu \neq \lambda$ and $\mathbf{t}_\lambda \subseteq (\bigcup_{\mu \in P \backslash Q} \mathbf{t}_\mu)^\perp$ since $\lambda \in P \cap Q$. Thus $\mathcal{Q}^\perp \cap \mathbf{t}_\lambda = \mathcal{Q}(\lambda)^\perp \cap \mathbf{t}_\lambda = \mathcal{R}(\lambda)^\perp \cap \mathbf{t}_\lambda \cap \mathcal{X}(\lambda)^\perp$. From Remark 3.8(2) we infer that $S[i] \in {}^\perp(\bigcup_{j=1}^{l} \mathcal{W}_j \cap \mathcal{X}(\lambda)^\perp) = \bigcap_{j=1}^{l} {}^\perp(\mathcal{W}_j \cap \mathcal{X}(\lambda)^\perp) = \bigcap_{j=1}^{l} {}^\perp(\mathcal{W}_j \cap \mathcal{X}_j^\perp)$, keeping in mind that $\mathcal{W}_j \subseteq \mathcal{W}_k^\perp \subseteq \mathcal{X}_k^\perp$ whenever $j \neq k$.

By assumption, we further know that $S \notin \mathcal{U}$ is a composition factor of $\tau^- Y$. So there is $1 \leq j \leq l$ such that $S \in \{S_{n_j+1}, \cdots, S_{n_j+m_j}\}$. Notice that not all modules in $\mathrm{Add}Y \cap \mathcal{W}_j$ can lie on the ray starting in S_{n_j}, because otherwise $\mathcal{X}_j = \widetilde{\mathcal{X}}_{\mathcal{W}_j} = \emptyset$ and the assumption $S[i] \in {}^\perp(\mathcal{W}_j \cap \mathcal{X}_j{}^\perp) = {}^\perp\mathcal{W}_j$ would yield a contradiction. In particular, this entails $m_j > 1$. So we can apply Proposition 3.11 (1) below and obtain $i \leq h < m_j$ such that $S[h] \in \mathcal{X}(\lambda) \subseteq \mathrm{Add}Y$.

(2) Assume that $\lambda \in Q \setminus P$. By construction $\mathcal{X}(\lambda) = \bigcup_{j=1}^{l} \mathcal{X}_j \subseteq \bigcup_{j=1}^{l} \mathcal{W}_j$, so its wide closure ${}^\perp(\mathcal{X}(\lambda)^\perp) \cap \mathrm{mod}R$ is contained in add $\bigcup_{j=1}^{l} \mathcal{W}_j$. As different tubes are Hom- and Ext-orthogonal, $\mathcal{M} \cap \mathbf{t}_\lambda = {}^\perp(\mathcal{X}(\lambda)^\perp) \cap \mathbf{t}_\lambda \subseteq \mathrm{add} \bigcup_{j=1}^{l} \mathcal{W}_j$. Since $S[i] \in \mathcal{M} \cap \mathbf{t}_\lambda$, there is $1 \leq j \leq l$ such that $S[i] \in \mathcal{W}_j$, thus $S \in \{S_{n_j}, \cdots, S_{n_j+m_j-1}\}$. Denote $\mathcal{W}_0 = \mathcal{W}_{\tau S_{n_j}[m_j+1]}$. Note that $\mathcal{W}_0 \subseteq \mathcal{W}_k^\perp \subseteq \mathcal{X}_k^\perp$ whenever $j \neq k$. Hence $S[i] \in {}^\perp(\mathcal{X}(\lambda)^\perp) \subseteq {}^\perp(\mathcal{W}_0 \cap \mathcal{X}_j^\perp)$. Now we can apply Proposition 3.11 (2) below and obtain $i \leq h \leq m_j$ such that $S[h] \in \mathcal{X}(\lambda) \subseteq \mathrm{Add}Y$. □

Lemma 3.10. *Let $\mathcal{W} = \mathcal{W}_{U_1[m]}$ be a wing rooted in the vertex $U_1[m] \in \mathrm{Add}Y$. Set $U_2 = \tau^{-1}U_1, \cdots, U_{m+1} = \tau^{-1}U_m$. Denote further $\mathcal{A} = {}^\perp(\mathcal{W} \cap \widetilde{\mathcal{X}}_{\mathcal{W}}{}^\perp)$. Then the following holds true.*

(1) $U_1[m] \in \mathcal{W} \cap \widetilde{\mathcal{X}}_{\mathcal{W}}{}^\perp$.

(2) Assume that $m > 1$, and that there are a subset $\mathcal{X} \subseteq \mathbf{t}_\lambda$ and integers $1 \leq j < m$ and $0 < n \leq m - j$ such that $\mathcal{W}' := \mathcal{W}_{U_j[n]} \subset \mathcal{X}^\perp$ and $\widetilde{\mathcal{X}}_{\mathcal{W}} = \widetilde{\mathcal{X}}_{\mathcal{W}'} \cup \mathcal{X}$. Then $\mathcal{A} \subseteq {}^\perp(\mathcal{W}' \cap \widetilde{\mathcal{X}}_{\mathcal{W}'}{}^\perp)$.

Proof. (1) is obvious. For (2) note that $\mathcal{W}' \cap \widetilde{\mathcal{X}}_{\mathcal{W}'}{}^\perp \subseteq \mathcal{W} \cap \widetilde{\mathcal{X}}_{\mathcal{W}'}{}^\perp \cap \mathcal{X}^\perp = \mathcal{W} \cap \widetilde{\mathcal{X}}_{\mathcal{W}}{}^\perp$. Then $\mathcal{A} \subseteq {}^\perp(\mathcal{W}' \cap \widetilde{\mathcal{X}}_{\mathcal{W}'}{}^\perp)$. □

Proposition 3.11. *Let assumptions and notation be as in Lemma 3.10.*

(1) Assume that $m > 1$ and that there are $i \in \mathbb{N}$ and $S \in \{U_2, \cdots, U_m, U_{m+1}\}$ such that $S[i] \in \mathcal{A} = {}^\perp(\mathcal{W} \cap \widetilde{\mathcal{X}}_{\mathcal{W}}{}^\perp)$. Then $S \neq U_{m+1}$ and there is $i \leq h < m$ such that $S[h] \in \mathrm{Add}Y$.

(2) Denote $\mathcal{W}_0 = \mathcal{W}_{\tau U_1[m+1]}$, and assume that there are $i \in \mathbb{N}$ and $S \in \{U_1, U_2, \cdots, U_m\}$ such that $S[i] \in \mathcal{B} = {}^\perp(\mathcal{W}_0 \cap \mathcal{X}_{\mathcal{W}}{}^\perp)$. Then there is $i \leq h \leq m$ such that $S[h] \in \mathrm{Add}Y$.

