Nonlinear Anal. RWA 93 (2027) 104672

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

The master equation in a bounded domain with absorption

Luca Di Persio [a], Matteo Garbelli [a,*], Michele Ricciardi [b]

[a] Department of Computer Science, University of Verona, Strada le Grazie 15, Verona, 37134, Italy
[b] Department of Economics, LUISS - Guido Carli University of Rome, Viale Romania 32, Roma, 00197, Italy

ARTICLE INFO

Keywords:
Master equation
First order MFG
Dirichlet conditions
Absorption

ABSTRACT

We analyze the Master Equation within Mean Field Games (MFG) theory considering a bounded domain with homogeneous Dirichlet conditions. Concerning the N-players differential game, the player's dynamic ends when touching the boundary. We analyze the well-posedness of the Master Equation and the regularity of its solutions for a suitable class of parabolic equations.

1. Introduction

Mean Field Games (MFG) theory was introduced in 2006 by J.-M. Lasry and P.-L. Lions ([1–4]) to describe the asymptotic behavior of differential games with a large number of players, also called agents. A similar definition was given in the same year by Caines, Huang and Malhamé [5].

In the description of a game with N players, each agent chooses his own strategy to minimize a certain functional cost, which also depends on the strategies of the other agents. The structure of this problem becomes really intricate when the number of players is large, and this leads us to study the asymptotic behavior of the Nash equilibria for $N \gg 1$, under some symmetry conditions for the agents and their dynamics. The system introduced to describe the macroscopic behavior of the optimal strategies for $N \to +\infty$ is called *Mean Field Games system*, and it is composed of a backward Hamilton-Jacobi-Bellman equation for the value function u of the single agent coupled with a forward Fokker-Planck equation satisfied by m, the distribution law of the population. A typical form of this system is the following:

$$\begin{cases} -u_t - \mathrm{tr}(a(x)D^2u) + H(t,x,Du) = F(t,x,m), \\ m_t - \sum_{i,j} \partial^2_{i,j}(a_{ij}(x)m) - \mathrm{div}(mH_p(t,x,Du)) = 0, \\ m(0) = m_0, \qquad u(T) = G(x,m(T)). \end{cases} \tag{1}$$

Here, H is called the *Hamiltonian* of the system, whereas a is a uniformly elliptic matrix, which corresponds to the second-order part of the infinitesimal generator associated with the SDE

$$dX_t = b(t,X_t,\alpha_t)dt + \sqrt{2}\sigma(X_t)dB_t, \tag{2}$$

which represents the controlled dynamic of the generic player.

Lions proved, in his lectures at Collége de France (some lectures are still available in [6]), that the solutions of the MFG system are the trajectories of an infinite-dimensional PDE, called *Master Equation*, which summarizes the MFG system in a unique equation. The

* Corresponding author.
E-mail address: matteo.garbelli@univr.it (M. Garbelli).

https://doi.org/10.1016/j.nonrwa.2026.104672
Received 28 October 2025; Received in revised form 22 May 2026; Accepted 29 May 2026
Available online 4 June 2026
1468-1218/© 2026 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

importance of the Master Equation relies on the so-called *convergence problem*: a rigorous convergence result of the Nash equilibria in the N-players game towards the optimal strategies in the MFG presents many difficulties, due to the lack of compactness properties of (1). Hence, most of the results in the literature related to the convergence problem are proved by means of the Master Equation.

So far, the Master Equation and the convergence problem have been studied in various frameworks. However, most of the literature mainly considers the case where the players' state space is the torus $\mathbb{T}^d$, hence a periodic framework, or the whole space $\mathbb{R}^d$.

Nevertheless, in many applications it is crucial to consider situations where the dynamic of the players is confined in a bounded domain, and boundary conditions become unavoidable issues we have to deal with. See, for example, the models developed in [7–11].

In this setting, several different boundary behaviors can arise depending on how the controlled dynamics (2) and the Mean Field Games system are modeled. The most common situations can be summarized as follows:

(i) *Invariant domains.* The drift and diffusion coefficients in the dynamics (2) are constructed in such a way that the trajectories of the agents remain inside the domain for any admissible control. This setting leads to Mean Field Games with *invariance conditions* on the boundary, which have been studied, e.g., in [12].

(ii) *State constraints.* Not all the controls guarantee the required confinement, but the optimal control is chosen in such a way that the optimal trajectories stay inside the domain. This situation is referred to as Mean Field Games with *state constraints*. A comprehensive result of first-order Mean Field Games (without diffusion σ in (2)) has been studied in [13,14]. See also [15–18] for the first-order case, and [19] for a result in the second-order case.

(iii) *Reflecting dynamics.* In this case, the process X_t is forced to remain in the domain by means of a reflection term acting on the boundary (which may represent, in the applications, a physical wall or barrier). The associated Mean Field Games system is then coupled with *Neumann boundary conditions* for both the HJB and the Fokker-Planck equations.

(iv) *Absorbing boundaries.* In this last case, the process can exit the domain, but the game stops as soon as the trajectory hits the boundary. Here, the Mean Field Games system (1) is complemented by *Dirichlet boundary conditions*, corresponding to the *absorption* of the players at the boundary.

The last two cases have been widely studied for the MFG system. See, for example, [20] for a very comprehensive result of Dirichlet and Neumann boundary conditions with minimal assumptions on the data, [21] for forward-forward Mean Field Games with Dirichlet or Neumann Boundary conditions, and [22] for Neumann boundary conditions in multi-population MFG. See also [23] for a result on first-order Mean Field Games with Neumann boundary conditions.

As regards the Master Equation, the results in a bounded domain are significantly fewer. The papers [24,25] provide a well-posedness result on the Master Equation and the study of the convergence problem in the case of Neumann boundary conditions, but we are not aware of results related to the other scenarios previously described.

In our paper, we aim to study the well-posedness of the Master Equation in a bounded domain $\Omega \subset \mathbb{R}^d$ and in a framework of absorbing boundary conditions. This paper can be thought of as a first, crucial step in order to prove the related convergence problem for N-player games with absorbing boundary conditions, which will be presented in a forthcoming paper.

In this case, the Master Equation is defined from its characteristics, which are solutions of the MFG system (22) constrained with the following Dirichlet boundary conditions: for all $x \in \partial\Omega$ and $t \in [0,T]$,

$$u(t,x) = 0, \qquad m(t,x) = 0. \tag{3}$$

To be more precise, we consider the set $\mathcal{P}^{sub}(\Omega)$ of Borel sub-probability measures on Ω, and for $(t_0,x,m_0) \in (0,T)\times\Omega\times\mathcal{P}^{sub}(\Omega)$ we consider the solution (u,m) of the MFG system (1), with boundary conditions (3) and with initial condition $m(t_0) = m_0$. Then we define the function U as

$$U : [0,T]\times\Omega\times\mathcal{P}^{sub}(\Omega), \qquad U(t_0,x,m_0) := u(t_0,x). \tag{4}$$

If we compute, at least formally, the equation satisfied by U, we obtain an infinite-dimensional equation, called the Master Equation. In this case, the Master Equation takes the following form:

$$\begin{cases} -\partial_t U(t,x,m) - \mathrm{tr}\big(a(x)D_x^2 U(t,x,m)\big) + H\big(t,x,D_x U(t,x,m)\big) \\ \quad - \displaystyle\int_\Omega \mathrm{tr}\big(a(y)D_y D_m U(t,x,m,y)\big)\,dm(y) \\ \quad + \displaystyle\int_\Omega D_m U(t,x,m,y)\cdot H_p(t,y,D_x U(t,y,m))\,dm(y) = F(t,x,m) \\ \text{in } [0,T]\times\Omega\times\mathcal{P}^{sub}(\Omega)\,, \\ U(T,x,m) = G(x,m) \text{in } \Omega\times\mathcal{P}^{sub}(\Omega)\,, \\ U(t,x,m) = 0 \text{ for } (t,x,m)\in[0,T]\times\partial\Omega\times\mathcal{P}^{sub}(\Omega)\,, \\ \dfrac{\delta U}{\delta m}(t,x,m,y) = 0 \text{for } (t,x,m,y)\in[0,T]\times\Omega\times\mathcal{P}^{sub}(\Omega)\times\partial\Omega\,, \end{cases} \tag{5}$$

where $D_m U$ and $\frac{\delta U}{\delta m}$ are two derivatives, considered with respect to the measure variable m, whose precise definition will be given later.

In the literature, an equation like (5) is often called the *First order Master Equation*, or *Master Equation without common noise*. This terminology reflects the fact that, in the underlying mean field model, the representative agent evolves according to the dynamics

(2), where the stochastic term dB_t accounts for an *idiosyncratic noise*, that is, a source of randomness affecting each individual independently of the others. In contrast, when a common source of uncertainty dW_t, shared by all agents, is included in the dynamics, one obtains the so-called *Second order Master Equation*, or *Master Equation with common noise*, which involves additional second-order terms in the measure variable.

After being introduced in [3], the Master Equation has been studied in many papers, almost always in the periodic case $\Omega = \mathbb{T}^d$, or in the whole space $\Omega = \mathbb{R}^d$ (except for the already cited articles [24,25]). In [26], Buckdahn, Li, and Peng proved the well-posedness of the first-order Master equation without coupling terms by means of probabilistic arguments, while Chassagneux, Crisan, and Delarue, in [27], provided a first exhaustive existence and uniqueness result of solution, still without common noise. Moreover, Gangbo and Swiech, in [28], gave a short-time existence for the Master Equation in the presence of common noise.

The most relevant result was certainly given in [29], where Cardaliaguet, Delarue, Lasry, and Lions proved the well-posedness of the Master Equation, with and without common noise, in a periodic setting.

Moreover, Carmona and Delarue in [30] derived convergence results in the whole space, while in [31,32], Delarue, Lacker and Ramanan used the Master Equation to analyse the large deviation problem as well as the central limit theorem. Concerning the major-minor problem, Cardaliaguet, Cirant and Porretta studied a convergence result in [33]. For finite state problems, we refer to the works of Bayraktar and Cohen in [34] and by Cecchin and Pelino in [35]. See also the work of Bertucci, Lasry and Lions [36], who studied the Master Equation for the Finite State Space Planning Problem. More recently, in the setting of MFG with exhaustible resources, i.e. the continuum limit of a dynamic game of exhaustible resources modelling Cournot competition between producers, see e.g. [37,38] for more details, Graber and Sircar studied the Master Equation for a MFG of controls with absorption in [39]. Differently from the classical setting studied in this paper where the interaction is through the mean of the state variable, in [39] the authors studied a model of Cournot competition between producers whose states depend on the empirical measure over controls.

Other important papers about the Master Equation and the convergence problem are given by [40–52].

Concerning the model with absorption, the MFG system has already been studied in the literature. See, for example, [10], where the authors studied the N-player game and the MFG by means of probabilistic arguments, working directly on the SDEs describing the dynamics and introducing renormalized empirical measures to take care of the loss of mass due to the players leaving the domain. Those results are also used to prove that the optimal strategies of the MFG provide $\epsilon-$approximated Nash equilibria in the N-players game. Some generalizations of the described results were given in [9], where a model of bank run is considered within the framework of an MFG model with absorption and common noise, and in [53], where a dependence on past absorptions is allowed. See also [54] for an elliptic first-order case. Similar problems, such as minimal-time MFGs where agents want to leave a given bounded domain through its boundary (or a part of it) in minimal time, with application to crowd motion, have been studied in [55]. Similarly, in [56–59], optimal stopping MFGs were studied for a model where a representative agent chooses both the optimal control and the optimal time to exit the game.

As already mentioned, the main goal of this paper is to establish the well-posedness of the first-order Master Eq. (5) in a bounded domain with absorbing (Dirichlet) boundary conditions. To the best of our knowledge, this is the first existence result for the Master Equation in such a framework. The natural setting for this problem is the space $\mathcal{P}^{sub}(\Omega)$ of sub-probability measures, which provides the appropriate structure to describe the dissipation of mass caused by absorption at the boundary.

First, we discuss the boundary conditions in (5). The condition $U(t,x,m) = 0$ for $x \in \partial\Omega$ reflects the fact that a player starting from the boundary is immediately absorbed and the corresponding value vanishes. The condition

$$\frac{\delta U}{\delta m}(t,x,m,y) = 0 \qquad \text{for } y \in \partial\Omega \tag{6}$$

has a different role. It states that an infinitesimal perturbation of the distribution located at the boundary produces no variation in the value function of a player that is still active inside the domain. Once a player reaches the boundary, that player exits the game and no longer contributes to the interactions governing the active population. Therefore, the value of a player still in Ω may depend on the mass that remains inside the domain, but it should not depend on the exact boundary position of the mass corresponding to players that have already been absorbed. The condition on $\delta U/\delta m$ is the infinitesimal expression of this fact.

At a formal level, assume that U admits a smooth extension to $[0,T]\times\overline{\Omega}\times\mathcal{P}(\overline{\Omega})$. If two measures on $\overline{\Omega}$ coincide on every Borel subset of Ω and differ only by a redistribution of mass supported on $\partial\Omega$, then they must yield the same value for a player that is still in the game. In particular, $U(t,x,\delta_y) = U(t,x,\delta_{\tilde{y}})$ whenever $y, \tilde{y} \in \partial\Omega$. This is the PDE counterpart of the fact that absorbed players no longer affect the running or terminal interactions of the surviving agents.

The N-player counterpart of this boundary condition will be discussed in detail in the forthcoming paper on the convergence problem mentioned above.

We also want to underline that the existence proof for the Master Equation relies on several estimates for the associated Mean Field Games system, and in particular on a careful study of a Fokker-Planck equation with Dirichlet boundary conditions in a very general form. As a byproduct, we refine some regularity estimates for such equations that may be of independent interest.

The function U is defined as in (4), and various estimates, such as global bounds and global Lipschitz regularity, are established. Notably, one of the key challenges in demonstrating that U satisfies (5) is establishing its C^1 continuity with respect to m. This step necessitates a meticulous analysis of the linearized mean field game system (refer to [24,29]) to establish strong regularity of U in both the spatial and measure variables.

It is worth highlighting that these regularity estimates demand strong regularity not only in the spatial domain but also in the measure variable.

In the spatial domain, the regularity is derived in [29] through differentiation of the equation with respect to x. However, in the case of Dirichlet boundary conditions, as well as in general for any boundary conditions, such methods are not directly applicable. Instead, we obtain these bounds by employing a distinct set of space-time estimates that require careful handling.

Moreover, it is essential to note that regularity estimates for the Dirichlet parabolic equation necessitate compatibility conditions between initial and boundary data. Unfortunately, these compatibility conditions cannot always be guaranteed within this context. Consequently, we extend the estimates obtained in [29] through an in-depth investigation of the regularity of solutions to the Fokker-Planck equation.

A noteworthy novelty in this work is the meticulous study of the Fokker-Planck equation and the linearized Mean Field Games system. This study is conducted within negative-order Hölder spaces, and the well-posedness is established via approximation using smooth functions. However, it is important to emphasize that smooth functions are not dense in the dual of Hölder spaces, as pointed out in [48]. As a result, we must confine our analysis to appropriate subsets of these dual spaces. Notably, these ideas can also be applied to fix the issues present in the Neumann case [24].

The immediate continuation of the present work is the convergence problem for the absorbing N-player game, announced at the beginning of this Introduction. The results proved here are tailored for that objective: the well-posedness of the Master Equation and the Lipschitz estimate of $\delta U/\delta m$ with respect to the sub-probability variable provide the stability ingredients needed to compare the finite-player Nash system with the limiting mean field game dynamics.

In a longer-term perspective, a further natural direction is the second-order Master Equation with common noise in a bounded domain with absorption. We expect the sub-probability framework and the boundary condition in the measure variable to remain the correct structural ingredients in that case as well. However, the common-noise term introduces second-order derivatives in the measure variable and requires a substantially finer analysis of the stochastic mean field game system, together with new compatibility arguments at the boundary. For this reason, the present paper is restricted to the first-order case.

The article is organized as follows: in Section 2, we introduce the basic notations and hypotheses we will need throughout the article, and we state the two main results we want to prove. In Section 3 we give a formal proof for the well-posedness of the Master Equation, which becomes rigorous provided some strong regularity results of the function U. In Sections 4 and 5 we state the latter regularity results for U: in particular, Section 4 is devoted to the study of the Fokker-Planck equation and the Mean Field Games, whereas in Section 5 we analyze a MFG linearized system which provides a suitable regularity for the function U.

