Theoretical Computer Science 935 (2022) 128–143

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Closed subsets in Bishop topological groups

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ARTICLE INFO

Article history:
Received 2 May 2022
Received in revised form 20 August 2022
Accepted 5 September 2022
Available online 8 September 2022

Keywords:
Constructive topological algebra
Bishop topological groups
Closed sets

ABSTRACT

We introduce the notion of a Bishop topological group i.e., a group X equipped with a Bishop topology of functions F such that the group operations of X are Bishop morphisms with respect to F. A closed subset in the neighborhood structure of X induced by its Bishop topology F is defined in a positive way i.e., not as the complement of an open subset in X. The corresponding closure operator, although it is not topological, in the classical sense, does not involve sequences. As countable choice (CC) is avoided, and in agreement with Richman's critique on the use of CC in constructive mathematics, the fundamental facts on closed subsets in Bishop topological groups shown here have a clear algorithmic content. We work within Bishop's informal system of constructive mathematics BISH, without countable choice, equipped with inductive definitions with rules of countably many premises.

1. Introduction

The constructive non-viability of the notion of topological space is corroborated by the fact that many classical topological phenomena, like the duality between open and closed sets, are compatible only with classical logic. In a straightforward, constructive translation of general topology we cannot accept that the set-theoretic complement of a closed set is open. E.g., $\{0\}$ is a closed subset $\mathbb{R}$, with respect to the topology on $\mathbb{R}$ induced by its standard metric, while its complement cannot be accepted constructively as open, since that would imply the implication $\neg(x = 0) \Rightarrow (x > 0 \vee x < 0)$, which is (constructively) equivalent to the constructively unacceptable principle of Markov (see [6], p. 15). The standard use of negative definitions in classical topology does not permit a smooth translation of classical topology to a constructive framework.

In [3], chapter 3, Bishop defined a *neighborhood space* $\mathcal{N} := (X, I, \nu)$, where X, I are sets, and $(\nu_i)_{i\in I}$ is a family of subsets of X indexed by I (see [35] for an elaborate study of this notion) that satisfies the following covering (NS_1) and neighborhood-condition (NS_2):

(NS_1) $\bigcup_{i\in I} \nu(i) = X$.
(NS_2) $\forall_{x\in X}\forall_{i,j\in I}\big[x \in \nu(i) \cap \nu(j) \Rightarrow \exists_{k\in I}\big(x \in \nu(k)\ \&\ \nu(k) \subseteq \nu(i) \cap \nu(j)\big)\big]$.

If $C \subseteq X$, its *interior* C° is defined by

$$C^\circ := \{x \in X \mid \exists_{i\in I}(x \in \nu(i)\ \&\ \nu(i) \subseteq C)\}.$$

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https://doi.org/10.1016/j.tcs.2022.09.004

A subset O of X is called ν-*open*, if $O \subseteq O^{\circ}$. An ν-closed set C is not defined negatively, as the complement of a ν-open set, but *positively* by the condition $\overline{C} \subseteq C$, where

$$\overline{C} := \{x \in X \mid \forall_{i\in I}(x \in \nu(i) \Rightarrow \nu(i) \between C)\},$$

and if A, B are subsets of X, then $A \between B :\Leftrightarrow \exists_{y\in X}(y \in A \cap B)$. If (Y, J, μ) is a neighborhood space, a function $h : X \to Y$ is *neighborhood-continuous*, if $h^{-1}(\mu(j))$ is ν-open, for every $j \in J$. The concept of neighborhood space was proposed as a *set-theoretic alternative* to the notion of topological space, and it is a formal topology in the sense of Sambin [45], [46].

In [3], chapter 3, Bishop also defined the notion of *function space* $\mathcal{F} := (X, F)$, where X is a set and F is a subset of $\mathbb{F}(X)$, the real-valued functions on X, that satisfies the closure conditions of the set $\mathrm{Bic}(\mathbb{R})$ of Bishop-continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Bishop called F a *topology* (of functions) on X. The set $\mathrm{Bic}(\mathbb{R})$ of Bishop-continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ is the canonical topology of functions on $\mathbb{R}$. Bishop also defined inductively[1] the *least topology* of functions on X that includes a given subset F_0 of $\mathbb{F}(X)$. The concept of function space was proposed as a *function-theoretic alternative* to the notion of topological space.

In [5], p. 77, Bishop and Bridges expressed in a clear way the superiority of the function-theoretic notion of function space to the set-theoretic notion of neighborhood space. As Bridges and Palmgren remark in [8], "little appears to have been done" in the theory of neighborhood spaces. Ishihara has worked in [16] (and with co-authors in [15]) on their connections to the apartness spaces of Bridges and Vîță (see [7]), and in [17] on their connections to Bishop's function spaces, while in [18] Ishihara and Palmgren studied the notion of quotient topology in neighborhood spaces.