Proof. (1) Since $\mathcal{A} \subseteq {}^\perp U_1[m]$ by Lemma 3.10, we have $S[i] \in \mathcal{W}$ and $S \in \{U_2, \cdots, U_m\}$. Observe that not all modules in $\mathrm{Add}Y \cap \mathcal{W}$ can lie on the ray starting in U_1, as we have already seen in the proof of Proposition 3.9.

We proceed by induction on m. When $m = 2$, we know by the observation above that $\mathcal{W} \cap \mathrm{Add}Y = \{U_2, U_1[2]\}$. Then $\mathcal{A} = {}^{\perp}(\mathcal{W} \cap U_2^{\perp})$ does not intersect $\{U_2[t] | \ t \geq 2\}$. Thus $S = U_2 \in \mathrm{Add}Y$.

Let $m > 2$. We distinguish two cases.

Case (i). Suppose that none of the modules $U_2[m-1], U_3[m-2], \cdots, U_m$ on the right border of the wing $\mathcal{W}$ belongs to $\mathrm{Add}Y$. By the definition of a branch module, we know that $\mathrm{Add}Y$ contains precisely m modules in $\mathcal{W}$: these are $U_1[m]$ and $m-1$ modules in $\mathcal{W}' = \mathcal{W}_{U_1[m-1]}$. We obtain $\widetilde{\mathcal{X}}_{\mathcal{W}} = \widetilde{\mathcal{X}}_{\mathcal{W}'}$. By Lemma 3.10 (here $\mathcal{X} = \emptyset$), we have $\mathcal{A} \subseteq {}^{\perp}(\mathcal{W}' \cap \widetilde{\mathcal{X}}_{\mathcal{W}'}{}^{\perp})$. By induction assumption, the claim holds true for all $S \in \{U_2, \cdots, U_m\}$.

Case (ii). Suppose that one of the modules $U_2[m-1], U_3[m-2], \cdots, U_m$ belongs to $\mathrm{Add}Y$. Choose $U_k[h] \in \mathrm{Add}Y$ of maximal regular length. By the definition of a branch module, the module $U_1[t_1]$ with $t_1 = m - h - 1 = k - 2$ must lie in $\mathrm{Add}Y$. Furthermore, $\mathrm{Add}Y$ contains precisely m modules in $\mathcal{W}$: these are t_1 modules from $\mathcal{W}_1 := \mathcal{W}_{U_1[t_1]}$, together with h modules from $\mathcal{W}_2 := \mathcal{W}_{U_k[h]}$ and $U_1[m]$.

By induction assumption and Lemma 3.10 (here $\mathcal{W}' = \mathcal{W}_1$ and $\mathcal{X} = \mathcal{X}_{\mathcal{W}_2}$), the claim holds true for $S \in \{U_2, \cdots, U_{t_1}, U_{k-1}\}$. Moreover, the claim is clear for $S = U_k$.

We again have two cases:

Case (i). Suppose that none of $U_{k+1}[h-1], U_{k+2}[h-2], \cdots, U_m$ belongs to $\mathrm{Add}Y$. Then $U_k[h-1] \in \mathrm{Add}Y$. The claim holds true for all $S \in \{U_{k+1}, \ldots, U_m\}$ again by Lemma 3.10 (here $\mathcal{W}' = \mathcal{W}_{U_k[h-1]}$ and $\mathcal{X} = \widetilde{\mathcal{X}}_{\mathcal{W}_1} \cup \{U_k[h]\}$) and induction assumption.

Case (ii). Suppose that one of $U_{k+1}[h-1], U_{k+2}[h-2], \cdots, U_m$ belongs to $\mathrm{Add}Y$. Choose $U_{k_2}[h_2] \in \mathrm{Add}Y$ of maximal regular length. Then $U_k[t_2]$ with $t_2 = h - h_2 - 1$ must lie in $\mathrm{Add}Y$, and the modules in $\mathcal{W}_2 \cap \mathrm{Add}Y$ different from the vertex are distributed in the wings $\mathcal{W}_3 := \mathcal{W}_{U_k[t_2]}$ and $\mathcal{W}_4 := \mathcal{W}_{U_{k_2}[h_2]}$. By Lemma 3.10 (here $\mathcal{W}' = \mathcal{W}_3$ and $\mathcal{X} = \widetilde{\mathcal{X}}_{\mathcal{W}_1} \cup \mathcal{X}_{\mathcal{W}_4} \cup \{U_k[h]\}$) and induction assumption, we obtain the claim for $S \in \{U_{k+1}, \cdots, U_{k_2-1}\}$. Moreover, the claim is clear if $S = U_{k_2}$. We proceed in this way, obtaining

$$2 \leq k = k_1 < k_2 < k_3 < \cdots \leq m,$$

and we keep distinguishing the cases $(i), (ii)$. After a finite number, say s, steps, either we reach $k := k_s$ where case (i) occurs, and then we are done, or we reach $k := k_s \leq m$ such that for $h := h_s$ with $k + h = m + 1$ we have $U_k[h] \in \mathrm{Add}Y$ and all modules on the coray of U_m of regular length $< h$ are in $\mathrm{Add}T$. Notice that in the step before, it only remained to prove the claim for $S \in \{U_k, \cdots, U_m\}$. But here the claim is trivial, as the intersection of the ray of S with the coray of U_m is in $\mathrm{Add}T$.

(2) Observe that $\tau U_1[m+1] \in \mathcal{W}_0 \cap \mathcal{X}_{\mathcal{W}}{}^{\perp}$, hence $\mathcal{B} \subseteq {}^{\perp}\tau U_1[m+1]$. So, $S[i] \in \mathcal{W}$. Notice that the claim is clear for $S = U_1$. So, we can assume $m > 1$ and $S \in \{U_2, \cdots, U_m\}$. We distinguish two cases.

(i) Suppose that none of the modules $U_m, U_{m-1}[2], \cdots, U_2[m-1]$ belongs to $\mathrm{Add}Y$. Then $U_1[m-1] \in \mathrm{Add}Y$. Let $\mathcal{W}' = \mathcal{W}_{U_1[m-1]}$. Then $\mathcal{X}_{\mathcal{W}} = \widetilde{\mathcal{X}}_{\mathcal{W}'} \cup \{U_1[m]\}$. It follows that $\mathcal{B} \subseteq {}^{\perp}(\mathcal{W}' \cap \widetilde{\mathcal{X}}_{\mathcal{W}'}{}^{\perp})$. By statement (1), the claim holds for $S \in \{U_2, U_3, \cdots, U_m\}$.