2. Notation and assumptions

Let $T > 0$ and $\Omega \subset \mathbb{R}^d$ be the closure of an open, bounded and connected set, with the boundary of class $C^{2+\alpha}$, for some $\alpha > 0$; we denote with Q_T the set $Q_T := [0,T] \times \Omega$. We call $d(\cdot)$ the oriented distance function from the boundary of Ω, defined as

$$d(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & x \in \Omega\,, \\ -\operatorname{dist}(x, \partial\Omega) & x \notin \Omega\,. \end{cases} \tag{7}$$

Thanks to [60], we have $d(\cdot) \in C^{2+\alpha}$ in a neighbourhood of the boundary. Since we are only interested in the local character of d near $\partial\Omega$, when we write $d(\cdot)$ in this paper, we mean a $C^{2+\alpha}$ function coinciding with d in a neighbourhood of the boundary.

2.1. Hölder spaces

In what follows, we briefly recall basic notions about the Banach function spaces used throughout the paper, see, e.g., [24,61], for more details.

For $n \geq 0$ and $\alpha \in (0,1)$, $C^{n+\alpha}(\Omega)$ is defined as the space of functions n-times differentiable, with derivatives α-Hölder continuous. Being $\phi \in C^{n+\alpha}(\Omega)$, its norm is defined in the following way:

$$\|\phi\|_{n+\alpha} := \sum_{|\ell| \leq n} \left\| D^l \phi \right\|_\infty + \sum_{|\ell|=n} \sup_{x \neq y} \frac{|D^\ell \phi(x) - D^\ell \phi(y)|}{|x-y|^\alpha}\,.$$

Similarly, the parabolic space $C^{\frac{n+\alpha}{2}, n+\alpha}(Q_T)$ consists of functions ϕ admitting derivatives $D_t^r D_x^s \phi$, with $2r + s \leq n$, and with norm

$$\|\phi\|_{\frac{n+\alpha}{2}, n+\alpha} := \sum_{2r+s \leq n} \left\| D_t^r D_x^s \phi \right\|_\infty + \sum_{2r+s=n} \sup_t \left\| D_t^r D_x^s \phi(t,\cdot) \right\|_\alpha + \sum_{0<n+\alpha-2r-s<2} \sup_x \left\| D_t^r D_x^s \phi(\cdot,x) \right\|_{\frac{n+\alpha-2r-s}{2}}\,.$$

In order to work with Dirichlet boundary conditions, we define the spaces $C^{n+\alpha,D}(\Omega)$ and $C^{\frac{n+\alpha}{2}, n+\alpha, D}(Q_T)$ as the subspaces of functions belonging to $C^{n+\alpha}(\Omega)$, resp. to $C^{\frac{n+\alpha}{2}, n+\alpha}(Q_T)$, vanishing at $\partial\Omega$. When there is no risk of confusion, we will omit the set Ω or Q_T.

Analogously, we define the spaces $C^{0,\alpha}$, $C^{\alpha,0}$, $C^{1,2+\alpha}$. For the sake of completeness, also because of its relevance throughout the paper, let us specify the norm equipping the latter functional space:

$$\|\phi\|_{1,2+\alpha} := \|\phi\|_\infty + \|\phi_t\|_{0,\alpha} + \|D_x \phi\|_\infty + \left\| D_x^2 \phi \right\|_{0,\alpha}\,.$$

Eventually, we have to work with suitable subsets of the dual spaces of $C^{n+\alpha}$ and $C^{n+\alpha,D}$. In particular, we denote as $C^{-(n+\alpha)}$ the set

$$C^{-(n+\alpha)} := \left\{ \rho \in \left(C^{(n+\alpha)}\right)' \,\middle|\, \langle \rho, \phi \rangle = \sum_{|\gamma|\le n} \int_\Omega \partial^\gamma \phi(x)\, \rho_\gamma(dx) \quad \forall \phi \in C^{n+\alpha}, \quad (\rho_\gamma)_\gamma \text{ measures on } \Omega \right\}.$$

In the same way we define the set $C^{-(n+\alpha),D}$ and $C^{-\frac{\alpha}{2},-\alpha}$. The norms on these spaces are inherited by the classical dual spaces norm:

$$\|\rho\|_{-(n+\alpha)} = \sup_{\|\phi\|_{n+\alpha}\le 1} \langle \rho, \phi \rangle, \qquad \|\rho\|_{-(n+\alpha),D} = \sup_{\|\phi\|_{n+\alpha,D}\le 1} \langle \rho, \phi \rangle,$$

$$\|\rho\|_{-\left(\frac{\alpha}{2},\alpha\right)} = \sup_{\|\phi\|_{\frac{\alpha}{2},\alpha}\le 1} \langle \rho, \phi \rangle.$$

The importance of these subspaces is given by the following result.

Lemma 2.1. *The space $C^{-(n+\alpha)}$ is a norm-closed subset of $\left(C^{n+\alpha}\right)'$. Moreover, if $\rho \in C^{-n}$, then there exist a sequence $\{\rho_k\}_k \subset C^{2+\alpha,D}$ such that $\rho_k \to \rho$ in $C^{-(n+\alpha)}$.*

Proof. Thanks to [48, Lemma 2.6], we know that $C^{-(n+\alpha)}$ is a norm-closed subset of $\left(C^{n+\alpha}\right)'$. Let $\rho \in C^{-n}$. Then there exist measures $\{\rho_\gamma\}_{|\gamma|\le n}$ such that $\rho = \sum_{|\gamma|\le n} \partial^\gamma \rho_\gamma(dx)$. Defining $\rho_\gamma(\mathbb{R}^d \setminus \Omega) = 0$, we can consider $\{\rho_\gamma\}_\gamma$ as measures on $\mathbb{R}^d$.

Let $\{n_\varepsilon\}_\varepsilon$ be smooth mollifiers. We define $\rho_\gamma^k := \rho_\gamma * \eta_{\frac{1}{k}}$ and $\tilde{\rho}_k := \sum_{|\gamma|\le n} \partial^\gamma \rho_\gamma^k(dx)$. Thanks to [48, Lemma 2.3, Lemma 2.7], we have $\tilde{\rho}_k \in C^\infty$ and $\tilde{\rho}_k \to \rho$ in $C^{-(n+\beta)}$ for all $\beta > 0$.

Then, we consider a nonnegative smooth function $\xi_k(s) \in C^\infty(\mathbb{R})$ such that $\xi_k(s) = 1$ for $|s| \ge \frac{1}{k}$ and $\xi_k(0) = 0$. We define $\rho_k(x) = \tilde{\rho}_k(x)\, \xi_k(d(x))$. Then we have $\rho_k \in C^{2+\alpha,D}$ and the convergence to ρ in $C^{-(n+\alpha)}$ is preserved. □

2.2. Subprobability measures and generalized wasserstein measure

Let us start the present subsection with a *proper* notion of distance for elements belonging to $\mathcal{P}^{sub}(\Omega)$. Since the usual Wasserstein distance $W_p(\mu, \nu)$ is defined only when the two measures μ, ν have the same mass, in what follows we rely on the so-called *generalized Wasserstein distance* which allows computing distance between measures with different masses, see e.g. [62,63] for a complete description and [64,65] for the corresponding probabilistic interpretation based on unbalanced optimal transport.

According to [63], we also introduce the following definition, which is the one we will use throughout the article.

Definition 2.2. Let $m_1, m_2 \in \mathcal{P}^{sub}(\Omega)$ be two Borel sub-probability measures on Ω. We call the generalized Wasserstein distance between m_1 and m_2, and we write $\mathbf{d}_1(m_1, m_2)$ the quantity

$$\mathbf{d}_1(m_1, m_2) := \sup_{Lip(\phi)\le 1,\, \|\phi\|_{C^0}\le 1} \int_\Omega \phi(x) d(m_1 - m_2)(x), \tag{8}$$

We remark that $\mathbf{d}_1(m_1, m_2)$ is equivalent to the well-known *flat metric* over the space of Radon measures with finite mass on Ω.

Furthermore, we define a suitable derivation of U with respect to the (sub)measure m as it appears in the Master Eq. (5). Namely, we have:

Definition 2.3. Let $U : \mathcal{P}^{sub}(\Omega) \to \mathbb{R}$. We say that U is of class C^1 if there exists a continuous map $K : \mathcal{P}^{sub}(\Omega) \times \Omega \to \mathbb{R}$ such that, for all $m_1, m_2 \in \mathcal{P}^{sub}(\Omega)$ we have

$$\lim_{s\to 0} \frac{U(m_1 + s(m_2 - m_1)) - U(m_1)}{s} = \int_\Omega K(m_1, x)(m_2(dx) - m_1(dx)). \tag{9}$$

We define $\dfrac{\delta U}{\delta m}(m, x) := K(m, x)$.

Remark 2.4. Condition (9) defines uniquely the derivative K. Conversely, in the classical setting, where $U : \mathcal{P}(\Omega) \to \mathbb{R}$, see e.g. [24,29], the function K is defined up to an additive constant, since, for $c \in \mathbb{R}$ and $m_1, m_2 \in \mathcal{P}(\Omega)$, $\int_\Omega c\, d(m_2 - m_1) = 0$. Hence, in those cases a normalization condition is needed to uniquely identify K.

Let us note that, exploiting Eq. (9), we have the following equality for $m_1, m_2 \in \mathcal{P}^{sub}(\Omega)$,

$$U(m_2) - U(m_1) = \int_0^1 \int_\Omega \frac{\delta U}{\delta m}(m_1 + s(m_2 - m_1), x)(m_2(dx) - m_1(dx)) ds.$$

which turns out to be very useful in our computations. Moreover, we note that, if $\frac{\delta U}{\delta m}$ is C^1 in the space variable, we have

$$|U(m_2) - U(m_1)| \le \sup_m \left(\left\| D_x \frac{\delta U}{\delta m}(m, \cdot) \right\|_\infty + \left\| \frac{\delta U}{\delta m}(m, \cdot) \right\|_\infty \right) \mathbf{d}_1(m_1, m_2),$$

suggesting us define the *intrinsic derivative* of U with respect to m, which directly appears in the Master Eq. (5).

Definition 2.5. Let $U : \mathcal{P}^{sub}(\Omega) \to \mathbb{R}$. If U is of class C^1 and $\frac{\delta U}{\delta m}$ is of class C^1 with respect to the last variable, we define the intrinsic derivative $D_m U : \mathcal{P}^{sub}(\Omega) \times \Omega \to \mathbb{R}^d$ as

$$D_m U(m, x) := D_x \frac{\delta U}{\delta m}(m, x).$$

2.3. Assumptions

We require the following hypotheses throughout the paper:

Hypotheses 2.6. Let $0 < \alpha < 1$. Assume that

(i) $\|a(\cdot)\|_{1+\alpha} < \infty$ and $\exists\, \mu > \lambda > 0$ s.t. $\forall \xi \in \mathbb{R}^d$
$$\lambda|\xi|^2 \le \langle a(x)\xi, \xi\rangle \le \mu|\xi|^2\,;$$

(ii) $H : [0,T]\times\Omega\times\mathbb{R}^d \to \mathbb{R}$, $G : \Omega\times\mathcal{P}^{sub}(\Omega) \to \mathbb{R}$ and $F : [0,T]\times\Omega\times\mathcal{P}^{sub}(\Omega) \to \mathbb{R}$ are smooth functions with H locally Lipschitz with respect to the last variable;

(iii) $\exists\, C > 0$ s.t.
$$0 < H_{pp}(t,x,p) \le C I_{d\times d}\,;$$

(iv) F is increasing in the last variable, i.e.
$$\int_\Omega \big(F(t,x,m) - F(t,x,m')\big)d(m-m')(x) \ge 0\,;$$
moreover
$$\sup_{m\in\mathcal{P}^{sub}(\Omega)} \left(\|F(\cdot,\cdot,m)\|_{\frac{\alpha}{2},\alpha} + \left\| \frac{\delta F}{\delta m}(\cdot,\cdot,m,\cdot)\right\|_{\frac{\alpha}{2},\alpha,2+\alpha}\right) + \mathrm{Lip}\Big(\frac{\delta F}{\delta m}\Big) \le C_F\,,$$
with
$$\mathrm{Lip}\Big(\frac{\delta F}{\delta m}\Big) := \sup_{m_1\neq m_2} \left(\|m_1 - m_2\|^{-1}_{-(2+\alpha),D} \left\| \frac{\delta F}{\delta m}(\cdot,\cdot,m_1,\cdot) - \frac{\delta F}{\delta m}(\cdot,\cdot,m_2,\cdot)\right\|_{\frac{\alpha}{2},\alpha,2+\alpha}\right);$$

(v) G satisfies the same estimates as F with α and $1+\alpha$ replaced by $2+\alpha$, i.e.
$$\sup_{m\in\mathcal{P}^{sub}(\Omega)} \left(\|G(\cdot,m)\|_{2+\alpha} + \left\| \frac{\delta G}{\delta m}(\cdot,m,\cdot)\right\|_{2+\alpha,2+\alpha}\right) + \mathrm{Lip}\Big(\frac{\delta G}{\delta m}\Big) \le C_G\,,$$
with
$$\mathrm{Lip}\Big(\frac{\delta G}{\delta m}\Big) := \sup_{m_1\neq m_2} \left(\|m_1 - m_2\|^{-1}_{-(2+\alpha),D} \left\| \frac{\delta G}{\delta m}(\cdot,m_1,\cdot) - \frac{\delta G}{\delta m}(\cdot,m_2,\cdot)\right\|_{2+\alpha,2+\alpha}\right);$$

(vi) For every $m \in \mathcal{P}^{sub}(\Omega)$ and every $\rho \in C^{-(2+\alpha),D}(\Omega)$, we assume
$$\left\langle \left\langle \frac{\delta F(t,\cdot,m,\cdot)}{\delta m}, \rho\right\rangle_y, \rho\right\rangle_x \ge 0\,,$$
$$\left\langle \left\langle \frac{\delta G(\cdot,m,\cdot)}{\delta m}, \rho\right\rangle_y, \rho\right\rangle_x \ge 0\,,$$
for every $t \in [0,T]$.

(vii) Moreover, we require the following Dirichlet boundary conditions:
$$\frac{\delta F}{\delta m}(t,x,m,y)_{|y\in\partial\Omega} = 0\,, \qquad \frac{\delta G}{\delta m}(x,m,y)_{|y\in\partial\Omega} = 0\,, \qquad G(x,m)_{|x\in\partial\Omega} = 0\,,$$
for all $m \in \mathcal{P}^{sub}(\Omega)$.

We note that hypothesis (vii) contains the standard Dirichlet compatibility condition on G and on $\delta G/\delta m$. The analogous boundary condition on $\delta F/\delta m$ is less standard. It is used at two decisive points of the proof. First, it allows us to propagate the boundary condition in the measure variable and to prove the second boundary condition for $\delta U/\delta m$ in the final boundary corollary in Section 5. Second, it is needed in Proposition 5.2 to ensure that the source terms of the linearized system are compatible with the Dirichlet framework.

Assumption (vi) is stronger than the classical Lasry–Lions monotonicity. In the present paper this stronger form is mainly required by the a priori estimates for the linearized system and by the identification of the boundary behavior of the intrinsic derivative. At the same time, it is structurally natural in the absorbing setting, since it expresses monotonicity with respect to signed perturbations of the surviving population.

A standard class of examples is given by couplings of the form
$$F(t,x,m) = f_0(t,x) + \int_\Omega K_F(t,x,y)\,m(dy), \qquad G(x,m) = g_0(x) + \int_\Omega K_G(x,y)\,m(dy),$$
where K_F and K_G are sufficiently smooth, positive semidefinite as kernels, and satisfy
$$K_F(t,x,y) = 0, \qquad K_G(x,y) = 0 \qquad \text{for } y \in \partial\Omega.$$
For such couplings, assumptions (vi) and (vii) are satisfied. We expect that part of the present regularity assumptions can be weakened, but this would require a different linear theory in lower regularity spaces. In particular, our proof uses the $C^{2+\alpha}$ regularity of the measure derivatives in order to control $D_y D_m U$ and to enforce the boundary compatibility conditions.

The conditions on $\delta F/\delta m$ and $\delta G/\delta m$ also have a clear interpretation in terms of the N-player game. Roughly speaking, if two configurations differ only through the positions of players that have already reached the boundary, then the interaction costs seen by the active players should remain unchanged. In that sense, the coupling depends only on the surviving population inside Ω, not on the boundary location of the absorbed players.