Bridges talked on Bishop's function spaces at the first workshop on formal topology in 1997, and revived the subject of function spaces in [9]. Motivated by Bridges's paper, Ishihara showed in [17] the existence of an adjunction between the category of neighborhood spaces and the category of Φ-closed pre-function spaces, where a pre-function space is an extension of the notion of a function space. In [26–34] and in [36,43] we try to develop the theory of function spaces, or *Bishop spaces*, as we call them. In [35,36] we also study the applications of the theory of set-indexed families of Bishop sets in the theory of Bishop spaces. In [13] connections between the theory of Bishop spaces and the theory of C-spaces of Escardó and Xu, developed in [50] and in [12], are studied.

A group X is a *topological group*, if there is a topology of open sets $\mathcal{T}$ on X such that the corresponding operations $+\colon X \times X \to X$ and $-\colon X \to X$ are continuous functions with respect to $\mathcal{T}$. The theory of topological groups is very well-developed, with numerous applications (see [2], [14] and [49]). A locally compact metric group is a group G equipped with a metric that makes its operations continuous. Based on a method of Cartan, and in combination with his integration theory of locally compact metric spaces, Bishop [3,5] defined the Haar measure on a locally compact metric group. The spectral theorem is used to establish the Fourier transform, and the Pontryagin duality theorem is shown. Bishop did not extend his theory of metric spaces to a general theory of some kind of constructive topological spaces. The reasons for that are explained in [43].

In analogy to the definition of a topological group, we call a group X, equipped with a Bishop topology of functions F, a *Bishop topological group*, if the corresponding group operations $+\colon X \times X \to X$ and $-\colon X \to X$ are Bishop morphisms with respect to F. The relation of Bishop topological groups to locally compact metric groups is described in Theorem 4.4. A *Bishop morphism* between Bishop spaces is the notion of arrow in the category of Bishop spaces that was introduced by Bridges in [9] and corresponds to the notion of a continuous function between topological spaces. Most of the concepts of the theory of Bishop spaces are function-theoretic i.e., they are determined by the Bishop topology of functions F on X. Each Bishop topology F generates a *canonical neighborhood structure*, a family of basic open sets in X, described in section 3. As explained above, a closed set C with respect to this neighborhood structure is defined positively, and independently from its set-theoretic complement. Generally we cannot show constructively that the set-theoretic complement $X \setminus C$ of a closed set C is open. What we show in Theorem 3.4 though, is that a positive notion of complement, determined by F, the *uniform* F-*complement* $X \setminus^{u}_{F} C$ of C, is the largest open set included in $X \setminus C$.

In the main core of this paper we prove some fundamental properties of the closed sets in Bishop topological groups. Using functions to describe general properties of sets, and working with the aforementioned positive notion of closed set gives us the opportunity to find constructive proofs with a clear computational content of results, which in many cases in the classical theory of topological groups depend on the use of classical negation. Moreover, our concepts and results avoid the use even of countable choice (CC). Although practitioners of Bishop-style constructive mathematics usually embrace CC, avoiding it, and using non-sequential or non-choice-based arguments instead, forces us to formulate "better" concepts and find "better" proofs. This standpoint was advocated first by Richman (see [44] and [48]).

The study of closed sets in the neighborhood structure induced by the Bishop topology of a Bishop topological group shows the fruitfulness of combining the two constructive proposals of Bishop to the classical topology of open sets. Moreover, the group-structure of a Bishop space X helps us "recover" part of the classical duality between closed and open sets. As Corollary 5.10 indicates, there are many cases of closed sets in a Bishop topological group for which we can show that their set-theoretic complement is open! We structure this paper as follows:

[1] This definition, together with the notion of the least algebra of Borel sets generated by a family of complemented subsets of X, relative to a given set of real-valued functions on X, are the main inductive definitions found in [3], both in chapter 3. The notion of the least algebra of Borel sets is avoided in [4] and [5], and the notion of the least topology is not developed neither in [3] nor in [5].

- In section 2 we include all definitions and facts on Bishop spaces that are necessary to the rest of the paper.
- In section 3 we give all definitions and results on the canonical neighborhood structure of a Bishop topology that are used in later sections. Theorem 3.4 is the result of this section that is most relevant to the study of closed sets in Bishop topological groups.
- In section 4 we introduce Bishop topological groups and we prove some of their basic properties. In Theorem 4.4 we relate (locally) compact metric groups, as these are defined by Bishop and Bridges in [5], with metric groups equipped with the corresponding canonical Bishop topology.
- In section 5, a central section of this work, we prove fundamental properties of closed sets in Bishop topological groups. As we work with functions and positively defined concepts, avoiding the use of choice, our proofs generate clear algorithms.
- In section 6 we briefly describe some important open questions and future tasks stemming from our work.

We work within Bishop's informal system of constructive mathematics BISH, without countable choice, equipped with inductive definitions with rules of countably many premises. A set-theoretic formal framework for this system[2] is Myhill's [23] CST* without countable choice, or CZF, equipped with a weak form of Aczel's regular extension axiom REA (see [1] and [24]). For a recent reconstruction of the theory of Bishop (non-inductive) sets within BISH see [32,35–37,40,42].