(ii) Suppose that one of the modules $U_m, U_{m-1}[2], \cdots, U_2[m-1]$ belongs to $\mathrm{Add}Y$. Choose $U_k[h] \in \mathrm{Add}Y$ of maximal regular length. Then $U_1[t_1]$ with $t_1 = m - h - 1 = k - 2$ must lie in $\mathrm{Add}Y$. So, $\mathrm{Add}Y$ contains precisely m modules in $\mathcal{W}$: these are the t_1 modules in $\mathcal{W}_1 = \mathcal{W}_{U_1[t_1]}$, together with h modules from $\mathcal{W}_2 = \mathcal{W}_{U_k[h]}$ and $U_1[m]$.

Notice that $\mathcal{W}_1 \cap \widetilde{\mathcal{X}}_{\mathcal{W}_1}{}^{\perp} \subseteq \mathcal{W}_0 \cap \widetilde{\mathcal{X}}_{\mathcal{W}_1}{}^{\perp} \cap \mathcal{X}_{\mathcal{W}_2}{}^{\perp} \cap U_1[m]^{\perp} = \mathcal{W}_0 \cap \mathcal{X}_{\mathcal{W}}{}^{\perp}$. Therefore $\mathcal{B} \subseteq {}^{\perp}(\mathcal{W}_1 \cap \widetilde{\mathcal{X}}_{\mathcal{W}_1}{}^{\perp})$ and the claim follows from statement (1) for $S \in \{U_2, \cdots, U_{k-1}\}$. Moreover, the claim is clear if $S = U_k$. So, it remains to verify the claim when $S \in \{U_{k+1}, \cdots, U_m\}$.

We again have two cases:

(i) Suppose none of $U_{k+1}[h-1], U_{k+2}[h-2], \cdots, U_m$ belongs to Add$Y$. Then $U_k[h-1] \in \mathrm{Add}Y$. It follows that $\mathcal{B} \subseteq {}^{\perp}(\mathcal{W}^{''} \cap \widetilde{\mathcal{X}}_{\mathcal{W}^{''}}{}^{\perp})$, here $\mathcal{W}^{''} = \mathcal{W}_{U_k[h-1]}$. Then the claim follows again from statement (1).

(ii) Suppose one of $U_{k+1}[h-1], U_{k+2}[h-2], \cdots, U_m$ belongs to Add$Y$. Choose $U_{k_2}[h_2] \in \mathrm{Add}Y$ of maximal regular length. Then $U_k[t_2]$ with $t_2 = h - h_2 - 1$ must lie in Add$Y$. It follows that $\mathcal{B} \subseteq {}^{\perp}(\widetilde{\mathcal{W}} \cap \widetilde{\mathcal{X}}_{\widetilde{\mathcal{W}}}{}^{\perp})$ where $\widetilde{\mathcal{W}} = \mathcal{W}_{U_k[t_2]}$. By statement (1), we obtain the claim for $S \in \{U_{k+1}, \cdots, U_{k_2-1}\}$. The claim is clear if $S = U_{k_2}$.

Continuing in this fashion and arguing as above, one obtains the claim for the remaining quasi-simple mosules $S \in \{U_{k_2+1}, \cdots, U_m\}$. □

Next, we have to show that the Prüfer modules corresponding to quasi-simples in $\mathcal{U}$ are filtered by modules from $\mathcal{M}$.

Proposition 3.12. *If $S \in \mathcal{U}$ belongs to a tube $\mathbf{t}_\lambda$ of rank r, then $S[r] \in \mathcal{M}$.*

Proof. The claim is clear for $\lambda \in P \setminus Q$ by construction of $\mathcal{M}$. So we assume that $\lambda \in P \cap Q$. Recall that the summands of Y are arranged in the wings $\mathcal{W}_1, \ldots, \mathcal{W}_l$ rooted in $S_{n_1}[m_1], \ldots, S_{n_l}[m_l]$. Since S is not a composition factor of $\tau^- Y$, it either lies in $\{S_{n_j}, S_{m_j+n_j+1}, \cdots, S_{n_{j+1}-1}\}$ for some $1 \leq j < l$, or in $\{S_{n_l}, S_{n_l+m_l+1}, \ldots, S_r, S_1, \ldots, S_{n_1-1}\}$. We can assume w.l.o.g. that S lies in $\{S_{n_1}, S_{m_1+n_1+1}, \cdots, S_{n_2-1}\}$. We will proceed case by case and express $S[r]$ as an iterated extension of modules from $\mathcal{R}(\lambda)$.

Case (i): $S = S_{n_1}$. We consider the following short exact sequences

$$0 \to S_{n_1}[n_2 - n_1] \to S_{n_1}[r] \to S_{n_2}[r + n_1 - n_2] \to 0$$

$$0 \to S_{n_2}[n_3 - n_2] \to S_{n_2}[r + n_1 - n_2] \to S_{n_3}[r + n_1 - n_3] \to 0$$

$$\vdots$$

$$0 \to S_{n_{l-1}}[n_l - n_{l-1}] \to S_{n_{l-1}}[r + n_1 - n_{l-1}] \to S_{n_l}[r + n_1 - n_l] \to 0$$

Since $S_{n_l}[r + n_1 - n_l]$ and all the first terms of these exact sequences are in $\mathcal{R}(\lambda)$, we obtain that $S_{n_1}[r] \in \mathcal{M}$.