3. Well-posedness of the master equation

We recall the definition of the function U, which turns out to be the solution of (5). Let $(t_0, m_0) \in [0,T] \times \mathcal{P}^{sub}(\Omega)$, and consider the solution (u, m) of the MFG system with initial condition $m(t_0) = m_0$, i.e.

$$\begin{cases} -u_t - \mathrm{tr}(a(x)D^2u) + H(t,x,Du) = F(t,x,m(t)), \\ m_t - \sum_{ij} \partial^2_{ij}(a_{ij}(x)m) - \mathrm{div}(mH_p(t,x,Du)) = 0, \\ m(t_0) = m_0, u(x,T) = G(x,m(T)), \\ u_{|\partial\Omega} = 0, \qquad m_{|\partial\Omega} = 0, \end{cases} \tag{10}$$

where, as always, a backward Hamilton-Jacobi-Bellman equation for the value function u of the generic player is coupled with a forward Fokker-Planck equation for the density m of the population. Then we define $U : [0,T] \times \Omega \times \mathcal{P}^{sub}(\Omega) \to \mathbb{R}$ as

$$U(t_0, x, m_0) = u(t_0, x), \tag{11}$$

and we want to prove that, under certain assumptions, U is the solution of the Master Eq. (5).

A formal proof of existence and uniqueness of solutions for (5) can be easily given directly using the very definition of U. The difficulty here relies on the proof of the C^1 character of U with respect to m. First, we establish the required regularity for U and a related compatibility condition based on the boundary conditions. The existence and uniqueness theorem are rigorously proved, as we see in detail in the following theorem.

Theorem 3.1. *Suppose hypotheses 2.6 are satisfied. Take U defined as in (11), suppose that $\frac{\delta U}{\delta m}$ exists, being also bounded in $C^{2+\alpha}_x \times C^{2+\alpha}_y$, uniformly in t and continuously in m, and that boundary conditions of (5) hold for U. Then U is the unique classical solution U of the Master Eq. (5).*

Proof. *Existence*. By a density argument, we only need to prove that U solves (5) at (t_0, x, m_0), where m_0 is a smooth and positive function satisfying $m_{0|\partial\Omega} = 0$ and $\int_\Omega m_0(x)dx \le 1$. Taking (u, m) as the solution of the MFG system (10) starting from m_0 at time t_0, by the definition of U, we have that for all $s > t_0$

$$U(s, x, m(s)) = u(s, x),$$

therefore, we can write

$$\partial_t U(t_0, x, m_0) = \lim_{h\to 0} \frac{U(t_0+h, x, m_0) - U(t_0+h, x, m(t_0+h))}{h} + u_t(t_0, x).$$

Again by the very definition of U:

$$u_t(t_0, x) = -\mathrm{tr}(a(x)D^2_x U(t_0, x, m_0)) + H(t_0, x, D_x U(t_0, x, m_0)) - F(t_0, x, m_0).$$

As regards the limit part, we define $m_s := (1-s)m(t_0) + sm(t_0+h)$. Since U is C^1 with respect to m, then:

$$-\lim_{h\to 0} \int_0^1 \int_\Omega \frac{\delta U}{\delta m}(t_0+h, x, m_s, y) \frac{(m(t_0+h, y) - m(t_0, y))}{h} dy ds =$$
$$-\int_\Omega \frac{\delta U}{\delta m}(t_0, x, m_0, y) \left(\sum_{ij} \partial^2_{ij}(a_{ij}(x)m(t_0, y)) + \mathrm{div}\big(m(t_0, y)H_p(t_0, y, Du(t_0, y))\big) \right) dy.$$

Integrating by parts, and using the boundary conditions of $\frac{\delta U}{\delta m}$ and m, we can rewrite the right-hand side as

$$\int_\Omega \left[H_p(t_0, y, D_x U(t_0, y, m_0)) D_m U(t_0, x, m_0, y) - \mathrm{tr}\Big(a(y) D_y D_m U(t_0, x, m_0, y)\Big) \right] dm_0(y),$$

therefore

$$\begin{aligned} \partial_t U(t, x, m) = &-\mathrm{tr}\big(a(x)D^2_x U(t, x, m)\big) + H\big(t, x, D_x U(t, x, m)\big) \\ &- \int_\Omega \mathrm{tr}\big(a(y) D_y D_m U(t, x, m, y)\big) dm(y) \\ &+ \int_\Omega D_m U(t, x, m, y) \cdot H_p(t, y, D_x U(t, y, m)) dm(y) - F(t, x, m), \end{aligned}$$

hence concluding the existence part.

Uniqueness. Suppose that there exists a second solution V of (5). We fix (t_0, m_0), with m_0 smooth s.t. $m_{0|\partial\Omega} = 0$, and we take $\tilde{m}$ as the solution of

$$\begin{cases} \tilde{m}_t - \sum_{ij} \partial^2_{ij}(a_{ij}(x)\tilde{m}) - \mathrm{div}\big(\tilde{m} H_p(t, x, D_x V(t, x, \tilde{m}))\big) = 0, \\ \tilde{m}(t_0) = m_0, \\ \tilde{m}_{|\partial\Omega} = 0 \end{cases}$$

which is well-posed since $D_x V$ is Lipschitz continuous with respect to the measure. Then we define $\tilde{u}(t,x) = V(t,x,\tilde{m}(t))$ and we want to prove that $\tilde{u}$ solves a Hamilton-Jacobi equation. Using the equations of V and $\tilde{m}$, we get

$$\begin{aligned}\tilde{u}_t(t,x) &= V_t(t,x,\tilde{m}(t)) + \int_\Omega \frac{\delta V}{\delta m}(t,x,\tilde{m}(t),y)\,\tilde{m}_t(t,y)\,dy \\ &= V_t(t,x,\tilde{m}(t)) + \int_\Omega \frac{\delta V}{\delta m}(t,x,\tilde{m}(t),y)\sum_{ij}\partial^2_{ij}(a_{ij}(x)\tilde{m}(t,y))\,dy \\ &+ \int_\Omega \frac{\delta V}{\delta m}(t,x,\tilde{m}(t),y)\,\mathrm{div}\big(\tilde{m}H_p(t,x,D_xV(t,x,\tilde{m}))\big)\,dy\,.\end{aligned}$$

Previous integrals can be easily estimated with an integration by parts, and taking into account the boundary condition for V. As regards the first, we have

$$\begin{aligned}&\int_\Omega \frac{\delta V}{\delta m}(t,x,\tilde{m}(t),y)\sum_{ij}\partial^2_{ij}(a_{ij}(x)\tilde{m}(t,y))\,dy = \\ &\quad\int_\Omega \mathrm{tr}(a(y)D_yD_mV(t,x,\tilde{m}(t),y))\,\tilde{m}(t,y)\,dy\,,\end{aligned}$$

while for the second

$$\begin{aligned}&\int_\Omega \frac{\delta V}{\delta m}(t,x,\tilde{m}(t),y)\,\mathrm{div}\big(\tilde{m}H_p(t,x,D_xV(t,x,\tilde{m}))\big)\,dy \\ &\quad= -\int_\Omega H_p(t,x,D_xV(t,x,\tilde{m}))D_mV(t,x,\tilde{m},y)\tilde{m}(t,y)dy\,.\end{aligned}$$

Previous estimates allow us to write

$$\begin{aligned}\tilde{u}_t(t,x) &= V_t(t,x,\tilde{m}(t)) + \int_\Omega \mathrm{tr}(a(y)D_yD_mV(t,x,\tilde{m},y))\,d\tilde{m}(y) \\ &\quad - \int_\Omega H_p(t,y,D_xV(t,y,\tilde{m}))D_mV(t,x,\tilde{m},y)\,d\tilde{m}(y) \\ &= -\mathrm{tr}(a(x)D^2\tilde{u}(t,x)) + H(t,x,D\tilde{u}(t,x)) - F(t,x,\tilde{m}(t))\,,\end{aligned}$$

hence $(\tilde{u},\tilde{m})$ is a solution of the MFG system (10). Then, by uniqueness of solutions for the MFG system, we get $(\tilde{u},\tilde{m}) = (u,m)$, implying that $V(t_0,x,m_0) = U(t_0,x,m_0)$, whenever m_0 is smooth. Therefore, by density, the uniqueness is proved. □

In the next two sections, we will prove the C^1 character of U with respect to m, the C^2 regularity in x and y of $\frac{\delta U}{\delta m}$ and the boundary conditions for U, hence justify the hypotheses previously given, also allowing us to then apply Theorem 3.1 and finally prove the well-posedness of (5).

4. The Fokker-Planck equation and the MFG system

This section has two goals. First, we establish well-posedness and stability for a Dirichlet Fokker–Planck equation with rough data in negative Hölder spaces. Second, we use those estimates to solve the mean field game system and to obtain the first regularity bounds for the master field U. The strategy is the following. The regularity lemma 4.1 gives the dual regularity for the backward test equation. The main proposition 4.3 and the stability corollary that follows it transfer those estimates to the Fokker–Planck equation by duality. These results then enter the fixed-point argument for the mean field game system and provide the analytic input for the linearized analysis developed in Section 5.

Let us first give the following technical lemma, providing a regularity result for a linear PDE with non-homogeneous drift.

Lemma 4.1. *Suppose a satisfies hypothesis (i) of 2.6 , let $q = \frac{d+2}{1-\alpha}$, $b, f \in L^q(Q_T)$, $\psi \in C^{1+\alpha,D}(\Omega)$, and z be the solution of*

$$\begin{cases} -z_t - \mathrm{tr}(a(x)D^2z) + b(t,x)\cdot Dz = f(t,x)\,, \\ z(T) = \psi\,, \\ z_{|\partial\Omega} = 0\,. \end{cases}$$

Then z satisfies, for a certain $C > 0$ depending on $\|a\|_{L^q}$ and $\|b\|_{L^q}$,

$$\|z\|_{\frac{1+\alpha}{2},1+\alpha} \le C\big(\|f\|_{L^q} + \|\psi\|_{1+\alpha}\big)\,. \tag{12}$$

Moreover, if $b, f \in C^{0,\alpha}(Q_T)$ and $\psi \in C^{2+\alpha,D}(\Omega)$, it holds

$$\|z\|_{1,2+\alpha} \le C\big(\|f\|_{0,\alpha} + \|\psi\|_{2+\alpha}\big)\,. \tag{13}$$

Proof. We exploit ideas stated in [24]. In particular, if $f \in C(Q_T)$, $b \in C(\Omega)$, then we conclude with Theorem 5.1.1 of [66]. While, in the general case, we write $z = z_1 + z_2$, where z_1 and z_2 respectively satisfy:

$$\begin{cases} -(z_1)_t - \mathrm{tr}(a(x)D^2 z_1) = 0\,, \\ z_1(T) = \psi\,, \\ z_{1|\partial\Omega} = 0\,, \end{cases}$$

and

$$\begin{cases} -(z_2)_t - \mathrm{tr}(a(x)D^2 z_2) + b \cdot Dz_2 = f - b \cdot Dz_1\,, \\ z_2(T) = 0\,, \\ z_{2|\partial\Omega} = 0\,. \end{cases}$$

For z_1, we apply Theorem 5.1.11 of [66] to obtain $\|z_1\|_{\frac{1+\alpha}{2},1+\alpha} \le C\|\psi\|_{1+\alpha}$. For z_2, by Corollary of Theorem IV.9.1 in [61], we have

$$\|z_2\|_{\frac{1+\alpha}{2},1+\alpha} \le C\|f - bDz_1\|_{L^q} \le C\big(\|f\|_{L^q} + \|\psi\|_{1+\alpha}\big)\,,$$

concluding the first part. Then, if $b, f \in C^{0,\alpha}(Q_T)$ and $\psi \in C^{2+\alpha,D}$, by (12), we have $f - b \cdot Dz \in C^{0,\alpha}$, allowing to apply Theorem 5.1.13 in [66] to conclude. □

4.1. The Fokker-Planck equation

From now on, q will denote the quantity $q := \frac{d+2}{1-\alpha}$, and p will be the conjugate exponent of q, i.e. $p := \frac{d+2}{d+1+\alpha}$. We start from the linear Fokker–Planck equation because it is the central analytic ingredient for both the nonlinear mean field game system and the linearized problem. The main difficulty is that the natural data belong to negative Hölder spaces, while the Dirichlet boundary condition prevents a direct density argument in the full dual space. Our strategy is to define solutions by duality, prove the key a priori estimate through backward test equations, and then pass to the limit inside the intrinsic negative Hölder spaces introduced in the subsection on sub-probability measures.

The present section is focused on the study of the following Fokker-Planck equation

$$\begin{cases} \mu_t - \mathrm{div}(a(x)D\mu) - \mathrm{div}(\mu b) = \mathrm{div}(c)\,, \\ \mu(t_0) = \mu_0\,, \\ \mu_{|\partial\Omega} = 0\,, \end{cases} \tag{14}$$

where $c \in L^1(Q_T)$, $\mu_0 \in C^{-(1+\alpha)}$, and $b \in L^q(Q_T)$. Let us underline that the latter equation plays a fundamental role in studying both the Mean Field Games system and the linearized one.

The main difficulty here lies in the low regularity of both data and coefficients. Hence, the main idea is to start with the regular case, where we have the existence and uniqueness of solutions, to then obtain estimates that we will then exploit to pass to the limit in the general case.

Let us start by giving a suitable definition of a solution for the system (14):

Definition 4.2. Let $c \in L^1$, $\mu_0 \in C^{-1}$, $b \in L^q$. We say that a function $\mu \in C([0,T]; C^{-(1+\alpha),D}) \cap L^1(Q_T)$ is a weak solution of (14) if, for all $t \in (t_0, T]$ and ϕ satisfying in $[t_0, t] \times \Omega$ the following linear equation

$$\begin{cases} -\phi_t - \mathrm{div}(aD\phi) + bD\phi = \psi\,, \\ \phi(t) = \xi\,, \\ \phi_{|\partial\Omega} = 0\,, \end{cases} \tag{15}$$

with $\psi \in L^\infty(\Omega)$ and $\xi \in C^{1+\alpha,D}$, it holds true:

$$\langle \mu(t), \xi\rangle + \int_{t_0}^{t}\int_\Omega \mu(s,x)\psi(s,x)\,dxds = \langle \mu_0, \phi(t_0,\cdot)\rangle - \int_{t_0}^{t}\int_\Omega c(s,x)\cdot D\phi(s,x)\,dxds\,, \tag{16}$$

where $\langle\cdot,\cdot\rangle$ denotes, respectively, the duality between $C^{-(1+\alpha),D}$ and $C^{1+\alpha,D}$, $C^{-(1+\alpha)}$ and $C^{1+\alpha}$.

Note that Lemma 4.1 implies $\phi \in C^{\frac{1+\alpha}{2},1+\alpha}$, therefore the terms on the right-hand side are well-defined and the definition is well-posed.

Proposition 4.3. *Let $c \in L^1$, $\mu_0 \in C^{-1}$, $b \in L^q$. Then there exists a unique solution for (14), which satisfies, for a certain C depending on $\|a\|_{L^q}$ and $\|b\|_{L^q}$,*

$$\sup_{t\in[t_0,T]} \|\mu(t)\|_{-(1+\alpha),D} + \|\mu\|_{L^p} \le C\big(\|\mu_0\|_{-(1+\alpha)} + \|c\|_{L^1}\big)\,, \tag{17}$$

Moreover, if $\mu_0^n \to \mu_0$ *in* $C^{-(1+\alpha)}$, $b^n \to b$ *in* L^q, $c^n \to c$ *in* L^1, *we have* $\mu^n \to \mu$ *in* $C([0,T]; C^{-(1+\alpha),D}) \cap L^p(Q_T)$, *where* μ^n *and* μ *are the solutions related to* (μ_0^n, b^n, f^n) *and* (μ_0, b, f).

The choice of the space C^{-1} motivated by the initial datum in Eq. 10 enables a smooth approximation scheme within the framework of dual Hölder spaces, while maintaining uniform estimates in the stronger $C^{-(1+\alpha),D}$ norm.

Proof. Without loss of generality, we can consider $t_0 = 0$.

Existence: smooth case. Assume that f, b and μ_0 are smooth functions, with μ_0 satisfying $\mu_0(x)_{|x\in\partial\Omega} = 0$. Then, splitting the divergence terms in (14), we obtain a linear equation and the existence of solutions is guaranteed by [61,66].

Let ϕ be a solution of (15), with $\psi = 0$ and $\xi \in C^{1+\alpha,D}$. Using ϕ as a test function for μ, we get by Lemma 4.1

$$\begin{aligned}\langle \mu(t), \xi\rangle &= \langle \mu_0, \phi(0,\cdot)\rangle - \int_0^t \int_\Omega c(s,x)\cdot D\phi(s,x)\,dxds \\ &\le C\|\xi\|_{1+\alpha}\left(\|\mu_0\|_{-(1+\alpha)} + \|c\|_{L^1}\right).\end{aligned}$$

Passing to the *sup* for $\|\xi\|_{1+\alpha} \le 1$, we obtain

$$\sup_{t\in[0,T]} \|\mu(t)\|_{-(1+\alpha),D} \le C\left(\|\mu_0\|_{-(1+\alpha)} + \|c\|_{L^1}\right), \tag{18}$$

and we are left to prove the L^p bound for μ. To this end, we take ϕ as the solution of (15), with $t = T$, $\xi = 0$ and $\psi \in L^q$. Then, again by Lemma 4.1, we have

$$\begin{aligned}\int_0^T \int_\Omega \mu\psi\,dxds &= \langle \mu_0, \phi(0,\cdot)\rangle - \int_0^T \int_\Omega c(s,x)\cdot D\phi(s,x)\,dxds \\ &\le C\|\psi\|_{L^q}\left(\|\mu_0\|_{-(1+\alpha)} + \|c\|_{L^1}\right).\end{aligned}$$

Passing to the sup for $\|\mu\|_{L^q} \le 1$, we get

$$\|\mu\|_{L^p} \le C\left(\|\mu_0\|_{-(1+\alpha)} + \|c\|_{L^1}\right),$$

which concludes the proof in the regular case.