2. Fundamentals of Bishop spaces

We include here all definitions and facts on Bishop spaces that are necessary to the rest of the paper. For all proofs on Bishop spaces that are not given here, we refer to [27]. For all results on constructive analysis that are used here without proof, we refer to [5].

We denote by $\mathbb{F}(X,Y)$ the set of all functions from the set X to the set Y. Let also $\mathbb{F}(X)=\mathbb{F}(X,\mathbb{R})$, where $\mathbb{R}$ is the set of reals. If $a,b\in\mathbb{R}$, let $a\vee b:=\max\{a,b\}$ and $a\wedge b:=\min\{a,b\}$. Hence, $|a|=a\vee(-a)$. If $f,g\in\mathbb{F}(X)$, let $f=_{\mathbb{F}(X)} g :\Leftrightarrow \forall_{x\in X}\big(f(x)=_{\mathbb{R}} g(x)\big)$. If $f,g\in\mathbb{F}(X)$, $\varepsilon>0$ and $\Phi\subseteq\mathbb{F}(X)$, let

$$U(g,f,\varepsilon) :\Leftrightarrow \forall_{x\in X}\big(|g(x)-f(x)|\leq\varepsilon\big),$$
$$U(\Phi,f) :\Leftrightarrow \forall_{\varepsilon>0}\exists_{g\in\Phi}\big(U(g,f,\varepsilon)\big).$$

A set X is *inhabited*, if it has an element. We denote by $\overline{a}^X$, or by $\overline{a}$, or even by a, the constant function on X with value $a\in\mathbb{R}$, and by $\mathrm{Const}(X)$ their set. The set of functions of type $\mathbb{R}\to\mathbb{R}$ that are uniformly continuous on every[3] bounded subset of $\mathbb{R}$ is denoted by $\mathrm{Bic}(\mathbb{R})$.

Definition 2.1. A Bishop space is a pair $\mathcal{F}:=(X,F)$, where X is an inhabited set and F is an *extensional* subset of $\mathbb{F}(X)$ i.e., $\forall_{f,g\in\mathbb{F}(X)}\big([f\in F \ \&\ g=_{\mathbb{F}(X)} f]\Rightarrow g\in F\big)$, such that the following conditions hold:

(BS_1) $\mathrm{Const}(X)\subseteq F$.
(BS_2) If $f,g\in F$, then $f+g\in F$.
(BS_3) If $f\in F$ and $\phi\in\mathrm{Bic}(\mathbb{R})$, then $\phi\circ f\in F$.
(BS_4) If $f\in\mathbb{F}(X)$ and $U(F,f)$, then $f\in F$.

We call F a Bishop topology on X. If $\mathcal{G}:=(Y,G)$ is a Bishop space, a Bishop morphism from $\mathcal{F}$ to $\mathcal{G}$ is a function $h:X\to Y$ such that $\forall_{g\in G}\big(g\circ h\in F\big)$. We denote by $\mathrm{Mor}(\mathcal{F},\mathcal{G})$ the set of Bishop morphisms from $\mathcal{F}$ to $\mathcal{G}$. If $h\in\mathrm{Mor}(\mathcal{F},\mathcal{G})$, we say that h is *open*, if $\forall_{f\in F}\exists_{g\in G}\big(f=g\circ h\big)$. If $h\in\mathrm{Mor}(\mathcal{F},\mathcal{G})$ is a bijection and h^{-1} is a Bishop morphism, we call h a Bishop isomorphism.

A Bishop morphism $h\in\mathrm{Mor}(\mathcal{F},\mathcal{G})$ is a "continuous" function from $\mathcal{F}$ to $\mathcal{G}$. If $h\in\mathrm{Mor}(\mathcal{F},\mathcal{G})$ is a bijection, then $h^{-1}\in\mathrm{Mor}(\mathcal{G},\mathcal{F})$ if and only if h is open. Let $\mathcal{R}$ be the *Bishop space of reals* $(\mathbb{R},\mathrm{Bic}(\mathbb{R}))$. It is easy to show that if F is a topology on X, then $F=\mathrm{Mor}(\mathcal{F},\mathcal{R})$ i.e., an element of F is a real-valued "continuous" function on X. By condition (BS_3) a Bishop topology F on X is closed under the operations $|.|, {}^2, -$ of absolute value, square and minus, respectively, and hence it is an algebra and a lattice, using the following equalities:

$$f\cdot g=\frac{(f+g)^2-f^2-g^2}{2},$$
$$\lambda f=\overline{\lambda}f.$$

[2] Extensional Martin-Löf Type Theory or the theory of setoids within intensional Martin-Löf Type Theory are possible type-theoretic systems for this informal system (see [10]), although there choice, in the form of the distributivity of the Pi-type over the Sigma-type, is provable.
[3] This definition can be formulated predicatively using quantification only over $\mathbb{N}$.