Case (ii): $S = S_{n_1+m_1+1}$. Notice that $\mathcal{M}$, being extension closed, contains the wing $\mathcal{W}^{'}$ rooted in $S_{n_1+m_1+1}[n_2 - n_1 - m_1 - 1]$. Consider the exact sequence

$$0 \to S_{n_1+m_1+1}[n_2 - n_1 - m_1 - 1] \to S_{n_1+m_1+1}[n_3 - n_1 - m_1 - 1] \to S_{n_2}[n_3 - n_2] \to 0.$$

It follows that the middle term $S_{n_1+m_1+1}[m_3 - m_1 - n_1 - 1]$ is in $\mathcal{M}$. There is an exact sequence

$$0 \to S_{n_1+m_1+1}[n_3 - n_1 - m_1 - 1] \to S_{n_1+m_1+1}[r] \to S_{n_3}[r + n_1 + m_1 + 1 - n_3] \to 0.$$

Now we need to prove that $S_{n_3}[r + n_1 + m_1 + 1 - n_3] \in \mathcal{M}$. We have the following exact sequences

$$0 \to S_{n_3}[n_4 - n_3] \to S_{n_3}[r + n_1 + m_1 + 1 - n_3] \to S_{n_4}[r + n_1 + m_1 + 1 - n_4] \to 0$$

$$0 \to S_{n_4}[n_5 - n_4] \to S_{n_4}[r + n_1 + m_1 + 1 - n_4] \to S_{n_5}[r + n_1 + m_1 + 1 - n_5] \to 0$$

$$\vdots$$

$$0 \to S_{n_l}[r+n_1-n_l] \to S_{n_l}[r+n_1+m_1+1-n_l] \to S_{n_1}[m_1+1] \to 0$$

Since $S_{n_1}[m_1+1]$ and $S_{n_l}[r+n_1-n_l]$ are in $\mathcal{R}(\lambda)$, we have that $S_{n_l}[r+n_1+m_1+1-n_l] \in \mathcal{M}$. It follows that $S_{n_3}[r+n_1+m_1+1-n_3] \in \mathcal{M}$. So, $S_{n_1+m_1+1}[r] \in \mathcal{M}$.

Case (iii): $S = S_{n_2-1}$. Consider the exact sequence

$$0 \to S_{n_2-1} \to S_{n_2-1}[n_3-n_2+1] \to S_{n_2}[n_3-n_2] \to 0.$$

Since $S_{n_2}[n_3-n_2]$ and S_{n_2-1} are in $\mathcal{M}$, we obtain $S_{n_2-1}[n_3-n_2+1] \in \mathcal{M}$. There is an exact sequence

$$0 \to S_{n_2-1}[n_3-n_2+1] \to S_{n_2-1}[r] \to S_{n_3}[r-1+n_2-n_3] \to 0.$$

Now, we need to prove $S_{n_3}[r-1+n_2-n_3] \in \mathcal{M}$. We have a series of exact sequences

$$0 \to S_{n_3}[n_4-n_3] \to S_{n_3}[r-1+n_2-n_3] \to S_{n_4}[r-1+n_2-n_4] \to 0$$

$$\vdots$$

$$0 \to S_{n_{l-1}}[n_l-n_{l-1}] \to S_{n_{l-1}}[r-1+n_2-n_{l-1}] \to S_{n_l}[r-1+n_2-n_l] \to 0$$

$$0 \to S_{n_l}[r+n_1-n_l] \to S_{n_l}[r-1+n_2-n_l] \to S_{n_1}[n_2-n_1-1] \to 0.$$

Since $S_{n_l}[r+n_1-n_l]$ and $S_{n_1}[n_2-n_1-1]$ are in $\mathcal{R}(\lambda)$, we have $S_{n_l}[r-1+n_2-n_l] \in \mathcal{M}$. It follows that $S_{n_3}[r-1+n_2-n_3] \in \mathcal{M}$. So, $S_{n_2-1}[r] \in \mathcal{M}$.

The remaining cases $S \in \{S_{n_1+m_1+2}, \cdots, S_{n_2-2}\}$ are obtained similarly, keeping in mind that $\{S_{n_1}[i] \mid m_1 < i \leq n_2-n_1\} \subseteq \mathcal{M}$. □

We are now ready to prove that T arises from the universal localization $R \to R_{\mathcal{M}}$.

Theorem 3.13. *$T_{(Y,P)}$ is equivalent to $T_{\mathcal{M}} = R_{\mathcal{M}} \oplus R_{\mathcal{M}}/R$.*

Proof. By Lemma 3.4, we need to prove $\bigcap\limits_{S\in\mathcal{U}} S[\infty]^{\perp_1} \bigcap Y^{\perp_1} = \mathcal{M}^{\perp_1}$.

Firstly, $\mathcal{M} \subseteq \mathcal{S}$ by Proposition 3.9, hence $\bigcap\limits_{S\in\mathcal{U}} S[\infty]^{\perp_1} \cap Y^{\perp_1} = \mathcal{S}^{\perp_1} \subseteq \mathcal{M}^{\perp_1}$.

Secondly, if $S \in \mathcal{U}$ lies in a tube of rank r, then $\mathcal{M}$ contains $S[r]$ by Proposition 3.12, and we deduce inductively from the short exact sequence

$$0 \to S[r(n-1)] \to S[nr] \to S[r] \to 0$$

that $S[nr]$ belongs to $\mathcal{M}$ for all $n \in \mathbb{N}$. Since $S[\infty]$ is filtered by the $S[nr]$, $n \geq 1$, we conclude from Remark 3.3 that $\mathcal{M}^{\perp_1} \subseteq \bigcap\limits_{S\in\mathcal{U}} S[\infty]^{\perp_1}$.

Thirdly, we recall from [9, Proposition 4.2] that there is a decomposition $Y = \bigoplus\limits_{\lambda\in Q} \mathbf{t}_\lambda(Y)$ with $\mathbf{t}_\lambda(Y) \in \mathrm{add}\mathbf{t}_\lambda$. If $\lambda \in Q \setminus P$, then the indecomposable summands of $\mathbf{t}_\lambda(Y)$ have the form $S[i]$ such that there is $h \geq i$ with $S[h] \in \mathcal{X}(\lambda)$. In other words, they are submodules of a module from $\mathcal{X}(\lambda)$. Similarly, when $\lambda \in Q \cap P$, the indecomposable summands of $\mathbf{t}_\lambda(Y)$ are either submodules of a module from $\mathcal{X}(\lambda)$, or lie on a ray starting in some quasi-simple $S \in \mathcal{U}$.

Thus $\mathcal{M}^{\perp_1} \subseteq \bigcap\limits_{\lambda\in Q} \mathbf{t}_\lambda(Y)^{\perp_1} = Y^{\perp_1}$, again by Remark 3.3. This concludes the proof. □

Combining Theorem 3.13 and Proposition 3.2, we obtain the main result of this section.

Theorem 3.14. *For any pair (Y, P), the tilting module $T_{(Y,P)}$ is minimal if and only if the set $P \subset \mathbb{X}$ is non-empty.*

We correct a result from [9] which contained an error.

Corollary 3.15. (cf. [9, Corollary 5.10]) *Let T be a large tilting R-module. The following statements are equivalent.*

(i) There exists an injective pseudoflat ring epimorphism $\lambda : R \to S$ such that $S \oplus S/R$ is a tilting module equivalent to T.