Existence: general case. We proceed adapting and adjusting [24, Proposition 5.3]. By Lemma 2.1, we take smooth data μ_0^k, c_k, b_k, with $\mu_0^k(x) = 0$ for $x \in \partial\Omega$, and converging to μ_0, c, b, respectively in $C^{-(1+\alpha),D}$, L^1 and L^q.

We consider μ^k as the solution of (14). The previous convergences imply, for a certain $C > 0$,

$$\begin{aligned}&||\mu_0^k||_{-(1+\alpha)} \le C\|\mu_0\|_{-(1+\alpha)}, \\ &\|b_k\|_{L^q} \le C\|b\|_{L^q}, \\ &||c_k||_{L^1} \le C\|c\|_{L^1}.\end{aligned}$$

Considering the function $\mu^{k,h} := \mu^k - \mu^h$, which satisfies (14) with data $b = b_k$, $c = c_k - c_h + \mu_h(b_k - b_h)$, $\mu^0 = \mu_0^k - \mu_0^h$, then, by estimates (17), we have

$$\begin{aligned}&\sup_t ||\mu^{k,h}(t)||_{-(1+\alpha),D} + ||\mu^{k,h}||_{L^p} \\ &\quad\le C\left(||\mu_0^k - \mu_0^h||_{-(1+\alpha)} + ||c_k - c_h||_{L^1} + ||\mu_h(b_k - b_h)||_{L^1}\right),\end{aligned}$$

By the uniform bound of $\{\mu_k\}_k$ in L^p, the last term can be estimated as follows

$$\|\mu^h(b_k - b_h)\|_{L^1} \le C||b_k - b_h||_{L^q}, \tag{19}$$

and since $\{\mu_0^k\}_k$, $\{c_k\}_k$ and $\{b_k\}_k$ are Cauchy sequences, the right-hand side goes to 0 and $\{\mu^k\}_k$ is a Cauchy sequence, too. Then $\mu^k \to \mu$ in $C([0,T]; C^{-(1+\alpha),N}) \cap L^p(Q_T)$, for a certain μ also satisfying (17). We are then left to prove that μ satisfies (16), for ϕ satisfying (15). To this end, we consider the test function ϕ_k related to μ^k, so that

$$\langle \mu^k(t), \xi\rangle + \int_0^t \int_\Omega \mu^k(s,x)\psi(s,x)\,dxds = \langle \mu_0^k, \phi_k(0,\cdot)\rangle - \int_0^t \int_\Omega c_k(s,x)\cdot D\phi_k(s,x)\,dxds, \tag{20}$$

Concerning convergence of ϕ_k towards ϕ, since $\phi_k - \phi$ satisfies

$$\begin{cases} -(\phi_k - \phi)_t - \operatorname{div}(aD(\phi_k - \phi)) + b_k D(\phi_k - \phi) = (b_k - b)D\phi, \\ (\phi_k - \phi)(t) = 0, \\ (\phi_k - \phi)_{|\partial\Omega} = 0, \end{cases}$$

by Lemma 4.1, we have

$$||\phi_k - \phi||_{\frac{1+\alpha}{2},1+\alpha} \le C||(b_k - b)D\phi||_{L^q} \to 0,$$

allowing us to pass to the limit in (20), hence ending the existence part.

Uniqueness and stability. Let μ_1 and μ_2 be two solutions. Then $\mu := \mu_1 - \mu_2$ solves the same problem with $c = \mu_0 = 0$, and by (17) we have $\|\mu\|_{L^1} = \sup_{t\in[0,T]} \|\mu(t)\|_{-(1+\alpha),D} = 0$, concluding the uniqueness part.

For the stability part, we consider $c_n \to c$, $\mu_0^n \to \mu_0$ and $b_n \to b$. Then the function $\mu^n - \mu$ satisfies (14), with b, μ_0 and f respectively replaced by b^n, $\mu_0^n - \mu_0$, and $c^n - c + \mu(b^n - b)$. Hence, (17) implies

$$\begin{aligned}&\sup_{t\in[0,T]} \|\mu^n - \mu\|_{-(1+\alpha),D} + \|\mu^n - \mu\|_{L^p}\\ &\le C\left(||\mu_0^n - \mu_0||_{-(1+\alpha)} + ||c^n - c||_{L^1} + ||\mu(b^n - b)||_{L^1}\right).\end{aligned}$$

Proceeding as done for the existence in the general case, we derive the right-hand side convergence to 0, which implies $\mu^n \to \mu$ in $C([0,T]; C^{-(1+\alpha),D}) \cap L^p(Q_T)$, hence concluding the proof. □

As a direct consequence of Proposition 4.3, we can derive a further estimate in the space $C^{-(2+\alpha),D}$.

Corollary 4.4. *Suppose hypotheses of Proposition 4.3 are satisfied together with $b \in C^{\frac{\alpha}{2},\alpha}$. Then the function μ satisfies*

$$\sup_{t\in[0,T]} \|\mu(t)\|_{-(2+\alpha),D} + \|\mu\|_{-\left(\frac{\alpha}{2},\alpha\right)} \le C\left(\|\mu_0\|_{-(2+\alpha)} + \|c\|_{L^1}\right), \tag{21}$$

Proof. Since $b \in C^{\frac{\alpha}{2},\alpha}$, we can take ϕ as the solution of (15), with $\xi \in C^{2+\alpha,D}(\Omega)$, $\psi = 0$, so that, applying the results of [61,66], we have $\|\phi\|_{1+\frac{\alpha}{2},2+\alpha} \le C\|\xi\|_{2+\alpha}$, and (16) implies

$$\begin{aligned}&\langle \mu(t), \xi\rangle = \langle \mu_0, \phi(0,\cdot)\rangle - \int_0^T \int_\Omega c(s,x)\cdot D\phi(s,x)\,dx\,ds\\ &\le C\left(\|\mu_0\|_{-(2+\alpha)} + \|c\|_{L^1}\right)\|\xi\|_{2+\alpha}.\end{aligned}$$

Now, as in Proposition 4.3, we choose ϕ as the solution of (15), with $t = T$, $\xi = 0$ and $\psi \in C^{\frac{\alpha}{2},\alpha}$. Then, again by Lemma 4.1, we have

$$\begin{aligned}&\int_0^T \int_\Omega \mu\psi\,dx\,ds = \langle \mu_0, \phi(0,\cdot)\rangle - \int_0^T \int_\Omega c(s,x)\cdot D\phi(s,x)\,dx\,ds\\ &\le C\|\psi\|_{\frac{\alpha}{2},\alpha}\left(\|\mu_0\|_{-(2+\alpha)} + \|c\|_{L^1}\right).\end{aligned}$$

Passing to the sup in the two inequalities for $\|\psi\|_{\frac{\alpha}{2},\alpha} \le 1$ and $\|\xi\|_{2+\alpha} \le 1$, we get (21). □

Remark 4.5. Observe that the analogue of this Corollary in the Neumann case is given by [24, Corollary 5.4]. However, in order to adjust the issues raised in [48], we have refined the estimate (21), by proving a bound for $\mu \in C^{-\left(\frac{\alpha}{2},\alpha\right)}$. This estimate is crucial to prove stronger regularity results for the intrinsic derivative of U in the measure variable, $D_m U$. We stress the fact that similar adjustments can also be done in the Neumann case [24].

4.2. Mean field games system and lipschitz regularity of u

Now, we turn to the study of the MFG system (10) and its properties to derive essential results that we will then exploit to obtain some first regularity estimates of the solution U of the Master Equation.

In particular, the MFG system has the following form:

$$\begin{cases} -\partial_t u - \mathrm{tr}(a(x)D^2u) + H(t,x,Du) = F(t,x,m(t)),\\ \partial_t m - \sum_{i,j}\partial^2_{ij}(a_{ij}(x)m) - \mathrm{div}(mH_p(t,x,Du)) = 0,\\ u(T) = G(x,m(T)), \qquad m(0) = m_0,\\ u_{|\partial\Omega=0}, \qquad m_{|\partial\Omega=0}. \end{cases} \tag{22}$$

The first result is obtained by studying some regularity properties of (22) uniformly in m_0.

Proposition 4.6. *The system (22) has a unique classical solution (u,m), with $u \in C^{1+\frac{\alpha}{2},2+\alpha}$ and $m \in C([t_0,T]; \mathcal{P}^{sub}(\Omega))$. Moreover, $m(t)$ has a positive density for $t > t_0$, and the following estimates hold:*

$$\|m\|_{\frac{\alpha}{2},-(1+\alpha),D} + \|m\|_{L^p} + \|u\|_{1+\frac{\alpha}{2},2+\alpha} \le C. \tag{23}$$

Furthermore, if (u_1,m_1) and (u_2,m_2) are two solutions of (22), with $m_1(t_0) = m_{01}$, $m_2(t_0) = m_{02}$, then it holds

$$\begin{aligned}&\|u_1 - u_2\|_{1,2+\alpha} + \|m_1 - m_2\|_{L^p(Q_T)} \le C\mathbf{d}_1(m_{01},m_{02}),\\ &\sup_{t\in[0,T]}\|m_1(t) - m_2(t)\|_{-(1+\alpha),D} \le C\mathbf{d}_1(m_{01},m_{02}).\end{aligned} \tag{24}$$

Proof. Step 1: Existence and uniqueness of solutions. We want to apply Schauder's fixed-point Theorem. Let X be the convex set, closed for the uniform distance, defined as follows:

$$X := \left\{\gamma \in C([t_0,T]; \mathcal{P}^{sub}(\Omega)) \text{ s.t. } \|\gamma\|_{\frac{\alpha}{2},-(1+\alpha),D} \le M \ \ \forall s,t \in [t_0,T]\right\}, \tag{25}$$

where M will be specified later. We define a map $\Phi : X \to X$ as follows.
For $\gamma \in X$, let $u = u_\gamma$ be the solution of the HJB equation:

$$\begin{cases} -u_t - \mathrm{tr}(a(x)D^2u) + H(t,x,Du) = F(t,x,\gamma(t)), \\ u(T) = G(x,\gamma(T)), \\ u_{|\partial\Omega} = 0. \end{cases} \tag{26}$$

Using hypotheses on F and G and Theorem V.6.1 of [61], we know that there exists a unique classical solution $u \in C^{1+\frac{\alpha}{2},\alpha}$. Moreover, by Taylor's formula, we can write $H(t,x,Du) = H(t,x,0) + V(t,x)\cdot Du$ for a certain $V \in L^\infty$, thanks to the global boundedness of Du, obtaining a linear equation satisfied by u. Then, exploiting both the Corollary of Theorem IV.9.1 and Theorem IV.5.2 of [61] we get

$$\|u\|_{1+\frac{\alpha}{2},2+\alpha} \le C\left(\|F(\cdot,\cdot,\gamma(\cdot))\|_{\frac{\alpha}{2},\alpha} + \|G(\cdot,\gamma(T))\|_{2+\alpha} \right),$$

Define $\Phi(\gamma) = m$, where $m \in C([t_0,T];\mathcal{P}^{sub}(\Omega))$ is the solution of the FP equation

$$\begin{cases} m_t - \sum_{i,j} \partial^2_{ij}(a_{ij}(x)m) - \mathrm{div}(mH_p(t,x,Du)) = 0, \\ m(t_0) = m_0, \\ m_{|\partial\Omega} = 0. \end{cases} \tag{27}$$

The existence of a unique solution for (27) is guaranteed by Proposition 4.3. Moreover, we know from (17) that

$$\sup_{t\in[t_0,T]} \|m(t)\|_{-(1+\alpha),D} + \|m\|_{L^p} \le C\|m_0\|_{-(1+\alpha)},$$

for a certain $p > 1$. Therefore, to check that $m \in X$, we are left with proving that, for some $C > 0$, it holds

$$\|m(t) - m(s)\|_{-(1+\alpha),D} \le C|t-s|^{\frac{\alpha}{2}}, \qquad \text{for all } t \ne s.$$

Subtracting the distributional formulation (16) in t and s, we have

$$\int_\Omega \xi(x)m(t,dx) - \int_\Omega \phi(s,x)m(s,dx) + \int_s^t \int_\Omega \psi(r,x)m(r,dx)dr = 0, \tag{28}$$

for each $\xi \in C^{1+\alpha,D}$, $\psi \in L^\infty$ and ϕ satisfying

$$\begin{cases} -\phi_t - \mathrm{tr}(a(x)D^2\phi) + H_p(t,x,Du)\cdot D\phi = \psi, \\ \phi(t) = \xi, \\ \phi_{|\partial\Omega} = 0. \end{cases} \tag{29}$$

We choose $\psi = 0$. Thanks to Lemma 4.1, we know that $\phi \in C^{1+\frac{\alpha}{2},1+\alpha}$ and its norm is bounded according to (12). Coming back to (28), we obtain

$$\begin{aligned} \int_\Omega \xi(x)(m(t,dx) - m(s,dx)) &= \int_\Omega (\phi(s,x) - \phi(t,x))m(s,dx) \\ &\le C\|\phi(t) - \phi(s)\|_{1+\alpha}\|m(s)\|_{-(1+\alpha),D} \le C|t-s|^{\frac{\alpha}{2}}\|m_0\|_{-(1+\alpha)}\|\xi\|_{1+\alpha}, \end{aligned} \tag{30}$$

and taking the sup over the
$\xi \in C^{1+\alpha,D}$ with $\|\xi\|_{1+\alpha} \le 1$,

$$\|m(t) - m(s)\|_{-(1+\alpha),D} \le C|t-s|^{\frac{\alpha}{2}}\|m_0\|_{-(1+\alpha)}. \tag{31}$$

Choosing $M = C\|m_0\|_{-(1+\alpha)}$, we have proved that $m \in X$.
Since X is convex and closed and $\Phi(X) \subseteq X$, to apply Schauder's theorem, we need to show that:

- $\Phi(X)$ is relatively compact;
- Φ is continuous.

In this way, the closure of the convex envelope of $\Phi(X)$, say $\hat{X} := \overline{\mathrm{inv}(\Phi(X))}$, is compact and convex, and for the closure and the convexity of X we have $\Phi(X) \subseteq \hat{X} \subseteq X$. So we can consider the restriction $\Phi_| : \hat{X} \to \hat{X}$, which satisfies the classical hypotheses of the Schauder's Theorem. Hence, the existence of a fixed point remains guaranteed.

We start proving the relatively compactness of $\Phi(X)$. Let $\{\gamma_n\}_n \subset X$ and let u_n and m_n be the related solutions. Applying Ascoli-Arzelà's Theorem we have $u_{n_k} \to u$ in $C^{1,2}$, for a certain subsequence $\{u_{n_k}\}_k$ and $u \in X$.

To prove the convergence of $\{m_{n_k}\}_n$, we take ϕ_{n_k} as the solution of (29) with Du replaced by Du_{n_k} and $\psi = 0$. The difference $\phi_{k,h} := \phi_{n_k} - \phi_{n_h}$ satisfies

$$\begin{cases} -(\phi_{k,h})_t - \mathrm{tr}(a(x)D^2\phi_{k,h}) + H_p(t,x,Du_{n_k})\cdot D\phi_{k,h} \\ = (H_p(t,x,Du_{n_h}) - H_p(t,x,Du_{n_k}))\cdot D\phi_{n_h}\,, \\ \phi_{k,h}(t) = 0\,, \\ \phi_{k,h|\partial\Omega} = 0\,, \end{cases} \tag{32}$$

then Lemma 4.1 implies

$$\begin{aligned} &\left\|\phi_{k,h}\right\|_{\frac{1+\alpha}{2},1+\alpha} \le C\left\|(H_p(t,x,Du_{n_h}) - H_p(t,x,Du_{n_k}))\cdot D\phi_{n_h}\right\|_\infty \\ &\le C\left\|Du_{n_h} - Du_{n_k}\right\|_\infty \le \omega(k,h)\,, \end{aligned} \tag{33}$$

where $\omega(k,h) \to 0$ when $k,h \to \infty$.