$$f \vee g := \max\{f, g\} = \frac{f + g + |f - g|}{2},$$
$$f \wedge g := \min\{f, g\} = -\max\{-f, -g\} = \frac{f + g - |f - g|}{2}.$$

If $\mathbb{F}^*(X)$ denotes the bounded elements of $\mathbb{F}(X)$, then $F^* := F \cap \mathbb{F}^*(X)$ is a Bishop topology on X. If $x =_X y$ is the given equality on X, a Bishop topology F on X *separates the points* of X, or F is *separating* (see [26]), if

$$\forall_{x,y\in X}\big[\forall_{f\in F}\big(f(x) =_{\mathbb{R}} f(y)\big) \Rightarrow x =_X y\big].$$

An apartness relation on X is a positively defined inequality on X. E.g., if $a, b \in \mathbb{R}$, then

$$a \neq_{\mathbb{R}} b :\Leftrightarrow |a - b| > 0 \Leftrightarrow a > b \vee a < b.$$

The *canonical apartness relation* on X induced by F is defined by

$$x \neq_F y :\Leftrightarrow \exists_{f\in F}\big(f(x) \neq_{\mathbb{R}} f(y)\big).$$

If f witnesses the apartness $f(x) \neq_{\mathbb{R}} f(y)$, we write $f\colon x \neq_F y$. In Proposition 5.1.2. of [27] we show that $a \neq_{\mathbb{R}} b \Leftrightarrow a \neq_{\mathrm{Bic}(\mathbb{R})} b$.

Definition 2.2. Turning the definitional clauses $(\mathrm{BS}_1) - (\mathrm{BS}_4)$ into inductive rules, the least topology $\bigvee F_0$ generated by a set $F_0 \subseteq \mathbb{F}(X)$, called a subbase of $\bigvee F_0$, is defined by the following inductive rules:

$$\frac{f_0 \in F_0}{f_0 \in \bigvee F_0}, \quad \frac{f \in \bigvee F_0,\ g \in \mathbb{F}(X),\ g =_{\mathbb{F}(X)} f}{g \in \bigvee F_0}, \quad \frac{a \in \mathbb{R}}{\overline{a} \in \bigvee F_0},$$
$$\frac{f, g \in \bigvee F_0}{f + g \in \bigvee F_0}, \quad \frac{f \in \bigvee F_0 \quad \phi \in \mathrm{Bic}(\mathbb{R})}{\phi \circ f \in \bigvee F_0},$$
$$\frac{g_1 \in \bigvee F_0 \,\&\, U(g_1, f, \frac{1}{2}),\ g_2 \in \bigvee F_0 \,\&\, U(g_2, f, \frac{1}{2^2}),\ g_3 \in \bigvee F_0 \,\&\, U(g_3, f, \frac{1}{2^3}), \ldots}{f \in \bigvee F_0}.$$

The above rules induce the corresponding induction principle $\mathrm{Ind}_{\bigvee F_0}$ on $\bigvee F_0$.

Clearly, $\mathrm{Bic}(\mathbb{R}) = \bigvee\{\mathrm{id}_{\mathbb{R}}\}$. If $h\colon X \to Y$ and $G = \bigvee G_0$, then one can show inductively i.e., with the use of $\mathrm{Ind}_{\bigvee G_0}$, that $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \Leftrightarrow \forall_{g_0\in G_0}\big(g_0 \circ h \in F\big)$. We call this property the $\bigvee$-*lifting of morphisms*.

Definition 2.3. If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are given Bishop spaces, their *product* is the structure $\mathcal{F} \times \mathcal{G} = (X \times Y, F \times G)$, where

$$F \times G := \bigvee\{f \circ \pi_1 \mid f \in F\} \cup \{g \circ \pi_2 \mid g \in G\} =: \bigvee_{f\in F}^{g\in G} f \circ \pi_1, g \circ \pi_2,$$

and π_1, π_2 are the projections of $X \times Y$ to X and Y, respectively. If $A \subseteq X$, the relative Bishop topology $F_{|A}$ on A induced by F is defined by

$$F_{|A} := \bigvee\{f_{|A} \mid f \in F\}.$$

It is straightforward to show that $\mathcal{F} \times \mathcal{G}$ satisfies the universal property for products and that $F \times G$ is the least topology which turns the projections π_1, π_2 into morphisms. If F_0 is a subbase of F and G_0 is a subbase of G, then we show inductively that

$$\bigvee F_0 \times \bigvee G_0 = \bigvee\{f_0 \circ \pi_1 \mid f_0 \in F_0\} \cup \{g_0 \circ \pi_2 \mid g_0 \in G_0\} =: \bigvee_{f_0\in F_0}^{g_0\in G_0} f_0 \circ \pi_1, g_0 \circ \pi_2.$$

Consequently, $\mathrm{Bic}(\mathbb{R}) \times \mathrm{Bic}(\mathbb{R}) = \bigvee \mathrm{id}_{\mathbb{R}} \circ \pi_1, \mathrm{id}_{\mathbb{R}} \circ \pi_2 = \bigvee \pi_1, \pi_2$.