(ii) T is equivalent to a tilting module $T_{\mathcal{U}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ where $\mathcal{U} \subset \mathbf{t}$ is a set of finite dimensional indecomposable regular (not necessarily quasi-simple) modules.

3.5 Minimal cotilting modules

The large cotilting modules have been classified in [14]. They are also parametrized by the pairs (Y, P) where Y is a branch module and P is a subset of $\mathbb{X}$. More precisely, the following modules form a complete irredundant list of all large cotilting left R-modules, up to equivalence (cf. [1, §8.5]):

$$\begin{aligned} C_{(Y,P)} = Y \oplus G \oplus \bigoplus &\{\text{all } S[\infty] \text{ in } {}^{\perp_1}Y \text{from tubes } \mathbf{t}_x, x \notin P\} \oplus \\ &\prod\{\text{all } S[-\infty] \text{ in } Y^{\perp_1} \text{from tubes } \mathbf{t}_\lambda, \lambda \in P\} \end{aligned}$$

In fact, the duality $D = \mathrm{Hom}_k(-, k)$ induces a bijection between (equivalence classes of) tilting right R-modules and cotilting left R-modules, which restricts to a bijection between minimal tilting and minimal cotilting modules (see e.g. [1, §6.1 and 6.2] and Theorem 2.12). As shown in [9, Appendix], this bijection maps $T_{(Y,P)}$ to $C_{(Y,P)}$.

Notice that a cotilting module is large if and only if it has an infinite dimensional indecomposable direct summand [9, Lemma 2.6].

Theorem 3.16. *A large cotilting module is a minimal cotilting module if and only if it has an adic module as a direct summand.*

Proof. By Theorems 2.12 and 3.14, a large cotilting module C is minimal if and only if $C = C_{(Y,P)}$ is equivalent to $D(T_{(Y,P)})$ for a tilting module $T = T_{(Y,P)}$ given by a pair (Y, P) with $P \neq \emptyset$. Notice that $C_{(Y,P)}$ can be rewritten as follows

$$C_{(Y,P)} = Y \oplus G \oplus \bigoplus\{\text{all } S[\infty] \text{ in } {}^{\perp_1}Y \text{from tubes } \mathbf{t}_\lambda, \lambda \notin P\} \oplus \prod_{S \in \mathcal{U}} S[-\infty]$$

and $P \neq \emptyset$ if and only if $\mathcal{U} \neq \emptyset$, which amounts to C having an adic direct summand. □

4 Ascent of minimality

The behaviour of (co)silting modules under ring extensions has been studied in [12]. In particular, it was shown that over commutative rings every silting module extends to a silting module.

Theorem 4.1. [12, Theorems 2.2 and 2.7] *Let $\lambda : R \to S$ be a ring homomorphism, and let T be a silting module with respect to a projective presentation σ. Then $T \otimes_R S$ is a silting S-module with respect to $\sigma \otimes_R S$ if and only if $T \otimes_R S$, viewed as an R-module, lies in the silting class* Gen$T$.

The latter condition is verified whenever R and S are commutative rings.

Of course, if $\lambda : R \to S$ is surjective with kernel I, then every silting right R-module T verifies the condition $T \otimes_R S \simeq T/TI \in \mathrm{Gen}T$ and therefore extends to a silting S-module $T \otimes_R S$, cf. [12, Corollary 2.4]. In general, however, the condition in the theorem above can fail, even when T is minimal.

Example 4.2. Let R be the Kronecker algebra, and let T be a silting R-module. Then the R-module $T \otimes_R S$ lies in $\mathrm{Gen}T$ for every homological ring epimorphism $\lambda : R \to S$ if and only if T is not equivalent to the simple projective module P_1.

The statement follows from the classification results reviewed in Section 3.2. We proceed in several steps.

Step 1. Let $T = P_1$, and let $id_R \neq \lambda : R \to S$ be an injective universal localization. Then the associated bireflective subcategory $\mathcal{X}$ coincides with $\mathrm{Add}P_i$ for some $i \geq 2$, or with $\mathrm{Add}Q_{i+1}$ for some $i \geq 1$, or with $\mathcal{U}^{\perp}$ for a non-empty set $\mathcal{U}$ of simple regular modules.

Notice that in all cases there is a non-trivial map from T to a module in $\mathcal{X}$. This is clear in the first two cases, and in the third we can for instance choose the embedding of P_1 in the generic module $G \in \mathcal{X}$. It follows that the $\mathcal{X}$-reflection $T \to T \otimes_R S$ is non-trivial.

We claim that $T \otimes_R S \neq 0$ does not lie in $\mathrm{Gen}T$. Indeed, no non-trivial module in $\mathcal{X}$ can belong to $\mathrm{Gen}T = \mathrm{Add}P_1$. Again, this is clear in the first two cases, and in the third we observe that by the Auslander-Reiten formula $\mathrm{Ext}^1_R(S, P_1) \cong D\mathrm{Hom}_R(P_1, S) \neq 0$ for every simple regular module S.

Step 2. Let now $T = Q_1$, and let $\lambda : R \to S$ and $\mathcal{X}$ be as above. Then there are no nontrivial maps from T to $\mathcal{X}$, which is again clear in the first two cases, and in the third it follows from the fact that $\mathrm{Hom}_R(S, Q_1) \neq 0$ for every simple regular module S. We conclude that the $\mathcal{X}$-reflection $T \to T \otimes_R S = 0$ is trivial, and $T \otimes_R S$ lies in $\mathrm{Gen}T$.

Step 3. By the discussion above, it remains to prove the if-part of the statement, and w.l.o.g. we can assume that λ is injective, and T is a tilting module. We have one of the following cases.

Case (i). λ is the universal localization at $P_{i+1}, i \geq 2$, and $\mathcal{X} = \mathrm{Add}P_i$.

If $\mathrm{Hom}_R(T, P_i) = 0$, then the $\mathcal{X}$-reflection $T \to T \otimes_R S = 0$ is trivial, and the claim holds true. If $\mathrm{Hom}_R(T, P_i) \neq 0$, then T must be preprojective, because the non-preprojective silting modules all belong to the class $\mathrm{Gen}\mathbf{L} = {}^{\perp_0}\mathbf{p}$. More precisely, T must be equivalent to $P_j \oplus P_{j+1}$ for some $j \leq i$, hence it generates P_i, and therefore also $T \otimes_R S$ which belongs to $\mathcal{X} = \mathrm{Add}P_i$.

Case (ii). λ is the universal localization at $Q_i, i \geq 1$, and $\mathcal{X} = \mathrm{Add}Q_{i+1}$.