Using (28) with (m_{n_k}, ϕ_{n_k}) and (m_{n_h}, ϕ_{n_h}), for $k,h \in \mathbb{N}$, $s = t_0$, subtracting the two equalities, we get

$$\sup_{t\in[t_0,T]}\left\|m_{n_k}(t) - m_{n_h}(t)\right\|_{-(1+\alpha),D} \le \omega(k,h) \implies \exists\, m \in X \text{ s.t. } m_{n_k} \to m \text{ in } X\,,$$

hence concluding the compactness part. The continuity is an easy consequence of the previous arguments.

In particular, we apply Schauder's theorem and obtain a classical solution of the problem (22). The estimate (23) follows from the above estimates for (26) and (27).

We skip the uniqueness part, which is a standard argument, see, e.g., Proposition 3.3 of [24].

Step 2. Let (u_1, m_1) and (u_2, m_2) be two classical solutions of (22), with $m_1(t_0) = m_{01}$, $m_2(t_0) = m_{02}$. We take ϕ as the solution of (29) related to u_1, with $\psi = 0$, and we note that ϕ is also an admissible test function for the equation of m_2, since it satisfies, for $\psi = \left(H_p(t,x,Du_2) - H_p(t,x,Du_1)\right)\cdot D\phi \in L^\infty$,

$$\begin{cases} -\phi_t - \mathrm{tr}(a(x)D^2\phi) + H_p(t,x,Du_2)\cdot D\phi = \psi\,, \\ \phi(t) = \xi\,, \\ \phi_{|\partial\Omega} = 0\,. \end{cases} \tag{34}$$

Then, subtracting the weak formulations of m_1 and m_2 related to ψ, we find

$$\int_\Omega \xi(x)(m_1(t) - m_2(t))\,dx = \tag{35}$$

$$\int_{t_0}^t\int_\Omega (H_p(t,x,Du_1) - H_p(t,x,Du_2))D\phi\, m_2(s,x)dx\,ds + \langle\phi(0,\cdot), m_{01} - m_{02}\rangle\,. \tag{36}$$

By Lipschitz continuity of both ϕ and H_p, with respect to p, we get

$$\int_\Omega \xi(x)(m_1(t) - m_2(t))\,dx \le C\int_{t_0}^t\int_\Omega |Du_1 - Du_2|m_2(s,x)dx\,ds + C\mathbf{d}_1(m_{01},m_{02})\,, \tag{37}$$

We want to estimate the first term in the right-hand side with Young's inequality. To this end, we consider the quantities

$$\tilde{c} = \int_{t_0}^t\int_\Omega m_2(s,x)\,dx ds\,, \qquad \tilde{m}_2(s,x) = \frac{m_2(s,x)}{\tilde{c}}\,.$$

Then $\tilde{m}_2$ is a probability measure in $[0,t]\times\Omega$, and Young's inequality implies that

$$\begin{aligned} &\int_{t_0}^t\int_\Omega |Du_1 - Du_2|m_2(s,x)dx\,ds = \tilde{c}\int_{t_0}^t\int_\Omega |Du_1 - Du_2|\tilde{m}_2(s,x)dx\,ds \\ &\le \tilde{c}\left(\int_{t_0}^t\int_\Omega |Du_1 - Du_2|^2\tilde{m}_2(s,x)dx\right)^{\frac{1}{2}} \le C\left(\int_{t_0}^t\int_\Omega |Du_1 - Du_2|^2 m_2(s,x)dx\right)^{\frac{1}{2}}, \end{aligned}$$

since $\tilde{c}$ is bounded thanks to the L^p bound of m_2.

Using the Lasry-Lions monotonicity argument (see Lemma 3.1.2 of [29]), we have

$$\begin{aligned} &\int_{t_0}^T\int_\Omega |Du_1 - Du_2|^2(m_1(t,dx) + m_2(t,dx))dt \le \\ &\le C\int_\Omega (u_1(0,x) - u_2(0,x))(m_{01}(dx) - m_{02}(dx)) \\ &\le C\left\|u_1 - u_2\right\|_{\frac{1+\alpha}{2},1+\alpha}\mathbf{d}_1(m_{01},m_{02})\,. \end{aligned}$$

Hence, we obtain

$$\int_\Omega \xi(x)(m_1(t)-m_2(t))\,dx \le C\left(\|u_1-u_2\|^{\frac{1}{2}}_{\frac{1+\alpha}{2},1+\alpha} \mathbf{d}_1(m_{01},m_{02})^{\frac{1}{2}} + \mathbf{d}_1(m_{01},m_{02}) \right),$$

and finally, taking the sup over the ξ with $\|\xi\|_{-(1+\alpha),D} \le 1$ and over $t \in [0,T]$,

$$\sup_{t\in[0,T]} \|m_1(t)-m_2(t)\|_{-(1+\alpha),D} \tag{38}$$

$$\le C\left(\|u_1-u_2\|^{\frac{1}{2}}_{\frac{1+\alpha}{2},1+\alpha} \mathbf{d}_1(m_{01},m_{02})^{\frac{1}{2}} + \mathbf{d}_1(m_{01},m_{02}) \right). \tag{39}$$

Let us now call $u := u_1 - u_2$, then u solves:

$$\begin{cases} -u_t - \mathrm{tr}(a(x)D^2u) + V(t,x)Du = f(t,x), \\ u(T) = g(x), \qquad u_{|\partial\Omega} = 0, \end{cases}$$

where

$$V(t,x) = \int_0^1 H_p(t,x,\lambda Du_1(t,x) + (1-\lambda)Du_2(t,x)d\lambda;$$

$$f(t,x) = \int_0^1 \int_\Omega \frac{\delta F}{\delta m}(t,x,m_\lambda(t),y)(m_1(t,dy)-m_2(t,dy))d\lambda;$$

$$g(x) = \int_0^1 \int_\Omega \frac{\delta G}{\delta m}(x,m_\lambda(T),y)(m_1(T,dy)-m_2(T,dy))d\lambda,$$

where m_λ is defined as follows

$$m_\lambda(\cdot) := \lambda m_1(\cdot) + (1-\lambda)m_2(\cdot).$$

To apply Lemma 4.1, we estimate the regularity of the data:

$$\begin{aligned} \sup_{t\in[0,T]} \|f(t,\cdot)\|_\alpha &\le \sup_{t\in[0,T]} \int_0^1 \left\| \frac{\delta F}{\delta m}(t,\cdot,m_\lambda(t),\cdot) \right\|_{\alpha,(1+\alpha,D)} d\lambda \, \|m_1(t)-m_2(t)\|_{-(1+\alpha),D} \\ &\le C \sup_{t\in[0,T]} \|m_1(t)-m_2(t)\|_{-(1+\alpha),D}, \end{aligned}$$

analogously

$$\|g(\cdot)\|_{2+\alpha} \le C \sup_{t\in[0,T]} \|m_1(t)-m_2(t)\|_{-(1+\alpha),D}. \tag{40}$$

So, Eq. (13) implies

$$\|u_1-u_2\|_{1,2+\alpha} \le C \sup_{t\in[0,T]} \|m_1(t)-m_2(t)\|_{-(1+\alpha),D}. \tag{41}$$

Coming back to (38), we have

$$\sup_{t\in[0,T]} \|m_1(t)-m_2(t)\|_{-(1+\alpha),D} \le C\left(\left(\sup_{t\in[0,T]} \|m_1(t)-m_2(t)\|_{-(1+\alpha),D} \right)^{\frac{1}{2}} \mathbf{d}_1(m_{01},m_{02})^{\frac{1}{2}} + \mathbf{d}_1(m_{01},m_{02}) \right),$$

hence, by a generalized Young's inequality:

$$\sup_{t\in[0,T]} \|m_1(t)-m_2(t)\|_{-(1+\alpha),D} \le C\mathbf{d}_1(m_{01},m_{02}). \tag{42}$$

Plugging this estimate in (41), we finally obtain

$$\|u_1-u_2\|_{1,2+\alpha} \le C\mathbf{d}_1(m_{01},m_{02}).$$

For the L^p inequality, we consider $m := m_1 - m_2$. Then m solves the equation

$$\left\{ m_t - \sum_{i,j} \partial^2_{ij}(a_{ij}(x)m) - \mathrm{div}(m(H_p(t,x,Du_1))) = \mathrm{div}(m_2(H_p(t,x,Du_2) - H_p(t,x,Du_1))),\ m(t_0) = m_{01}-m_{02},\ \left[m_1-m_2\right]_{|\partial\Omega} = 0, \right.$$

i.e. m is a solution of (14) with $f = \mathrm{div}(m_2(H_p(t,x,Du_2) - H_p(t,x,Du_1)))$, $\mu_0 = m_{01}-m_{02}$, $b = H_p(t,x,Du_1)$. Then estimates (17) imply

$$\|m_1-m_2\|_{L^p(Q_T)} \le C\left(\|\mu_0\|_{-(1+\alpha)} + \|f\|_{L^1} \right).$$

We estimate the right-hand side term. As regards μ_0 we have

$$\|\mu_0\|_{-(1+\alpha)} = \sup_{\|\phi\|_{1+\alpha}\le 1} \int_\Omega \phi(x)(m_{01}-m_{02})(dx) \le C\mathbf{d}_1(m_{01},m_{02}).$$

For the f term, we argue in the following way:

$$\begin{aligned}\|f\|_{L^1} &= \int_0^T \sup_{\|\phi\|_{W^{1,\infty}}\le 1}\left(\int_\Omega H_p(x,Du_2)-H_p(x,Du_1)D\phi\, m_2(t,dx)\right)dt\\ &\le C\left\|u_1-u_2\right\|_{\frac{1+\alpha}{2},1+\alpha}\le C\mathbf{d}_1(m_{01},m_{02}),\end{aligned}$$

which allows us to conclude. □

The previous proposition (4.6) allows us to state that

$$\begin{aligned}&\sup_{t\in[0,T]}\sup_{m\in\mathcal{P}^{sub}(\Omega)}\|U(t,\cdot,m)\|_{2+\alpha}\le C,\\ &\sup_{t\in[0,T]}\sup_{m_1\neq m_2}\left[\left(\mathbf{d}_1(m_1,m_2)\right)^{-1}\left\|U(t,\cdot,m_1)-U(t,\cdot,m_2)\right\|_{2+\alpha}\right]\le C,\end{aligned}\tag{43}$$

which are two initial regularity results for the function U.

5. Linearized system and differentiability of U with respect to the measure

This section is devoted to the study of the following *linearized MFG system*:

$$\begin{cases}-z_t-\mathrm{tr}(a(x)D^2z)+H_p(t,x,Du)Dz=\dfrac{\delta F}{\delta m}(t,x,m(t))(\rho(t))+h(t,x),\\ \rho_t-\displaystyle\sum_{ij}\partial^2_{ij}(a_{ij}(x)\rho)-\mathrm{div}(\rho(H_p(t,x,Du)))+\\ -\mathrm{div}(mH_{pp}(t,x,Du)Dz+c)=0,\\ z(T,x)=\dfrac{\delta G}{\delta m}(x,m(T))(\rho(T))+z_T(x),\qquad \rho(t_0)=\rho_0,\\ z_{|\partial\Omega}=0,\qquad \rho_{|\partial\Omega}=0,\end{cases}\tag{44}$$

where $z_T\in C^{2+\alpha,D}$, $\rho_0\in C^{-(1+\alpha)}$, $h\in C^{0,\alpha}([t_0,T]\times\Omega)$, $c\in L^1([t_0,T]\times\Omega)$, and where we define for F (and for G)

$$\frac{\delta F}{\delta m}(t,x,m(t))(\rho(t)):=\left\langle\frac{\delta F}{\delta m}(t,x,m(t),\cdot),\rho(t)\right\rangle,$$

where the duality is between $C^{-(1+\alpha),D}$ and $C^{(1+\alpha),D}$.

The study of this system plays a crucial role in proving the C^1 character of U in terms of m. In particular, if we define the couple (v,μ) as the solution of (44) with $h=c=z_T=0$ and $\mu(t_0)=\mu_0$, we obtain

$$v(t_0,x)=\left\langle\frac{\delta U}{\delta m}(t_0,x,m_0,\cdot),\mu_0\right\rangle.\tag{45}$$

Let us start by giving a suitable definition of a solution for the previous system.

Definition 5.1. A couple (z,ρ) is a solution of (44) if $z\in C^{1,2+\alpha}$, is a classical solution of the linear HJB equation and $\rho\in C([0,T];C^{-(1+\alpha),D}(\Omega))\cap L^1(Q_T)$ solves the Fokker-Planck equation, accordingly with Definition 4.2.

Let us underline that $c\in L^1([t_0,T]\times\Omega)\implies \mathrm{div}(c)\in L^1([t_0,T];W^{-1,\infty}(\Omega))$. Therefore, the well-posedness of the Fokker-Planck equation is included in Proposition 4.3.

The following Proposition states existence, uniqueness and regularity for the problem defined by Eq. (44). The proof relies on results previously obtained for the Fokker-Planck equation, and it proceeds similarly to the Neumann case [24]. We report it here for the sake of completeness.

Proposition 5.2. *Let hypotheses 2.6 hold for a certain $0<\alpha<1$, and let $\rho_0\in C^{-1}$. Then there exists a unique solution $(z,\rho)\in C^{1,2+\alpha}\times\left(C([0,T];C^{-(1+\alpha),D}(\Omega))\cap L^1(Q_T)\right)$ for the system (44), satisfying*

$$\|z\|_{1,2+\alpha}+\sup_t\|\rho(t)\|_{-(1+\alpha),D}+\|\rho\|_{L^p}\le CM,\tag{46}$$

where C depends on H, p is defined as in Section 4.1, and M is given by

$$M:=\|z_T\|_{2+\alpha}+\|\rho_0\|_{-(1+\alpha)}+\|h\|_{0,\alpha}+\|c\|_{L^1}.\tag{47}$$

Moreover, the solution (v,μ) related to $h=c=z_T=0$ satisfies

$$\|v\|_{1,2+\alpha}+\sup_{t\in[0,T]}\|\mu(t)\|_{-(2+\alpha),D}+\|\mu\|_{-\left(\frac{\alpha}{2},\alpha\right)}\le C\|\mu_0\|_{-(2+\alpha)}.\tag{48}$$

Observe that the analogue of this Theorem in the Neumann case is given by [24, Propositions 5.8,5.11]. However, even in this case, we have refined the estimate (48) in order to adjust the issues raised in [48]. See also Remark 4.5.

Proof. As always, we can assume $t_0=0$ without loss of generality. The main idea is to apply Schaefer's Theorem.

Step 1: Definition of the map $\mathbf{\Phi}$ satisfying Schaefer's Theorem. We set $X := C([0,T]; C^{-(1+\alpha),D})$, endowed with the norm

$$\|\phi\|_X := \sup_{t\in[0,T]} \|\phi(t)\|_{-(1+\alpha),D}\,.$$

For $\rho \in X$, we consider the classical solution z of the following equation

$$\begin{cases} -z_t - \mathrm{tr}(a(x)D^2 z) + H_p(t,x,Du)Dz = \frac{\delta F}{\delta m}(t,x,m(t))(\rho(t)) + h(t,x)\,, \\ z(T) = \frac{\delta G}{\delta m}(x,m(T))(\rho(T)) + z_T\,, \\ z|_{\partial\Omega} = 0\,. \end{cases} \tag{49}$$

We note that, from Hypotheses 2.6, we have

$$\frac{\delta F}{\delta m}(t,x,m(t),y)|_{y\in\partial\Omega} = 0, \quad \frac{\delta G}{\delta m}(x,m(T),y)|_{y\in\partial\Omega} = 0$$

Hence, compatibility conditions are satisfied for Eq. (49) and, from *Theorem 5.1.21* of [66], z satisfies

$$\begin{aligned} \|z\|_{1,2+\alpha} &\le C\left(\|z_T\|_{2+\alpha} + \sup_{t\in[0,T]} \|\rho(t)\|_{-(2+\alpha),D} + \|h\|_{0,\alpha} \right) \\ &\le C\left(M + \sup_{t\in[0,T]} \|\rho(t)\|_{-(1+\alpha),D} \right), \end{aligned} \tag{50}$$

where we also use hypothesis (vii) of 2.6, for the boundary condition of $\frac{\delta G}{\delta m}$.

Then we define $\mathbf{\Phi}(\rho) := \tilde{\rho}$, where $\tilde{\rho}$ is the solution in the sense of Definition 4.2 to:

$$\begin{cases} \tilde{\rho}_t - \mathrm{div}(a(x)D\tilde{\rho}) - \mathrm{div}(\tilde{\rho}(H_p(t,x,Du))) - \mathrm{div}(mH_{pp}(t,x,Du)Dz + c) = 0 \\ \tilde{\rho}(0) = \rho_0 \\ \tilde{\rho}|_{\partial\Omega} = 0 \end{cases}. \tag{51}$$

Thanks to Proposition 4.3, we have $\tilde{\rho} \in X$. We want to prove that the map $\mathbf{\Phi}$ is continuous and compact.