Corollary 2.4. *Let $\mathcal{H} = (Z, H), \mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ be Bishop spaces.*

(i) *If $h_1 : Z \to X$, $h_2 : Z \to Y$, the map $\langle h_1, h_2\rangle : Z \to X \times Y$, defined by $z \mapsto (h_1(z), h_2(z))$, is in $\mathrm{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G})$ if and only if $h_1 \in \mathrm{Mor}(\mathcal{H}, \mathcal{F})$ and $h_2 \in \mathrm{Mor}(\mathcal{H}, \mathcal{G})$.*

(ii) *If* $e_1 : X \to Z$, $e_2 : Y \to Z$, *then the map* $e_1 \times e_2 : X \times Y \to Z \times Z$, *defined by* $(x, y) \mapsto (e_1(x), e_2(y))$, *is in* $\mathrm{Mor}(\mathcal{F} \times \mathcal{G}, \mathcal{H} \times \mathcal{H})$ *if and only if* $e_1 \in \mathrm{Mor}(\mathcal{F}, \mathcal{H})$ *and* $e_2 \in \mathrm{Mor}(\mathcal{G}, \mathcal{H})$.

Proposition 2.5. *Suppose that* $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$, $\mathcal{H} = (Z, H)$ *are Bishop spaces,* $x \in X$, $y \in Y$, $\phi : X \times Y \to \mathbb{R} \in F \times G$ *and* $\Phi : X \times Y \to Z \in \mathrm{Mor}(\mathcal{F} \times \mathcal{G}, \mathcal{H})$.

(i) $i_x : Y \to X \times Y$, $y \mapsto (x, y)$, *and* $i_y : X \to X \times Y$, $x \mapsto (x, y)$, *are open morphisms.*
(ii) $\phi_x : Y \to \mathbb{R}$, $y \mapsto \phi(x, y)$, *and* $\phi_y : X \to \mathbb{R}$, $x \mapsto \phi(x, y)$, *are in* G *and* F, *respectively.*
(iii) $\Phi_x : Y \to Z$, $y \mapsto \Phi(x, y)$, *and* $\Phi_y : X \to Z$, $x \mapsto \Phi(x, y)$, *are in* $\mathrm{Mor}(\mathcal{G}, \mathcal{H})$ *and* $\mathrm{Mor}(\mathcal{F}, \mathcal{H})$, *respectively.*

Proof. (i) We show it only for i_y. By the $\mathcal{F}$-lifting of morphisms we have that $i_y \in \mathrm{Mor}(\mathcal{F}, \mathcal{F} \times \mathcal{G}) \Leftrightarrow \forall_{f \in F}((f \circ \pi_1) \circ i_y \in F)$ & $\forall_{g \in G}((g \circ \pi_2) \circ i_y \in F)$. If $f \in F$, then $(f \circ \pi_1) \circ i_y = f$, which shows also that i_y is open, while if $g \in G$, then $(g \circ \pi_2) \circ i_y = \overline{g(y)} \in F$.
(ii) We show it only for ϕ_y. We have that $\phi_y = \phi \circ i_y$, since $(\phi \circ i_y)(x) = \phi(x, y) = \phi_y(x)$, for each $x \in X$. Since $i_y \in \mathrm{Mor}(\mathcal{F}, \mathcal{F} \times \mathcal{G})$ and $\phi \in F \times G$, we get that $\phi \circ i_y = \phi_y \in F$.
(iii) The proof is similar to the proof of (ii). Actually, (ii) is a special case of (iii). □

3. The neighborhood structure of a Bishop topology

Here we give all definitions and basic results on the canonical neighborhood structure of a Bishop topology that are used subsequently. *In this section* F *is a Bishop topology on* X *and* G *is a Bishop topology on* Y. The *neighborhood structure* $N(F)$ *on* X *induced by* F is the family

$$N(F) := (U(f))_{f \in F}$$

of subsets of X, where, for every $f \in F$,

$$U(f) := \{x \in X \mid f(x) > 0\}.$$

The covering condition (NS_1) follows trivially from the equality $U(\overline{a}^X) = X$, where $a > 0$, and the neighborhood-condition (NS_2) follows from the equality $U(f) \cap U(g) = U(f \wedge g)$, for every $f, g \in F$. Consequently, if $C \subseteq X$, from the general definition of the interior and the closure of a subset of a neighborhood space we get

$$C^\circ = \big\{x \in X \mid \exists_{f \in F}\big(f(x) > 0 \,\&\, U(f) \subseteq C\big)\big\},$$
$$\overline{C} = \big\{x \in X \mid \forall_{f \in F}\big(f(x) > 0 \Rightarrow \exists_{c \in C}\big(f(c) > 0\big)\big)\big\}.$$

Impredicatively speaking, C° is the largest open set included in C and $\overline{C}$ is the smallest closed set[4] including C. The closure operator $A \mapsto \overline{A}$ is not topological, in the classical sense, as we cannot show constructively that the union of two closed sets is closed, in general (see also [5], p. 79). If $A, B \subseteq X$ and F is a Bishop topology on X, then it is straightforward to show that (i) $A \subseteq \overline{A}$, (ii) $\overline{\overline{A}} \subseteq \overline{A}$, (iii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$, and (iv) $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. The inverse inclusion $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ cannot be shown constructively. Next we define the various notions of complements of subsets that we are going to use.