As above we can assume w.l.o.g. that $\mathrm{Hom}_R(T, Q_{i+1}) \neq 0$. If T is preinjective, then it must be equivalent to $Q_j \oplus Q_{j+1}$ for some $j \geq i$, hence it generates Q_{i+1}, and therefore also $T \otimes_R S$ which belongs to $\mathcal{X} = \mathrm{Add}Q_{i+1}$.

Now assume that T is not preinjective. We know from the classification that the non-preinjective silting modules all belong to the class $\mathbf{q}^{\perp_0}$ of modules without non-trivial maps from the preinjective component $\mathbf{q} = \{Q_1, Q_2, \cdots\}$, which by the Auslander-Reiten-formula coincides with the class ${}^{\perp_1}\mathbf{q}$. In particular, $\mathrm{Ext}^1_R(T, Q_{i+1}) = 0$, and Q_{i+1} belongs to the tilting class $\mathrm{Gen}T$. Hence T generates $T \otimes_R S$.

Case (iii). λ is the universal localization at a set $\mathcal{U} \neq \emptyset$ of simple regulars, and $\mathcal{X} = \mathcal{U}^{\perp}$.

Again we assume w.l.o.g. that $\mathrm{Hom}_R(T, \mathcal{X}) \neq 0$. Notice that this implies that T is not preinjective since $\mathcal{U}^{\perp} \subset \mathbf{q}^{\perp_0}$. Moreover, $\mathcal{U}^{\perp} \subset {}^{\perp_0}\mathbf{p}$, which by the Auslander-Reiten formula coincides with the class $\mathbf{p}^{\perp_1}$. Hence $\mathrm{Ext}^1_R(T, T \otimes_R S) = 0$ and $T \otimes_R S$ is T-generated whenever T is preprojective. Moreover, $T \otimes_R S \in {}^{\perp_0}\mathbf{p} = \mathrm{Gen}\mathbf{L}$ is also T-generated when T is equivalent to the Lukas tilting module.

It remains to check the case when $T = R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R$ for a non-empty set $\mathcal{V}$ of simple regular modules. Then $\text{Gen}T = \mathcal{V}^{\perp_1}$ consists of the modules X with $\text{Ext}^1_R(V, X) = 0$ for all $V \in \mathcal{V}$. Pick a module $V \in \mathcal{V}$. If V also belongs to $\mathcal{U}$, then $\text{Ext}^1_R(V, T \otimes_R S) = 0$. If V does not belong to $\mathcal{U}$, then it belongs to $\mathcal{U}^\perp = \mathcal{X}$, as different tubes are Hom- and Ext-orthogonal. Since $R_{\mathcal{V}} \to R_{\mathcal{V}} \otimes_R S$ is an $\mathcal{X}$-reflection, we infer that $\text{Hom}_R(R_{\mathcal{V}} \otimes_R S, V) \cong \text{Hom}_R(R_{\mathcal{V}}, V)$, and by the Auslander-Reiten formula $\text{Ext}^1_R(V, R_{\mathcal{V}} \otimes_R S) \cong \text{Ext}^1_R(V, R_{\mathcal{V}}) = 0$. We conclude that $R_{\mathcal{V}} \otimes_R S$ is T-generated, and so are $R_{\mathcal{V}}/R \otimes_R S$ and $T \otimes_R S = (R_{\mathcal{V}} \otimes_R S) \oplus (R_{\mathcal{V}}/R \otimes_R S)$.

Now we return to the commutative case. If $\lambda : R \to S$ is a ring epimorphism and R is commutative, then so is S by [22]. It follows from Theorem 4.1 that every silting module extends to a silting module along λ. In fact, when R is commutative and hereditary, and λ is pseudoflat, also minimality is preserved. Indeed, S is then hereditary as well by [11, p.324], and all silting modules over commutative hereditary rings are minimal, as we observe next.

Remark 4.3. Let R be a commutative hereditary ring, and let T be a silting module. Denote by $C = T^+$ the corresponding cosilting module. As discussed in [3, paragraph after Proposition 6.5], there is a flat epimorphism $\lambda : R \to S$ such that $S^+ \oplus \text{Ker}\lambda^+$ is a cosilting module equivalent to C. Moreover, by Theorem 2.12, we also have a minimal silting module $T' = S \oplus \text{Coker}\lambda$. Now T and T' are both mapped to C under the silting-cosilting-bijection $T \mapsto T^+$ established in [2, Corollary 3.6], hence they are equivalent. This shows that T is a minimal silting module.

We now want to determine further conditions ensuring that minimality is preserved by ring extensions. Let us first explore how to relax the assumption hereditary.

Lemma 4.4. *Let $\lambda : R \to S$ be a pseudoflat ring epimorphism such that S_R has projective dimension at most one. Then $T = S \oplus \text{Coker}\lambda$ is a minimal silting module.*

Proof. We know from [3, Example 4.15] that S_R has a presilting presentation. By the proof of [7, Proposition 1.3], the R-module homomorphism λ can be lifted to a triangle $R \overset{\phi}{\to} \sigma_0 \to \sigma_1 \to R[1]$ in the derived category $D(\text{Mod}R)$ such that T is a silting module with respect to the projective presentation $\sigma = \sigma_0 \oplus \sigma_1$. It follows that ϕ is a left $\text{Add}\sigma$-approximation. It remains to prove that ϕ is left minimal. Consider a morphism $g \in \text{End}_{D(\text{Mod}R)}\sigma_0$ such that $g\phi = \phi$.

$$\begin{array}{ccc} R & \xrightarrow{\phi} & \sigma_0 \\ {\scriptstyle\phi}\downarrow & \nearrow_{g} & \\ \sigma_0 & & \end{array}$$

Applying the cohomology functor $\text{H}^0(-)$ to the diagram, we obtain the following commutative diagram.

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & S \\ {\scriptstyle\lambda}\downarrow & \nearrow_{\text{H}^0(g)} & \\ S & & \end{array}$$

where $\text{H}^0(g)$ is an isomorphism since λ is left minimal. Since $\text{H}^i(\sigma_0) = 0$ for any $i \neq 0$, we infer that g is an isomorphism. It follows that T is a minimal silting R-module. $\square$

The push-out (or coproduct) of ring epimorphisms will be an important tool for our considerations.