For the compactness, let $\{\rho_n\}_n \subset X$ be a subsequence with $\|\rho_n\|_X \le C$ for a certain $C > 0$. We consider for each n the solutions z_n and $\tilde{\rho}_n$ of (49) and (51) associated to ρ_n.

Using (50), we have $\|z_n\|_{1,2+\alpha} \le C_1$, where C_1 depends on C. Then, thanks to Ascoli-Arzelà's Theorem, $\exists z$ s.t. $z_n \to z$ up to subsequences at least in $C([0,T]; C^1(\Omega))$.

Using the pointwise convergence of Dz_n and the L^p boundedness of m stated in Proposition 4.6, we immediately obtain

$$mH_{pp}(t,x,Du)Dz_n + c \to mH_{pp}(t,x,Du)Dz + c \quad \text{in } L^1(Q_T)\,,$$

which immediately implies

$$\mathrm{div}(mH_{pp}(t,x,Du)Dz_n + c) \to \mathrm{div}(mH_{pp}(t,x,Du)Dz + c) \quad \text{in } L^1(W^{-1,\infty})\,.$$

Hence, stability results proved in Proposition 4.3 prove that $\tilde{\rho}_n \to \tilde{\rho}$ in X, where $\tilde{\rho}$ is the solution related to Dz. This proves the compactness result.

The continuity of Φ can be proved using the same computations as for the compactness.

Finally, in order to apply Schaefer's theorem, we have to prove that

$$\exists M > 0 \text{ s.t. } \rho = \sigma\mathbf{\Phi}(\rho) \text{ and } \sigma \in [0,1] \Rightarrow \|\rho\|_X \le M\,.$$

We will prove in the next step that, if $\rho = \sigma\mathbf{\Phi}(\rho)$, then the couple (z,ρ) satisfies (46). This allows us to apply Schaefer's theorem and also gives us the desired estimate (46), since each solution (z,ρ) of the system satisfies $\rho = \sigma\mathbf{\Phi}(\rho)$ with $\sigma = 1$.

Step 2: Estimate of ρ and z. Let $(\rho,\sigma) \in X \times [0,1]$ such that $\rho = \sigma\mathbf{\Phi}(\rho)$. Then the couple (z,ρ) satisfies

$$\begin{cases} -z_t - \mathrm{tr}(a(x)D^2 z) + H_p(t,x,Du)Dz = \frac{\delta F}{\delta m}(t,x,m(t))(\rho(t)) + h(t,x) \\ \rho_t - \mathrm{div}(a(x)D\rho) - \mathrm{div}(\rho H_p(t,x,Du)) - \sigma\mathrm{div}(mH_{pp}(t,x,Du)Dz + c) = 0 \\ z(T,x) = \frac{\delta G}{\delta m}(x,m(T))(\rho(T)) + z_T(x) \quad \rho(0) = \sigma\rho_0 \\ z|_{\partial\Omega} = 0 \quad \rho|_{\partial\Omega} = 0 \end{cases}.$$

We want to use z as a test function for the equation of ρ. This is allowed since z satisfies (15) with

$$\psi = \frac{\delta F}{\delta m}(t,x,m(t))(\rho(t)) + h(t,x) \in L^\infty(\Omega), \quad \xi = \frac{\delta G}{\delta m}(x,m(T))(\rho(T)) + z_T(x) \in C^{1+\alpha,D}$$

From the weak formulation of ρ, we obtain:

$$\begin{aligned} &\int_\Omega \big(\rho(T,x)z(T,x) - \sigma\rho_0(x)z(0,x)\big)dx = -\sigma\int_0^T\int_\Omega \langle c, Dz\rangle dxdt \\ &- \int_0^T\int_\Omega \rho(t,x)\Big(\frac{\delta F}{\delta m}(t,x,m(t))(\rho(t)) + h\Big)dxdt - \sigma\int_0^T\int_\Omega m\langle H_{pp}(t,x,Du)Dz, Dz\rangle dxdt\,. \end{aligned}$$

Using the terminal condition of z and the monotonicity of F and G (vi) in 2.6, we get a first estimate:

$$\begin{aligned}\sigma\int_0^T\int_\Omega m\langle H_{pp}(t,x,Du)Dz,Dz\rangle dxdt &\le \sup_{t\in[0,T]}\|\rho(t)\|_{-(2+\alpha),D}\|z_T\|_{2+\alpha}+\|\rho\|_{L^p}\|h\|_\infty\\ &\quad+\|z\|_{1,2+\alpha}\left(\|\rho_0\|_{-(2+\alpha),D}+\|c\|_{L^1}\right)\\ &\le M\left(\sup_{t\in[0,T]}\|\rho(t)\|_{-(1+\alpha),D}+\|\rho\|_{L^1}+\|z\|_{1,2+\alpha}\right).\end{aligned}\tag{52}$$

We already know an initial estimate on z in (50). Now we need to estimate ρ.
Using (17) from Proposition 4.3 we obtain

$$\sup_{t\in[0,T]}\|\rho\|_{-(1+\alpha),D}+\|\rho\|_{L^p}\le C\left(\left\|\sigma m H_{pp}(t,x,Du)Dz\right\|_{L^1}+\|c\|_{L^1}+\|\rho_0\|_{-(1+\alpha)}\right)\tag{53}$$

As regards the first term in the right-hand side, we can use Hölder's inequality and (52) to obtain

$$\begin{aligned}&\left\|mH_{pp}(t,x,Du)Dz\right\|_{L^1}=\sigma\sup_{\substack{\|\phi\|_\infty\le1\\ \phi\in L^\infty(Q_T;\mathbb{R}^d)}}\int_0^T\int_\Omega m\langle H_{pp}(t,x,Du)Dz,\phi\rangle dxdt\\ &\le\sigma\left(\int_0^T\int_\Omega m\langle H_{pp}(t,x,Du)Dz,Dz\rangle dxdt\right)^{1/2}\left(\int_0^T\int_\Omega m\langle H_{pp}(t,x,Du)\phi,\phi\rangle dxdt\right)^{1/2}\\ &\le M^{1/2}\left(\sup_{t\in[0,T]}\|\rho(t)\|^{1/2}_{-(1+\alpha),D}+\|\rho\|^{1/2}_{L^1}+\|z\|^{1/2}_{1,2+\alpha}\right)\end{aligned}$$

Putting these estimates into (53) we obtain

$$\sup_{t\in[0,T]}\|\rho\|_{-(1+\alpha),D}+\|\rho\|_{L^p}\le C\left(M+M^{1/2}\left(\sup_{t\in[0,T]}\|\rho(t)\|^{1/2}_{-(1+\alpha),D}+\|\rho\|^{1/2}_{L^1}+\|z\|^{1/2}_{1,2+\alpha}\right)\right).$$

Using a generalized Young's inequality with suitable coefficients, we get

$$\sup_{t\in[0,T]}\|\rho\|_{-(1+\alpha),D}+\|\rho\|_{L^p}\le C\left(M+M^{1/2}\|z\|^{1/2}_{1,2+\alpha}\right).\tag{54}$$

This gives us an initial estimate for ρ, depending on the estimate of z.
Coming back to (50), (54) implies

$$\|z\|_{1,2+\alpha}\le C\left(M+M^{1/2}\|z\|^{1/2}_{1,2+\alpha}\right).$$

Using a generalized Young's inequality with suitable coefficients, this implies

$$\|z\|_{1,2+\alpha}\le CM\,.$$

Plugging this estimate in (54), we finally obtain

$$\|z\|_{1,2+\alpha}+\sup_{t\in[0,T]}\|\rho\|_{-(1+\alpha),D}+\|\rho\|_{L^p}\le CM\,.$$

This concludes the existence part.

Step 3. Uniqueness. Let (z_1,ρ_1) and (z_2,ρ_2) be two solutions of (44). Then the couple $(z,\rho):=(z_1-z_2,\rho_1-\rho_2)$ satisfies the following linear system:

$$\begin{cases}-z_t-\mathrm{tr}(a(x)D^2z)+H_p(t,x,Du)Dz=\frac{\delta F}{\delta m}(t,x,m(t))(\rho(t))=0\,,\\ \rho_t-\mathrm{div}(a(x)D\rho)-\mathrm{div}(\rho H_p(t,x,Du))-\mathrm{div}(mH_{pp}(t,x,Du)Dz)=0\,,\\ z(T,x)=\frac{\delta G}{\delta m}(x,m(T))(\rho(T))\,,\quad \rho(t_0)=0\,,\\ z|_{\partial\Omega}=0\,,\quad \rho|_{\partial\Omega}=0\,,\end{cases}$$

i.e., a system of the form (44) with $h=c=z_T=\rho_0=0$. Then estimation (46) tells us that

$$\|z\|_{1,2+\alpha}+\sup_{t\in[0,T]}\|\rho\|_{-(1+\alpha),D}+\|\rho\|_{L^p}\le0\,,$$

and so $z=0$, $\rho=0$. This proves the estimate (46).

Concerning the second inequality (48), we consider the solution (v,μ) obtained above. Since μ satisfies $\mu=\sigma\Phi(\mu)$ with $\sigma=1$, we can use (50) with $z_T=h=0$ and obtain

$$\|v\|_{1,2+\alpha}\le C\sup_{t\in[0,T]}\|\mu(t)\|_{-(2+\alpha),D}\,.\tag{55}$$

We want to estimate the right-hand side. Using Corollary 4.4 (which addresses the regularity issues raised in [48]) we have

$$\sup_{t\in[0,T]}\|\mu(t)\|_{-(2+\alpha),D}+\|\mu\|_{-\left(\frac{\alpha}{2},\alpha\right)}\le C\left(\|\mu_0\|_{-(2+\alpha)}+\left\|mH_{pp}(t,x,Du)Dv\right\|_{L^1}\right).\tag{56}$$

The last term is estimated, as above, by

$$\left\| \sigma m H_{pp}(t,x,Du)Dv \right\|_{L^1} \leq C\left(\int_0^T \int_\Omega m\langle H_{pp}(t,x,Du)Dv, Dv\rangle dxdt \right)^{1/2}. \tag{57}$$

The right-hand side term can be bounded using (52) with $h = z_T = c = 0$:

$$\int_0^T \int_\Omega m\langle H_{pp}(t,x,Du)Dv, Dv\rangle dxdt \leq \|v\|_{1,2+\alpha} \left\| \mu_0 \right\|_{-(2+\alpha)}. \tag{58}$$

Hence, plugging estimates (57) and (58) into (56) we obtain

$$\sup_{t\in[0,T]} \|\mu(t)\|_{-(2+\alpha),D} + \|\mu\|_{-\left(\frac{\alpha}{2},\alpha\right)} \leq C\left(\left\| \mu_0 \right\|_{-(2+\alpha)} + \|v\|^{1/2}_{1,2+\alpha} \left\| \mu_0 \right\|^{1/2}_{-(2+\alpha),D} \right). \tag{59}$$

Coming back to (55) and using a generalized Young's inequality, we get

$$\|v\|_{1,2+\alpha} \leq C \left\| \mu_0 \right\|_{-(2+\alpha)},$$

and finally, substituting the last estimate into (59), we obtain (48) and conclude. □

Throughout the rest of the paper, we will denote with (v,μ) a solution to the system (44), with $h = c = z_T = 0$ and $\mu_0 := \rho_0$. We will refer to this system as the *pure linearized system*. Instead, the general system (44), with solution (z,ρ), will be called the *general linearized system*.

To prove $U \in C^1$ as well as the related representation formula (45), we have to prove that the pure linearized system has a fundamental solution, which is the content of the next proposition.

Proposition 5.3. *Let hypotheses 2.6 hold. Then there exists a function $K : [0,T]\times\Omega\times\mathcal{P}^{sub}(\Omega)\times\Omega \to \mathbb{R}$ such that, for any solution (v,μ), with initial data (t_0, m_0, μ_0), we have*

$$v(t_0,x) = {}_{-(1+\alpha)}\langle \mu_0, K(t_0,x,m_0,\cdot)\rangle_{1+\alpha} \tag{60}$$

Moreover, K is twice differentiable with respect to both x and y, and it holds

$$\sup_{(t,m)\in[0,T]\times\mathcal{P}^{sub}(\Omega)} \|K(t,\cdot,m,\cdot)\|_{2+\alpha,2+\alpha} \leq C, \tag{61}$$

Proof. Let $\mu_0 = \delta_y$ be the Dirac function at $y \in \Omega$. We define $K(t_0,x,m_0,y) = v(t_0,x;\delta_y)$, where $v(\cdot,\cdot;\mu_0)$ indicates the function v related to the pure linearized system with initial data μ_0. Exploiting (46), we have that K is twice differentiable w.r.t. x, and it holds

$$\left\| K(t_0,\cdot,m_0,y) \right\|_{2+\alpha} \leq C \left\| \delta_y \right\|_{-(1+\alpha)} = C$$

Furthermore, the linearity character of (44) implies

$$\frac{K(t_0,x,m_0,y+he_i) - K(t_0,x,m_0,y)}{h} = v(t_0,x;\Delta^i_h\delta_y),$$

where we use the notation $\Delta^i_h\delta_y = \frac{1}{h}(\delta_{y+he_i} - \delta_y)$. The linear character of the pure linearized system directly implies that the solution (v,μ) is stable with respect to the initial condition μ_0, allowing to pass to the limit and obtain

$$\frac{\partial K}{\partial y_i}(t_0,x,m_0,y) = v(t_0,x;-\partial_{y_i}\delta_y),$$

with the derivative that has to be intended in a distributional sense. As to prove the existence and second derivatives' bounds, we consider the incremental ratio

$$R^h_{i,j}(x,y) := \frac{\partial_{y_i} K(t_0,x,m_0,y+he_j) - \partial_{y_i} K(t_0,x,m_0,y)}{h}. \tag{62}$$

Hence, estimate (48) together with Lagrange Theorem imply that, for $|l| \leq 2$, we have

$$\begin{aligned}
&\left| D^l_x R^h_{i,j}(x,y) - D^l_x R^k_{i,j}(x,y) \right| = \left| D^l_x v\left(t_0,x;\Delta^j_h(-\partial_{y_i}\delta_y) - \Delta^j_k(-\partial_{y_i}\delta_y)\right) \right| \\
&\leq C \left\| \Delta^j_h(-\partial_{y_i}\delta_y) - \Delta^j_k(-\partial_{y_i}\delta_y) \right\|_{-(2+\alpha)} \\
&= \sup_{\|\phi\|_{2+\alpha}\leq 1} \left(\frac{\partial_{y_i}\phi(y+he_j) - \partial_{y_i}\phi(y)}{h} - \frac{\partial_{y_i}\phi(y+ke_j) - \partial_{y_i}\phi(y)}{k} \right) \\
&= \sup_{\|\phi\|_{2+\alpha}\leq 1} \left(\partial^2_{y_i y_j}\phi(y_{\phi,h}) - \partial^2_{y_i y_j}\phi(y_{\phi,k}) \right) \leq \sup_{\|\phi\|_{2+\alpha}\leq 1} |y_{\phi,h} - y_{\phi,k}|^\alpha \leq |h|^\alpha + |k|^\alpha,
\end{aligned}$$

for a certain $y_{\phi,h}$, resp. $y_{\phi,k}$ in the line segment between y and $y+he_j$ (resp. $y+ke_j$), and where we have used the same notation as seen above for $\Delta^j_h(-\partial_{y_i}\delta_y)$.

The latter proves that (62), together with its first and second derivative w.r.t x, are Cauchy sequences in h, implying that $D^l_x \frac{\delta U}{\delta m}$ is twice differentiable w.r.t. y and for all $|l| \leq 2$.