Definition 3.1. If $C \subseteq X$, besides the classical, negatively defined complement

$$X \setminus C := \{x \in X \mid \neg(x \in C)\},$$

we define positively the F-complement $X \setminus_F C$ and the uniform F-complement $X \setminus_F^u C$ of C by

$$X \setminus_F C := \big\{x \in X \mid \forall_{c \in C}(x \neq_F c)\big\},$$
$$X \setminus_F^u C := \big\{x \in X \mid \exists_{f \in F}\big(f : x \notin_F C\big)\big\},$$
$$f : x \notin_F C :\Leftrightarrow f(x) > 0 \,\&\, \forall_{c \in C}\big(f(c) = 0\big).$$

Clearly, $X \setminus_F^u C \subseteq X \setminus_F C \subseteq X \setminus C$.

Proposition 3.2. *Let* $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ *be Bishop spaces,* $f \in F$ *and* $h : X \to Y$.

[4] The notions of an open and closed set and of the interior and closure of a set mentioned here are always with respect to a given neighborhood structure on X induced by an extensional subset F of $\mathbb{F}(X)$.

(i) *$\mathcal{O}$ is open in $N(\mathrm{Bic}(\mathbb{R}))$ if and only if it is open in the standard neighborhood structure of $\mathbb{R}$.*
(ii) *If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then h is neighborhood-continuous.*
(iii) *If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, the inverse image of a closed set in Y under h is closed in X.*
(iv) *If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ and $A \subseteq X$, then $h(\overline{A}) \subseteq \overline{h(A)}$.*
(v) *The classical complement $X \setminus U(f)$ of $U(f)$ in X is closed in X, for every $f \in F$.*
(vi) *The zero set $\zeta(f) := \{x \in X \mid f(x) =_{\mathbb{R}} 0\}$ of f is closed.*

Proof. For (i)-(v) see [27], Proposition 4.4, and for (vi) see [27], Proposition 5.3.2. □

Proposition 3.3.

(i) *If O is open, its F-complement $X \setminus_F O$ is closed.*
(ii) *The cozero set $\mathrm{co}\zeta(f) := X \setminus_F \zeta(f) := \{x \in X \mid f(x) \neq_{\mathbb{R}} 0\}$ of $f \in F$ is open.*

Proof. (i) Let $x \in \overline{X \setminus_F O}$ i.e., $\forall_{f \in F}\big(f(x) > 0 \Rightarrow \exists_{w \in X \setminus_F O}\big(f(w) > 0\big)$. We show that $x \in X \setminus_F O$. Let $z \in O$. As $O \subseteq O^{\circ}$, there is $g \in F$ such that $g(z) > 0$ and $U(g) \subseteq O$. Suppose that $g(x) > 0$. Then $g(w) > 0$, for some $w \in X \setminus_F O$. Hence $w \in O$ too, which is a contradiction. By the constructively valid implication $\neg(a > 0) \Rightarrow a \leq 0$, for every $a \in \mathbb{R}$ (see [5], Lemma 2.18), we get $g(x) \leq 0$, hence $g(x) \leq 0 < g(z)$. Consequently $g : x \neq_F z$.
(ii) Let $x \in X$ with $f(x) \neq_{\mathbb{R}} 0$. Without loss of generality let $f(x) > 0$ (if $f(x) < 0$, we work with $|f| \in F$). If $y \in X$ such that $f(y) > 0$, then $y \in \mathrm{co}\zeta(f)$, hence $x \in U(f)$ and $U(f) \subseteq \mathrm{co}\zeta(f)$. □

One needs Markov's principle to show that the classical complement $X \setminus \zeta(f)$ is open. Notice that constructively, one cannot show, in general, that $X \setminus_F C$ is open, if C is closed. This can be shown though, for its uniform F-complement. Next we show that the uniform F-complement of an arbitrary subset is the largest open set included in its classical complement.