Lemma 4.5. [11, Proposition 5.2],[15, Lemma 6.2],[4, Lemma 4.1] *Let R be an arbitrary ring, and let $\lambda : R \to S$ and $\lambda^{'} : R \to S^{'}$ be ring epimorphisms with associated bireflective subcategories $\mathcal{X}$ and $\mathcal{X}^{'}$. Consider the push-out $S \sqcup_R S^{'}$ of λ and $\lambda^{'}$ in the category of rings*

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & S \\ {\scriptstyle \lambda^{'}}\downarrow & & \downarrow{\scriptstyle \mu^{'}} \\ S^{'} & \xrightarrow[\mu]{} & S \sqcup_R S^{'} \end{array} \tag{4.1}$$

Then μ and $\mu^{'}$ are ring epimorphisms and the bireflective subcategory associated to the composition $\mu\lambda^{'} = \mu^{'}\lambda$ is given by $\mathcal{X} \cap \mathcal{X}^{'}$. If λ is pseudoflat, so is μ. If λ is a universal localization, so is μ.

Lemma 4.6. *Let R be a commutative ring, and let $\lambda : R \to S$ and $\lambda^{'} : R \to S^{'}$ be ring epimorphisms. Then Coker$\lambda \otimes_R S' \cong$ Cokerμ, where $\mu : S' \to S \otimes_R S^{'}$ is given by the push-out of λ and $\lambda^{'}$.*

Proof. It is well known (see e.g. [15, Lemma 6.3]) that $S \sqcup_R S^{'} \cong S \otimes_R S^{'}$. We keep the notation of diagram (4.1), and set $\tilde{\lambda} = \mu^{'}\lambda = \mu\lambda^{'}$. We know from Lemma 4.5 that $\tilde{\lambda}$ is a ring epimorphism, and its corresponding bireflective subcategory is $\tilde{\mathcal{X}} = \mathcal{X} \cap \mathcal{X}^{'}$. Applying the functor $- \otimes_R S^{'}$ to the diagram (4.1), we obtain the following commutative diagram

$$\begin{array}{ccc} R \otimes_R S^{'} & \xrightarrow{\lambda \otimes_R S^{'}} & S \otimes_R S^{'} \\ {\scriptstyle \lambda^{'} \otimes_R S^{'}}\downarrow & \searrow^{\tilde{\lambda} \otimes_R S^{'}} & \downarrow{\scriptstyle \mu^{'} \otimes_R S^{'}} \\ S^{'} \otimes_R S^{'} & \xrightarrow[\mu \otimes_R S^{'}]{} & S \otimes_R S^{'} \otimes_R S^{'}. \end{array}$$

Since $\lambda^{'}$ is a ring epimorphism, the multiplication map $S^{'} \otimes_R S^{'} \to S^{'}$ is an isomorphism, and so is the map $\lambda' \otimes_R S'$. We claim that $\mu' \otimes_R S'$ is an isomorphism as well. To this end, we will show that both $\lambda \otimes_R S'$ and $\tilde{\lambda} \otimes_R S'$ are $\tilde{\mathcal{X}}$-reflections of $R \otimes_R S'$. Our claim will then follow from the uniqueness of reflections.

Let us consider the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & S \\ {\scriptstyle \psi_R}\downarrow & & \downarrow{\scriptstyle \psi_S} \\ R \otimes_R S^{'} & \xrightarrow{\lambda \otimes_R S^{'}} & S \otimes_R S^{'}. \end{array}$$

Recall that λ is an $\mathcal{X}$-reflection of R, and ψ_R, ψ_S are $\mathcal{X}^{'}$-reflections of R and S, respectively. Since $S \otimes_R S^{'} \in \tilde{\mathcal{X}}$, we conclude that $\lambda \otimes_R S'$ is an $\tilde{\mathcal{X}}$-reflection. We argue similarly for $\tilde{\lambda} \otimes_R S'$, using the commutative diagram below together with the fact that $S \otimes_R S^{'} \otimes_R S^{'} \simeq S \otimes_R S^{'} \in \tilde{\mathcal{X}}$

$$\begin{array}{ccc} R & \xrightarrow{\tilde{\lambda}} & S \otimes_R S^{'} \\ {\scriptstyle \psi_R}\downarrow & & \downarrow{\scriptstyle \psi_{S \otimes_R S^{'}}} \\ R \otimes_R S^{'} & \xrightarrow{\tilde{\lambda} \otimes_R S^{'}} & S \otimes_R S^{'} \otimes_R S^{'}. \end{array}$$

Next, we observe that the $\mathcal{X}'$-reflections $\psi_{S'}$ and $\psi_{S \otimes_R S'}$ are isomorphisms, because $S^{'}$ and $S \otimes_R S^{'}$

are in $\mathcal{X}^{'}$. So, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
R \otimes_R S^{'} & \xrightarrow{\lambda \otimes_R S^{'}} & S \otimes_R S^{'} & \longrightarrow & \mathrm{Coker}\lambda \otimes_R S^{'} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \lambda^{'} \otimes_R S^{'}} & & \downarrow{\scriptstyle \mu^{'} \otimes_R S^{'}} & & \downarrow & & \\
S^{'} \otimes_R S^{'} & \xrightarrow{\mu \otimes_R S^{'}} & S \otimes_R S^{'} \otimes_R S^{'} & \longrightarrow & \mathrm{Coker}\mu \otimes_R S^{'} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \psi^{-1}_{S^{'}}} & & \downarrow{\scriptstyle \psi^{-1}_{S \otimes_R S^{'}}} & & \downarrow & & \\
S^{'} & \xrightarrow{\mu} & S \otimes_R S^{'} & \longrightarrow & \mathrm{Coker}\mu & \longrightarrow & 0.
\end{array}$$

where the isomorphisms in the first two columns induce an isomorphism $\mathrm{Coker}\lambda \otimes_R S^{'} \simeq \mathrm{Coker}\mu$. □

Proposition 4.7. *Let R be a commutative ring, and let $T = S \oplus_R \mathrm{Coker}\lambda$ be a silting module arising from a ring epimorphism $\lambda : R \to S$. Assume $\lambda^{'} : R \to S^{'}$ is a ring epimorphism such that $S \otimes_R S^{'}$ has a presilting presentation over $S^{'}$. Then $T \otimes_R S^{'}$ is a silting $S^{'}$-module which arises from the ring epimorphism $\mu : S' \to S \otimes_R S^{'}$ given by the push-out of λ and $\lambda^{'}$.*

Proof. By Proposition 2.8, the ring epimorphism $\mu : S' \to S \otimes_R S^{'}$ induces a silting S'-module $S \otimes_R S^{'} \oplus \mathrm{Coker}\mu$, which by Lemma 4.6 is isomorphic to $T \otimes_R S'$. □

Corollary 4.8. *Let R be a commutative ring, and let $T = S \oplus_R \mathrm{Coker}\lambda$ be a silting module arising from a ring epimorphism $\lambda : R \to S$. Let $\lambda^{'} : R \to S^{'}$ be a ring epimorphism such that the projective dimension of $S \otimes_R S^{'}$ over $S^{'}$ is at most one.*

(1) If λ is a pseudoflat ring epimorphism, then $T \otimes_R S^{'}$ is a minimal silting $S^{'}$-module.