As to conclude estimate (61), we prove the estimate for the second order derivatives w.r.t. y, the first derivative being simpler. By estimate (48) on v, if $y, y' \in \Omega$, then

$$\begin{aligned}\left\| R^h_{i,j}(\cdot, y) - R^h_{i,j}(\cdot, y') \right\|_{2+\alpha} &= \left\| v\big(t_0, \cdot; \Delta^j_h(-\partial_{y_i}\delta_y + \partial_{y_i}\delta_{y'})\big) \right\|_{2+\alpha} \\ &\leq C \left\| \Delta^j_h(-\partial_{y_i}\delta_y + \partial_{y_i}\delta_{y'}) \right\|_{-(2+\alpha)}.\end{aligned}$$

Passing to the limit for $h \to 0$, we have

$$\Delta^j_h(-\partial_{y_i}\delta_y + \partial_{y_i}\delta_{y'}) \longrightarrow \partial_{y_j}\partial_{y_i}\delta_y - \partial_{y_j}\partial_{y_i}\delta_{y'} \text{ , in } C^{-(2+\alpha)}\,.$$

Then, by Ascoli-Arzelà and previously obtained convergence result for $R^h_{i,j}$, we have

$$\left\| \partial^2_{y_i y_j}\big(K(t_0, \cdot, m_0, y) - K(t, \cdot, m_0, y')\big) \right\|_{2+\alpha} \leq C \left\| \partial^2_{y_i y_j}(\delta_y - \delta_{y'}) \right\|_{-(2+\alpha)} \leq C|y - y'|^\alpha\,,$$

which proves (61). The representation formula (60) is an immediate consequence of the linear character of the equation and of the density of the Dirac functions generated set, hence concluding the proof. □

We are now ready to consider the main topic of this section: we want to prove that the function K is actually the derivative of U with respect to the measure. Let us underline that the differentiability with respect to the measure m will be the key for proving U is indeed a classical solution of the Master Eq. (5).

Theorem 5.4. *Suppose hypotheses 2.6 hold. Then the function U defined by (11) is differentiable with respect to m, with the derivative given by*

$$\frac{\delta U}{\delta m}(t, x, m, y) = K(t, x, m, y)\,.$$

Proof. Within the present proof, given two functions a_1, a_2, we define $a_{1+\tau} := \tau a_2 + (1-\tau)a_1$, for $\tau \in [0, 1]$.

We will prove a more general fact, the representation formula for U being then a direct consequence of it.

In particular, if (u_1, m_1) and (u_2, m_2) are two solutions of (22), with initial conditions (t_0, m_{01}) and (t_0, m_{02}), and (v, μ) is the solution of the pure linearized system related to (u_2, m_2), with initial condition $(t_0, m_{01} - m_{02})$, then a sort of first-order Taylor expansion of U with respect to m holds, namely:

$$\|u_1 - u_2 - v\|_{1,2+\alpha} + \sup_{t\in[0,T]} \|m_1(t) - m_2(t) - \mu(t)\|_{-(1+\alpha),D} \leq C\mathbf{d}_1(m_{01}, m_{02})^2\,. \tag{63}$$

As to prove above expansion, let us start defining $(z, \rho) = (u_1 - u_2 - v, m_1 - m_2 - \mu)$. Then (z, ρ) satisfies the general linearized system (44) related to (u_2, m_2), with data $h = h_1 + h_2$, $c = c_1 + c_2$ and z_T given by:

$$\begin{aligned}h_1 &= -\int_0^1 (H_p(t, x, Du_{1+s}) - H_p(t, x, Du_2)) \cdot D(u_1 - u_2)\, ds\,,\\ h_2 &= \int_0^1 \int_\Omega \Big(\frac{\delta F}{\delta m}(t, x, m_{1+s}(t), y) - \frac{\delta F}{\delta m}(t, x, m_2(t), y)\Big)(m_1(t) - m_2(t))(dy)ds,\\ c_1(t) &= (m_1(t) - m_2(t))H_{pp}(t, x, Du_2)(Du_1 - Du_2)\,,\\ c_2(t) &= m_1 \int_0^1 \big(H_{pp}(t, x, Du_{1+s}) - H_{pp}(t, x, Du_2)\big)(Du_1 - Du_2)\, ds\,,\\ z_T &= \int_0^1 \int_\Omega \Big(\frac{\delta G}{\delta m}(x, m_{1+s}(T), y) - \frac{\delta G}{\delta m}(x, m_2(T), y)\Big)(m_1(T) - m_2(T))(dy)ds\,.\end{aligned}$$

Applying (46) one has

$$\|z\|_{1,2+\alpha} + \sup_{t\in[0,T]} \|\rho(t)\|_{-(1+\alpha),D} \leq C\big(\|h\|_{0,\alpha} + \|c\|_{L^1} + \|z_T\|_{2+\alpha}\big)\,. \tag{64}$$

We want to estimate the right-hand side term to obtain the desired Taylor expansion. As regards h, exploiting Eq. (24) and Hölder norm properties, we have:

$$\begin{aligned}\|h_1\|_{0,\alpha} &= \left\| \int_0^1 \int_0^1 s\,\langle H_{pp}(t, x, Du_{1+rs})(Du_1 - Du_2), (Du_1 - Du_2)\rangle\, dr ds \right\|_{0,\alpha} \\ &\leq C\mathbf{d}_1(m_{01}, m_{02})^2\,,\end{aligned}$$

Analogously, again by eq. (24) and exploiting regularity of both F and G, the same estimate also holds for h_2 and z_T. For the function c, we can write

$$\begin{aligned}\|c_1\|_{L^1} &= \int_0^T \int_\Omega H_{pp}(t, x, Du_2)(Du_1 - Du_2)(m_1(t, dx) - m_2(t, dx))\, dt \\ &\leq C\|u_1 - u_2\|_{1,2+\alpha} \|m_1(t) - m_2(t)\| \leq C\mathbf{d}_1(m_{01}, m_{02})^2\,,\end{aligned}$$

then, proceeding as above, we have

$$\begin{aligned}\|c_2\|_{L^1} &= \int_0^1 \int_0^T \int_\Omega \big(H_{pp}(t,x,Du_{1+s}) - H_{pp}(t,x,Du_2)\big)(Du_1 - Du_2)m_1(t,dx)\,dtds \\ &\le C\|Du_1 - Du_2\|_\infty^2 \le C\mathbf{d}_1(m_0^1, m_0^2)^2\,.\end{aligned}$$

Substituting these estimates in (64), we obtain (63). Using the representation (60) for v, we get

$$\left\| U(t_0,\cdot,m_{01}) - U(t_0,\cdot,m_{02}) - \int_\Omega K(t_0,\cdot,m_{02},y)(m_{01}-m_{02})(dy)\right\|_\infty \le C\mathbf{d}_1(m_{01},m_{02})^2.$$

As a straightforward consequence, we have that U is differentiable with respect to m and

$$\frac{\delta U}{\delta m}(t,x,m,y) = K(t,x,m,y)\,,$$

hence concluding the proof. □

Using (61) we also obtain the following strong regularity estimate for $\frac{\delta U}{\delta m}$:

$$\sup_t \left\| \frac{\delta U}{\delta m}(t,\cdot,m,\cdot)\right\|_{2+\alpha,2+\alpha} \le C\,. \tag{65}$$

This gives sense to the Master Eq. (5), since the quantity D_yD_mU is now well-defined. However, to apply Theorem 3.1 we still need to prove the continuity of D_yD_mU in the measure variable.

In the next result we establish a Lipschitz bound for $\frac{\delta U}{\delta m}$ with respect to the measure m. This estimate plays a crucial role, as it yields the continuity of D_yD_mU with respect to m, a property that is essential for proving the existence of solutions to the Master Equation in the appropriate framework. Such regularity of D_yD_mU was not available in the Neumann case, as reported in []. We also point out that, by adapting the same ideas, an analogous result could be obtained in the Neumann setting as well, although we shall not pursue this direction in the present work.

Theorem 5.5. *Let Assumptions 2.6 hold. Then the derivative of the solution of the Master Equation $\frac{\delta U}{\delta m}$ is Lipschitz continuous with respect to the measure m:*

$$\sup_{t\in[0,T]}\sup_{m_1\neq m_2}(\mathbf{d}_1(m_1,m_2))^{-1}\left\|\frac{\delta U}{\delta m}(t,\cdot,m_1,\cdot) - \frac{\delta U}{\delta m}(t,\cdot,m_2,\cdot)\right\|_{2+\alpha,2+\alpha} \le C \tag{66}$$

where C depends on d, F, G, H and T.

Proof. We consider, for $i=1,2$, the solution (v_i,μ_i) of the linearized system (44) related to (u_i,m_i).

To avoid too heavy notations, we take $t_0=0$ and we define

$$H_i'(t,x) := H_p(t,x,Du_i(t,x))\,, H_i''(t,x) = H_{pp}(t,x,Du_i(t,x))\,,$$
$$F'(t,x,m,\mu) = \int_\Omega \frac{\delta F}{\delta m}(t,x,m,y)\,\mu(dy)\,, G'(x,m,\mu) = \int_\Omega \frac{\delta G}{\delta m}(x,m,y)\,\mu(dy)\,.$$

Then the couple $(z,\rho) := (v_1-v_2, \mu_1-\mu_2)$ satisfies the following linear system:

$$\begin{cases} -z_t - \mathrm{tr}(a(x)D^2z) + H_1'\cdot Dz = F'(t,x,m_1(t),\rho(t)) + h\,, \\ \rho_t - \sum_{i,j}\partial^2_{ij}(a_{ij}(x)\rho) - \mathrm{div}(\rho H_1') - \mathrm{div}(m_1H_1''Dz + c) = 0\,, \\ z(T,x) = G'(x,m_1(T),\rho(T)) + z_T\,, \qquad \rho(t_0) = 0\,, \\ z_{|\partial\Omega} = 0\,, \qquad \rho_{|\partial\Omega} = 0\,, \end{cases}$$

where

$$\begin{aligned} h(t,x) &= h_1(t,x) + h_2(t,x)\,, \\ h_1(t,x) &= F'(t,x,m_1(t),\mu_2(t)) - F'(t,x,m_2(t),\mu_2(t))\,, \\ h_2(t,x) &= \big(H_1'(t,x) - H_2'(t,x)\big)\cdot Dv_2(t,x)\,, \\ c(t,x) &= \mu_2(t)\big(H_1' - H_2'\big)(t,x) + \big[(m_1H_1'' - m_2H_2'')\big](t,x)\,, \\ z_T(x) &= G'(x,m_1(T),\mu_2(T)) - G'(x,m_2(T),\mu_2(T))\,. \end{aligned}$$

Applying (46) we obtain this estimate on z:

$$\|z\|_{1,2+\alpha} \le C\big(\|z_T\|_{2+\alpha} + \|h\|_{0,\alpha} + \|c\|_{L^1}\big)\,.$$

Now we estimate the terms in the right-hand side.

The term with z_T, thanks to (48) and the hypothesis *(v)* of 2.6, is immediately estimated:

$$\|z_T\|_{2+\alpha} \le \left\|\frac{\delta G}{\delta m}(\cdot,m_1(T),\cdot) - \frac{\delta G}{\delta m}(\cdot,m_2(T),\cdot)\right\|_{2+\alpha,2+\alpha}\|\mu_2(T)\|_{-(2+\alpha),D} \le C\mathbf{d}_1(m_{01},m_{02})\|\mu_0\|_{-(2+\alpha)}\,.$$

As regards the space estimate for h, we have

$$\|h(t,\cdot)\|_\alpha \le \left\|F'(\cdot,m_1(t),\mu_2(t)) - F'(\cdot,m_2(t),\mu_2(t))\right\|_\alpha + \left\|(H_1' - H_2')(t,\cdot)Dv_2(t,\cdot)\right\|_\alpha .$$

The first term is bounded as z_T :

$$\left\|F'(\cdot,m_1(t),\mu_2(t)) - F'(\cdot,m_2(t),\mu_2(t))\right\|_\alpha \le C\mathbf{d}_1(m_{01},m_{02})\|\mu_0\|_{-(2+\alpha)} .$$

The second term, using (24) and (48), can be estimated in this way:

$$\left\|(H_1' - H_2')(t,\cdot)Dv_2(t,\cdot)\right\|_\alpha \le C\|(u_1-u_2)(t)\|_{1+\alpha}\|v_2(t)\|_{1+\alpha} \le C\mathbf{d}_1(m_{01},m_{02})\|\mu_0\|_{-(2+\alpha)} .$$

In summary,

$$\|h\|_{0,\alpha} = \sup_{t\in[0,T]}\|h(t,\cdot)\|_\alpha \le C\mathbf{d}_1(m_{01},m_{02})\|\mu_0\|_{-(2+\alpha)} .$$

Finally, we estimate $\|c\|_{L^1}$. We have

$$\begin{aligned}\|c\|_{L^1} &= \int_0^T\int_\Omega (H_1' - H_2')(t,x)\,\mu_2(t,dx)\,dt + \int_0^T\int_\Omega H_1''(t,x)Dv_2(t,x)(m_1(t)-m_2(t))(dx)\,dt\\ &+ \int_0^T\int_\Omega (H_1' - H_2')(t,x)\,Dv_2(t,x)\,m_2(t,dx)\,dt \le C\|u_1-u_2\|_{\frac{1+\alpha}{2},1+\alpha}\|\mu_2\|_{-\left(\frac{\alpha}{2},\alpha\right)}\\ &+ C\|u_1\|_{\frac{1+\alpha}{2},1+\alpha}\|v_2\|_{\frac{1+\alpha}{2},1+\alpha}\|m_1-m_2\|_{L^1} + C\|u_1-u_2\|_{\frac{1+\alpha}{2},1+\alpha}\|v_2\|_{\frac{1+\alpha}{2},1+\alpha} .\end{aligned}$$

The first term in the right-hand side, thanks to (24) and (48), is bounded by

$$C\|u_1-u_2\|_{\frac{1+\alpha}{2},1+\alpha}\|\mu_2\|_{-\left(\frac{\alpha}{2},\alpha\right)} \le C\mathbf{d}_1(m_{01},m_{02})\|\mu_0\|_{-(2+\alpha)} .$$

The second and the third term are estimated in the same way, using (23), (24) and (48). Then

$$\|c\|_{L^1} \le C\mathbf{d}_1(m_{01},m_{02})\|\mu_0\|_{-(2+\alpha)} .$$

Putting together all these estimates, we finally obtain:

$$\|z\|_{1,2+\alpha} \le C\mathbf{d}_1(m_{01},m_{02})\|\mu_0\|_{-(2+\alpha)} .$$

Since

$$z(t_0,x) = \int_\Omega \left(\frac{\delta U}{\delta m}(t_0,x,m_1,y) - \frac{\delta U}{\delta m}(t_0,x,m_2,y)\right)\mu_0(dy),$$

we have proved (66). □

We stress the fact that in the bound (66) a $2+\alpha$ norm in the last variable is strongly required, to make this estimate valid also for the quantity D_yD_mU.

To conclude, we are still left proving the boundary condition for $\frac{\delta U}{\delta m}$ in (5) to verify all the hypotheses needed to apply Theorem 3.1. The latter will be the last result of this section.

Corollary 5.6. *If hypotheses 2.6 hold true, then we have the following boundary conditions for U:*

$$\begin{aligned}&\frac{\delta U}{\delta m}(t,x,m,y) = 0, \qquad \forall x\in\partial\Omega, y\in\Omega, t\in[0,T], m\in\mathcal{P}^{sub}(\Omega),\\ &\frac{\delta U}{\delta m}(t,x,m,y) = 0, \qquad \forall x\in\Omega, y\in\partial\Omega, t\in[0,T], m\in\mathcal{P}^{sub}(\Omega).\end{aligned}$$

Proof. The proof of the first boundary condition is trivial, since $\frac{\delta U}{\delta m}(t_0,x,m_0,y) = v(t_0,x;\delta_y)$ and v satisfies a Dirichlet boundary condition.

For the second condition, let us consider $y\in\partial\Omega$. To prove that $v(t_0,x;\delta_y)=0$, it is enough to check that $v=0$ solves

$$\begin{cases} -v_t - \mathrm{tr}(a(x)D^2v) + H_p(t,x,Du)\cdot Dv = \dfrac{\delta F}{\delta m}(t,x,m(t))(\mu(t)),\\ v(T,x) = \dfrac{\delta G}{\delta m}(x,m(T))(\mu(T)),\\ v_{|\partial\Omega} = 0,\end{cases} \tag{67}$$

where μ is the unique solution, in the sense of Definition 4.2, of

$$\begin{cases} \mu_t - \sum_{ij}\partial^2_{ij}(a_{ij}(x)\mu) - \mathrm{div}(\mu H_p(t,x,Du)) = 0,\\ \mu(t_0) = \mu_0,\\ \mu_{|\partial\Omega} = 0.\end{cases}$$

In this way we have $mH_{pp}(t,x,Du)Dv = 0$, and so the couple (v,μ) turns out to be a solution of

$$\begin{cases} -v_t - \mathrm{tr}(a(x)D^2v) + H_p(t,x,Du)Dv = \dfrac{\delta F}{\delta m}(t,x,m(t))(\rho(t))\,, \\ \mu_t - \displaystyle\sum_{ij}\partial^2_{ij}(a_{ij}(x)\mu) - \mathrm{div}\big(\mu H_p(t,x,Du) + mH_{pp}(t,x,Du)Dv\big) = 0, \\ v(T,x) = \dfrac{\delta G}{\delta m}(x,m(T))(\mu(T))\,, \qquad \mu(t_0) = \mu_0\,, \\ v_{|\partial\Omega} = 0\,, \qquad \mu_{|\partial\Omega} = 0\,, \end{cases}$$

which is exactly the pure linearized system.