Theorem 3.4. *If $C \subseteq X$, then $X \setminus_F^u C = (X \setminus C)^{\circ}$.*

Proof. We have already seen that $X \setminus_F^u C \subseteq X \setminus C$. First we show that $X \setminus_F^u C$ is open. If $x \in X \setminus_F^u C$, let $f \in F$ such that $f : x \notin_F C$. Clearly, $x \in U(f)$, and if $y \in X$ such that $y \in U(f)$, then $f : y \notin_F C$ i.e., $y \in X \setminus_F^u C$. As $X \setminus_F^u C$ is an open subset of $X \setminus C$, it is trivially a subset of its interior. Next we show the inverse inclusion i.e., if $x \in (X \setminus C)^{\circ}$, then $x \in X \setminus_F^u C$. By the definition of $(X \setminus C)^{\circ}$ there is $g \in F$ such that $g(x) > 0$ and $U(g) \subseteq (X \setminus C)^{\circ} \subseteq X \setminus C$. We claim that $g(c) \leq 0$, for every $c \in C$. Suppose that $g(c) > 0$, for some $c \in C$, i.e., $c \in U(g)$, hence $c \in X \setminus C$, which is a contradiction. Hence, by Lemma 2.18 in [5] again, we get $g(c) \leq 0$. Since $g \vee \overline{0}^X \in F$, we conclude that $g \vee \overline{0}^X : x \notin_F C$, hence $x \in X \setminus_F^u C$. □

Classically, one can show that the topology induced by F is always completely regular i.e., if C is closed in X and $x \notin C$, then there is $f \in F$ such that $f : x \notin_F C$ (see [27], Proposition 3.7.6). Constructively, by Theorem 3.4 we only show that $X \setminus_F^u C$ is the interior of $X \setminus C$. Although we cannot show in general that $X \setminus C$ is open, and hence $X \setminus C = X \setminus_F^u C$, we can replace this computationally dubious result by the computationally meaningful fact that $X \setminus_F^u C$ is the largest open set included in $X \setminus C$. The next result is used in the proofs of Theorem 5.6(ii) and Theorem 5.7(ii).

Proposition 3.5. *The following are equivalent:*

(i) *F separates the points of X.*
(ii) *The inequality $\neq_F$ generated by F is tight i.e., $\neg(x \neq_F y) \Rightarrow x =_X y$, for every $x, y \in X$.*
(iii) *The singleton $\{x\}$ is closed, for every $x \in X$.*

Proof. (i)⇒(ii) Let $\neg(x \neq_F y) :\Leftrightarrow \neg[\exists_{f \in F}\big(f(x) \neq_{\mathbb{R}} f(y)\big)]$, for some $x, y \in X$. We show that $\forall_{f \in F}\big(f(x) =_{\mathbb{R}} f(y)\big)$. Let $f \in F$ such that $f(x) \neq_{\mathbb{R}} f(y)$. By our hypothesis on x, y this is impossible, hence by the tightness of $\neq_{\mathbb{R}}$ we conclude that $f(x) =_{\mathbb{R}} f(y)$. As F separates the points of X, we conclude that $x =_X y$.
(ii)⇒(iii) Let $x, y \in X$ such that $\forall_{f \in F}\big(f(y) > 0 \Rightarrow f(x) > 0\big)$. We show that $y =_X x$, by showing that $\neg\big(y \neq_F x\big)$. Suppose that $y \neq_F x$ and, without loss of generality, let $g \in F$ such that $g(y) = 1$ and $g(x) = 0$. By the hypothesis on x we have that $g(y) > 0 \Rightarrow g(x) > 0$, and we get the required contradiction.
(iii)⇒(i) Let $x, y \in X$ such that $\forall_{f \in F}\big(f(x) =_{\mathbb{R}} f(y)\big)$. We show that $y \in \overline{\{x\}}$, hence $y = x$. Let $f \in F$ such that $f(y) > 0$. Since $f(x) = f(y)$, we get $f(x) > 0$. □

Proposition 3.6. *If C is closed in X and D is closed in Y, then $C \times D$ is closed in $X \times Y$.*

Proof. Let $(x, y) \in \overline{C \times D}$ i.e., if $h(x, y) > 0$, there is $(u, w) \in C \times D$ such that $h(u, w) > 0$, for every $h \in F \times G$. We show that $x \in \overline{C}$ and (similarly) $y \in \overline{D}$, hence $(x, y) \in C \times D$. Let $f \in F$ such that $f(x) > 0$. Since $(f \circ \pi_1)(x, y) > 0$ and $f \circ \pi_1 \in F \times G$, there is $(u, w) \in C \times D$ such that $(f \circ \pi_1)(u, w) := f(u) > 0$, hence there is $u \in C$ with $f(u) > 0$. □

4. Bishop topological groups

Definition 4.1. We call the structure $\boldsymbol{\mathcal{F}} := (X, +, 0, -; F)$ a Bishop topological group, if $\mathcal{X} := (X, +, 0, -)$ is a group and $\mathcal{F} := (X, F)$ is a Bishop space such that $+ : X \times X \to X \in \mathrm{Mor}(\mathcal{F} \times \mathcal{F}, \mathcal{F})$ and $- : X \to X \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$. If necessary, we may use the notations $+^X, 0^X$, and $-^X$ for the operations of the group $\mathcal{X}$. If $f \in F$, let $f_+ := f \circ +$ and $f_- := f \circ -$. We use bold letters to denote Bishop topological groups.