(2) If λ is a universal localization, then $T \otimes_R S^{'}$ arises from a universal localization.

Proof. By Lemma 4.5, the ring epimorphism μ is pseudoflat. Statement (1) then follows by combining Lemma 4.6 and 4.4. Statement (2) follows similarly by observing that μ is a universal localization. □

Let us now turn to the dual situation. We will see that minimality of cosilting modules is often preserved by extensions along arbitrary ring epimorphisms.

Proposition 4.9. [12, Remark 2.3] *Let $\lambda : R \to S$ be a ring homomorphism, and let C be a cosilting module with respect to a injective copresentation ω. Then $\mathrm{Hom}_R(S, C)$ is a cosilting S-module with respect to $\mathrm{Hom}_R(S, \omega)$ if and only if $\mathrm{Hom}_R(S, C)$, viewed as an R-module, lies in $\mathrm{Cogen}C$.*

Proposition 4.10. *Let R be a commutative ring and C be a minimal cosilting module arising from a ring epimorphism $\lambda : R \to S$. Assume that $\lambda^{'} : R \to S^{'}$ is a ring epimorphism such that $(S \otimes_R S^{'})^{+}$ has a precosilting copresentation over $S^{'}$. Then $\mathrm{Hom}_R(S^{'}, C)$ is a minimal cosilting $S^{'}$-module.*

Proof. The push-out of λ and λ' in (4.1) gives rise to a ring epimorphism $\mu : S' \to S \otimes_R S^{'}$ as in condition (i) of Theorem 2.10. Hence $(S \otimes_R S^{'})^{+} \oplus \mathrm{Ker}\mu^{+}$ is a minimal cosilting $S^{'}$-module. On the other hand, we infer from the shape of $C = S^{+} \oplus \mathrm{Ker}\lambda^{+}$ that $\mathrm{Hom}_R(S^{'}, C) = \mathrm{Hom}_R(S^{'}, S^{+}) \oplus \mathrm{Hom}_R(S^{'}, \mathrm{Ker}\lambda^{+}) \simeq \mathrm{Hom}_R(S^{'}, S^{+}) \oplus \mathrm{Hom}_R(S^{'}, (\mathrm{Coker}\lambda)^{+}) \simeq (S \otimes_R S^{'})^{+} \oplus (\mathrm{Coker}\lambda \otimes_R S^{'})^{+}$. Now recall from Lemma 4.6 that $\mathrm{Coker}\lambda \otimes_R S^{'} \simeq \mathrm{Coker}\mu$. We deduce that $\mathrm{Hom}_R(S^{'}, C) \simeq (S \otimes_R S^{'})^{+} \oplus \mathrm{Ker}\mu^{+}$, which concludes the proof. □

Corollary 4.11. *Let R be a commutative noetherian ring, or a commutative ring of weak global dimension at most one. Then all minimal cosilting modules extend to minimal cosilting modules along any pseudoflat ring epimorphism.*

Proof. Assume that R is commutative noetherian. Recall from [4, Proposition 4.5] that every pseudoflat ring epimorphism $R \to S$ is flat. By Remark 2.11, every minimal cosilting module C arises from a flat ring epimorphism $\lambda : R \to S$. If $\lambda' : R \to S'$ is a flat ring epimorphism, then S' is again commutative and noetherian by [22, Corollary 1.2 and Proposition 1.6]. We infer from Lemma 4.5 that the push-out of λ and λ' in (4.1) gives rise to a flat ring epimorphism $\mu : S' \to S \otimes_R S^{'}$. Then $(S \otimes_R S^{'})^+$ has a precosilting copresentation over $S^{'}$ by Remark 2.11, and the claim follows from Proposition 4.10.

Now assume that R is a commutative ring of weak global dimension at most one. Then every pseudoflat ring epimorphism $R \to S$ is homological, and S has weak global dimension at most one, because the functors Tor_i^R and Tor_i^S agree on S-modules for all $i \geq 1$ by [17, Theorem 4.4]. So, we can proceed as above. By Remark 2.11, every minimal cosilting module C arises from a homological ring epimorphism $\lambda : R \to S$. The push-out of λ with a homological ring epimorphism $\lambda' : R \to S'$ yields a homological ring epimorphism $\mu : S' \to S \otimes_R S^{'}$. Then $(S \otimes_R S^{'})^+$ has a precosilting copresentation over $S^{'}$, and the claim follows from Proposition 4.10. □

Recall that a ring R is said to be *semihereditary* if every finitely generated right ideal is projective. Moreover, a cosilting module is said to be *of cofinite type* if it is equivalent to the dual T^+ of a silting module T.

Corollary 4.12. *Over a commutative ring, every minimal cosilting module arising from a universal localization extends to a minimal cosilting module along any ring epimorphism.*

In particular, over a commutative (semi)hereditary ring, every cosilting module (of cofinite type) extends to a minimal cosilting module along any ring epimorphism.

Proof. Let C be a minimal cosilting module arising from a universal localization $\lambda : R \to S$, and let $\lambda' : R \to S'$ be an arbitrary ring epimorphism. We know from Lemma 4.5 that the push-out of λ and λ' in (4.1) gives rise to a universal localization $\mu : S' \to S \otimes_R S^{'}$. Then μ is a flat ring epimorphism by [4, Corollary 4.4]. By Remark 2.11, this implies that $(S \otimes_R S^{'})^+$ has a precosilting copresentation over $S^{'}$, and the claim follows from Proposition 4.10.

Now assume that R is semihereditary. As observed in [3, paragraph after Proposition 6.5], every cosilting module of cofinite type is a minimal cosilting module arising from a universal localization. The claim then follows from the first statement. Finally, if R is hereditary, all cosilting modules are of cofinite type by [3, Theorem 3.11]. □

Acknowledgements. Lidia Angeleri Hügel acknowledges partial support by Fondazione Cariverona, program "Ricerca Scientifica di Eccellenza 2018", project "Reducing complexity in algebra, logic, combinatorics - REDCOM". Weiqing Cao acknowledges support from China Scholarship Council (Grant No. 201906860022).

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