Due to the linearity character of (67), we only need to prove that

$$\frac{\delta F}{\delta m}(t,x,m(t))(\mu(t)) = \frac{\delta G}{\delta m}(x,m(T))(\mu(T)) = 0\,.$$

Thanks to the hypotheses 2.6, both $\frac{\delta F}{\delta m}(x,m(t),\cdot)$ and $\frac{\delta G}{\delta m}(x,m(T),\cdot)$ satisfy a Dirichlet boundary condition, being also elements of $C^{2+\alpha}$. Then, choosing $\phi(t,y)$ satisfying (15) with $\psi(t,y) = 0$ and $\xi(y) = \frac{\delta F}{\delta m}(t,x,m(t),y)$, we have

$$\begin{aligned} \frac{\delta F}{\delta m}(t,x,m(t))(\mu(t)) &= \left\langle \mu(t), \frac{\delta F}{\delta m}(t,x,m(t),\cdot)\right\rangle \\ &= \langle \mu_0, \phi(0,\cdot)\rangle = \langle \delta_y, \phi(0,\cdot)\rangle = \phi(0,y) = 0\,, \end{aligned}$$

since ψ satisfies a Dirichlet boundary condition. The same computations hold for $\frac{\delta G}{\delta m}$, therefore we have:

$$\frac{\delta U}{\delta m}(t_0,x,m_0,y) = \left\langle \frac{\delta U}{\delta m}(t_0,x,m_0,\cdot), \delta_y\right\rangle = v(t_0,x) = 0\,.$$

□

Declarations

- All authors contributed to the study conception and design. All authors read and approved the final manuscript.
- No funds, grants, or other support was received.
- All authors declare that they have no financial interests or conflicts of interest.

Acknowledgements

We would like to thank Espen R. Jakobsen for the useful suggestions. We also gratefully acknowledge the anonymous reviewers and the editor, whose constructive feedback significantly contributed to improving the initial version of this manuscript.

References

[1] J.-M. Lasry, P.L. Lions, Jeux à champ moyen. I. Le cas stationnaire, Compt. Rend. Math. 343 (9) (2006) 619–625. https://doi.org/10.1016/j.crma.2006.09.019
[2] J.-M. Lasry, P.L. Lions, Jeux à champ moyen. II. Horizon fini et contrôle optimal, Compt. Rend. Math. 343 (10) (2006) 679–684. https://doi.org/10.1016/j.crma.2006.09.018
[3] J.-M. Lasry, P.L. Lions, Mean field games, Jpn. J. Math. 2 (1) (2007) 229–260. https://doi.org/10.1007/s11537-007-0657-8
[4] J.-M. Lasry, P.-L. Lions, O. Guéant, Application of mean field games to growth theory, in: Paris-Princeton Lectures on Mathematical Finance 2010, 2003 of *Lecture Notes in Mathematics*, Springer, Berlin, 2011, pp. 41–81. https://doi.org/10.1007/978-3-642-14660-2_2
[5] M. Huang, P.E. Caines, R.P. Malhamé, Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle, Commun. Inf. Syst. 6 (3) (2006) 221–251. https://doi.org/10.4310/CIS.2006.v6.n3.a5
[6] P.L. Lions, Cours au Collège de France, 2006–2012, https://www.college-de-france.fr/site/pierre-louis-lions/.
[7] R. Aïd, R. Dumitrescu, P. Tankov, The entry and exit game in the electricity market: a mean-field approach, J. Econ. Dyn. Control 128 (2021) 104160. https://doi.org/10.1016/j.jedc.2021.104160
[8] M. Burger, M. Di Francesco, P.A. Markowich, M.T. Wolfram, Mean field games with nonlinear mobilities in pedestrian dynamics, Discrete Contin. Dyn. Syst. B 19 (5) (2014) 1311–1333. https://doi.org/10.3934/dcdsb.2014.19.1311
[9] M. Burzoni, L. Campi, Mean field games with absorption and common noise with a model of bank run, Stoch. Process. Appl. 164 (2023) 206–241. https://doi.org/10.1016/j.spa.2023.07.007
[10] L. Campi, M. Fischer, N-player games and mean field games with absorption, Ann. Appl. Probab. 28 (4) (2018) 2188–2242. https://doi.org/10.1214/17-AAP1349
[11] P. Grover, K. Bakshi, E.A. Theodorou, A mean-field game model for homogeneous flocking, Chaos 28 (6) (2018) 061103. https://doi.org/10.1063/1.5031448
[12] A. Porretta, M. Ricciardi, Mean field games under invariance conditions for the state space, Commun. Part. Differ. Eqs. 45 (2) (2020) 146–190. https://doi.org/10.1080/03605302.2019.1666281
[13] P. Cannarsa, R. Capuani, Existence and uniqueness for mean field games with state constraints, in: PDE Models for Multi-agent Phenomena, 28 of *Springer INdAM Ser.*, Springer, Cham, 2018, pp. 49–71.
[14] P. Cannarsa, R. Capuani, P. Cardaliaguet, Mean field games with state constraints: from mild to pointwise solutions of the PDE system, Calc. Var. Part. Differ. Eqs. 60 (3) (2021) Paper No. 108, 33. https://doi.org/10.1007/s00526-021-01936-4
[15] P. Cannarsa, R. Capuani, P. Cardaliaguet, $C^{1,1}$-smoothness of constrained solutions in the calculus of variations with application to mean field games, Math. Eng. 1 (1) (2019) 174–203. https://doi.org/10.3934/Mine.2018.1.174
[16] P. Cannarsa, W. Cheng, C. Mendico, K. Wang, Weak KAM approach to first-order Mean Field Games with state constraints, 2020. arXiv:2004.06505 [math.AP]
[17] R. Capuani, A. Marigonda, Constrained mean field games equilibria as fixed point of random lifting of set-valued maps, IFAC-Pap. 55 (30) (2022) 180–185. 25th International Symposium on Mathematical Theory of Networks and Systems MTNS 2022, https://doi.org/10.1016/j.ifacol.2022.11.049
[18] R. Capuani, A. Marigonda, M. Ricciardi, Random lift of set valued maps and applications to multiagent dynamics, Set-Valued Var. Anal. 31 (3) (2023) 28. https://doi.org/10.1007/s11228-023-00693-0

[19] A. Porretta, M. Ricciardi, Ergodic problems for second-order mean field games with state constraints, Commun. Pure Appl. Anal. 23 (5) (Wed May 01 04:00:00 UTC 2024) 620–644. https://doi.org/10.3934/cpaa.2024028

[20] A. Porretta, Weak solutions to Fokker-Planck equations and mean field games, Arch. Ration. Mech. Anal. 216 (1) (2015) 1–62. https://doi.org/10.1007/s00205-014-0799-9

[21] A. Festa, S. Gottlich, M. Ricciardi, Forward-forward mean field games in mathematical modeling with application to opinion formation and voting models, Dyn. Games Appl. (2024). https://doi.org/10.1007/s13235-024-00578-3

[22] M. Cirant, Multi-population mean field games systems with Neumann boundary conditions, J. Math. Pures Appl. 103 (5) (2015) 1294–1315. https://doi.org/10.1016/j.matpur.2014.10.013

[23] D.A. Gomes, M. Ricciardi, Time dependent first-order mean field games with Neumann boundary conditions, (2023). preprint on arXiv, arXiv:2203.10128

[24] M. Ricciardi, The master equation in a bounded domain with Neumann conditions, Commun. Partial Differ. Equ. 47 (5) (2022) 912–947. https://doi.org/10.1080/03605302.2021.2022343

[25] M. Ricciardi, The convergence problem in mean field games with Neumann conditions, SIAM J. Math. Anal. 55 (4) (2023) 3316–3343. https://doi.org/10.1137/22M1527313

[26] R. Buckdahn, J. Li, S. Peng, C. Rainer, Mean-field stochastic differential equations and associated PDEs, Ann. Probab. 45 (2) (2017) 824–878. https://doi.org/10.1214/15-AOP1076

[27] J.F. Chassagneux, D. Crisan, F. Delarue, A probabilistic approach to classical solutions of the master equation for large population equilibria, Mem. Am. Math. Soc. 282 (1383) (2022). https://doi.org/10.1090/memo/1383

[28] W. Gangbo, A. Swiech, Existence of a solution to an equation arising from the theory of mean field games, J. Differ. Equ. 259 (11) (2015) 6573–6643. https://doi.org/10.1016/j.jde.2015.08.001

[29] P. Cardaliaguet, F. Delarue, J.-M. Lasry, P.L. Lions, The master equation and the convergence problem in mean field games, 201 of *Ann. Math. Stud.*, Princeton University Press, 2019.

[30] R. Carmona, F. Delarue, Probabilistic theory of mean field games with applications I: mean field FBSDEs, control, and games, 83 of *Probability Theory and Stochastic Modelling*, Springer, 2018. https://doi.org/10.1007/978-3-319-58920-6

[31] F. Delarue, D. Lacker, K. Ramanan, From the master equation to mean field game limit theory: large deviations and concentration of measure, Ann. Probab. 48 (1) (2020) 211–263. https://doi.org/10.1214/19-AOP1355

[32] F. Delarue, D. Lacker, K. Ramanan, From the master equation to mean field game limit theory: a central limit theorem, Electron. J. Probab. 24 (51) (2019) 1–54. https://doi.org/10.1214/19-EJP311

[33] P. Cardaliaguet, M. Cirant, A. Porretta, Remarks on Nash equilibria in mean field game models with a major player, Proc. Am. Math. Soc. 148 (10) (2020) 4241–4255. https://doi.org/10.1090/proc/15033

[34] E. Bayraktar, A. Cohen, Analysis of a finite state many player game using its master equation, SIAM J. Control Optim. 56 (5) (2018) 3538–3568. https://doi.org/10.1137/17M1142235

[35] A. Cecchin, G. Pelino, Convergence, fluctuations and large deviations for finite state mean field games via the master equation, Stoch. Process. Appl. 129 (11) (2019) 4510–4555. https://doi.org/10.1016/j.spa.2018.12.002

[36] C. Bertucci, J.-M. Lasry, P.L. Lions, Master equation for the finite state space planning problem, Arch. Ration. Mech. Anal. 242 (1) (2021) 327–380. https://doi.org/10.1007/s00205-021-01687-8

[37] P.J. Graber, C. Mouzouni, On mean field games models for exhaustible commodities trade, ESAIM 26 (2020). Article 11 (A11). https://doi.org/10.1051/cocv/2019008

[38] P.J. Graber, Linear quadratic mean field type control and mean field games with common noise, with application to production of an exhaustible resource, Appl. Math. Optim. 74 (3) (2016) 459–486. https://doi.org/10.1007/s00245-016-9385-x

[39] P.J. Graber, R. Sircar, Master equation for Cournot mean field games of control with absorption, J. Differ. Equ. 343 (2023) 816–909. https://doi.org/10.1016/j.jde.2022.10.031

[40] E. Bayraktar, A. Cecchin, A. Cohen, F. Delarue, Finite state mean field games with Wright-Fisher common noise, J. Math. Pures Appl. 147 (2021) 98–162. https://doi.org/10.1016/j.matpur.2021.01.003

[41] R. Carmona, F. Delarue, The master field and the master equation, in: Probabilistic Theory of Mean Field Games with Applications II: Mean Field Games with Common Noise and Master Equations, Springer, Cham, 2018, pp. 239–321. https://doi.org/10.1007/978-3-319-56436-4_6

[42] A. Cecchin, P. Dai Pra, M. Fischer, G. Pelino, On the convergence problem in mean field games: a two state model without uniqueness, SIAM J. Control Optim. 57 (4) (2019) 2443–2466. https://doi.org/10.1137/18M1203407

[43] A. Cecchin, F. Delarue, Selection by vanishing common noise for potential finite state mean field games, Commun. Partial Differ. Equ. 47 (1) (2022) 89–168. https://doi.org/10.1080/03605302.2021.1979882

[44] M.F. Djete, Mean field games of controls: on the convergence of nash equilibria, Stoch. Process. Appl. 144 (2022) 416–463. https://doi.org/10.1016/j.spa.2021.12.005

[45] J. Doncel, N. Gast, B. Gaujal, Discrete mean field games: existence of equilibria and convergence, J. Dyn. Games 6 (3) (2019) 221–239. https://doi.org/10.3934/jdg.2019016

[46] W. Gangbo, A.R. Mészáros, Global well-posedness of master equations for deterministic displacement convex potential mean field games, Commun. Partial Differ. Equ. 46 (9) (2021) 1738–1779. https://doi.org/10.1080/03605302.2021.1914655

[47] W. Gangbo, A.R. Mészáros, C. Mou, J. Zhang, Mean field games master equations with non-separable hamiltonians and displacement monotonicity, Arch. Ration. Mech. Anal. 245 (2022) 733–780. https://doi.org/10.1007/s00205-022-01777-1

[48] R.E. Jakobsen, A. Rutkowski, The master equation for mean field game systems with fractional and nonlocal diffusions, J. Differ. Equ. 365 (2023) 361–410. https://doi.org/10.1016/j.jde.2023.12.009

[49] D. Lacker, A general characterization of the mean field limit for stochastic differential games, Probab. Theory Relat. Fields 165 (3-4) (2016) 581–648. https://doi.org/10.1007/s00440-015-0641-9

[50] D. Lacker, On the convergence of closed-loop Nash equilibria to the mean field game limit, Ann. Appl. Probab. 30 (4) (2020) 1693–1761. https://doi.org/10.1214/19-AAP1537

[51] S. Mayorga, Short time solution to the master equation of a first order mean field game, J. Differ. Equ. 268 (10) (2020) 6251–6318. https://doi.org/10.1016/j.jde.2019.11.070

[52] M. Nutz, Y. Zhang, A mean field competition, Math. Oper. Res. 44 (4) (2019) 1245–1263. https://doi.org/10.1287/moor.2018.0956

[53] L. Campi, M. Ghio, G. Livieri, N-player games and mean-field games with smooth dependence on past absorptions, Ann. l'Inst. Henri Poincaré (2021). https://doi.org/10.1214/20-AIHP1138

[54] R. Ferreira, D. Gomes, T. Tada, Existence of weak solutions to first-order stationary mean-field games withDirichlet conditions, Proc. Am. Math. Soc. 147 (11) (2019) 4713–4731. https://doi.org/10.1090/proc/14498

[55] G. Mazanti, F. Santambrogio, Minimal-time mean field games, Math. Models Methods Appl. Sci. 29 (08) (2019) 1413–1464. https://doi.org/10.1142/S0218202519500258

[56] C. Bertucci, Optimal stopping in mean field games, an obstacle problem approach, J. Math. Pures Appl. 120 (2018) 165–194. https://doi.org/10.1016/j.matpur.2018.03.005

[57] C. Bertucci, Monotone solutions for mean field games master equations: finite state space and optimal stopping, J. l'École Polytech. 8 (2021) 1099–1132. https://doi.org/10.5802/jep.167

[58] G. Bouveret, R. Dumitrescu, P. Tankov, Mean-field games of optimal stopping: a relaxed solution approach, SIAM J. Control Optim. 58 (4) (2020) 1795–1821. https://doi.org/10.1137/19M1261365

[59] P. Cardaliaguet, J. Jackson, P.E. Souganidis, Mean field control with absorption, (2025). arXiv preprint arXiv:2509.07877.

[60] M.C. Delfour, J.P. Zolesio, Shape analysis via oriented distance function, J. Funct. Anal. 123 (1) (1994) 129–201. https://doi.org/10.1006/jfan.1994.1087
[61] O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasi-linear Equations of Parabolic Type, 23 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, Providence, RI, 1968.
[62] B. Piccoli, F. Rossi, Generalized Wasserstein distance and its application to transport equations with source, Arch. Ration. Mech. Anal. 211 (1) (2014) 335–358. https://doi.org/10.1007/s00205-013-0669-x
[63] B. Piccoli, F. Rossi, On properties of the generalized Wasserstein distance, Arch. Ration. Mech. Anal. 222 (3) (2016) 1339–1365. https://doi.org/10.1007/s00205-016-1026-7
[64] A. Figalli, N. Gigli, A new transportation distance between non-negative measures, with applications to gradient flows with Dirichlet boundary conditions, J. Math. Pures Appl. 94 (2) (2010) 107–130. https://doi.org/10.1016/j.matpur.2009.11.005
[65] F. Santambrogio, Euclidean, metric, and wasserstein gradient flows: an overview, Bull. Math. Sci. 7 (1) (2017) 87–154. https://doi.org/10.1007/s13373-017-0101-1
[66] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Modern Birkhäuser Classics, Birkhäuser, Basel, Basel, 2012. https://doi.org/10.1007/978-3-0348-0557-5