By the definition of a Bishop morphism we get

$$+ \in \mathrm{Mor}(\mathcal{F} \times \mathcal{F}, \mathcal{F}) :\Leftrightarrow \forall_{f \in F}\big(f \circ + \in F \times F\big) :\Leftrightarrow \forall_{f \in F}\big(f_+ \in F \times F\big),$$

$$- \in \mathrm{Mor}(\mathcal{F}, \mathcal{F}) :\Leftrightarrow \forall_{f \in F}\big(f \circ - \in F\big) :\Leftrightarrow \forall_{f \in F}\big(f_- \in F\big).$$

Example 4.2 *(The additive group of reals).* The structure $\boldsymbol{\mathcal{R}} := (\mathbb{R}, +, 0, -; \mathrm{Bic}(\mathbb{R}))$ is a Bishop topological group. By the $\bigvee$-lifting of morphisms $+ \in \mathrm{Mor}(\mathcal{R} \times \mathcal{R}, \mathcal{R}) \Leftrightarrow \mathrm{id}_{\mathbb{R}} \circ + \in \mathrm{Bic}(\mathbb{R}) \times \mathrm{Bic}(\mathbb{R})$. If $x, y \in \mathbb{R}$, then $\big(\mathrm{id}_{\mathbb{R}} \circ +\big)(x, y) := x + y := (\pi_1 + \pi_2)(x, y)$ i.e., $\mathrm{id}_{\mathbb{R}} \circ + = \pi_1 + \pi_2 \in \mathrm{Bic}(\mathbb{R}) \times \mathrm{Bic}(\mathbb{R}) = \bigvee \pi_1, \pi_2$. Similarly, $- \in \mathrm{Mor}(\mathcal{R}, \mathcal{R}) \Leftrightarrow \mathrm{id}_{\mathbb{R}} \circ - \in \mathrm{Bic}(\mathbb{R})$. If $x \in \mathbb{R}$, then $\big(\mathrm{id}_{\mathbb{R}} \circ -\big)(x) := -x := -\mathrm{id}_{\mathbb{R}}(x)$ i.e., $\mathrm{id}_{\mathbb{R}} \circ - = -\mathrm{id}_{\mathbb{R}} \in \mathrm{Bic}(\mathbb{R})$.

Example 4.3 *(The trivial Bishop topological group).* If $\mathcal{X} := (X, +, 0, -)$ is a group, then $\mathrm{Const}(X)$ is the *trivial* Bishop topology on X that also makes the group $\mathcal{X}$ topological; if $a \in \mathbb{R}$, then $\overline{a}^X \circ + = \overline{a}^{X \times X} \in \mathrm{Const}(X \times X) = \mathrm{Const}(X) \times \mathrm{Const}(X)$, and $\overline{a}^X \circ - = \overline{a}^X \in \mathrm{Const}(X)$.

Next we describe the relation between the metric groups defined in [5] and Bishop topological groups. Within BISH, a metric space (X, d) is *compact*, if it is complete and totally bounded. A *compact metric group* is a compact metric space and a group such that its group operations are continuous. This means that $+$ is uniformly continuous on the compact metric space (with the product metric) $X \times X$, in symbols $+ \in C_u(X \times X, X)$ and $-$ is uniformly continuous on X, in symbols $- \in C_u(X, X)$. The canonical Bishop topology on a compact metric space X is the set $C_u(X)$ of uniformly continuous, real-valued functions on X.

Within BISH, an inhabited metric space (X, d) is *locally compact*, if every bounded subset of X is included in a compact subset of X.[5] A *locally compact metric group* is a locally compact metric space and a group such that its group operations are continuous. This means that $+$ is uniformly continuous on every bounded subset of (the locally compact metric space with the product metric) $X \times X$, in symbols $+ \in \mathrm{Bic}(X \times X, X)$, and $-$ is uniformly continuous on every bounded subset of X, in symbols $- \in \mathrm{Bic}(X, X)$. The canonical Bishop topology on a locally compact metric space X is the set $\mathrm{Bic}(X)$ of Bishop-continuous,[6] real-valued functions on X, where $f : X \to \mathbb{R} \in \mathrm{Bic}(X)$ if and only if f is uniformly continuous on every bounded subset of X (this notion is shown to be predicative in [27], p. 24).

Theorem 4.4. *Let X be a metric space and a group.*

(i) *If X is a compact metric group, then it is a Bishop topological group (in the canonical sense).*
(ii) *If X is compact and a Bishop topological group (in the canonical sense), then it is a compact metric group.*
(iii) *If X is locally compact and a Bishop topological group (in the canonical sense), then it is a locally compact metric group.*

Proof. (i) As $-$ is uniformly continuous, it is trivially a Bishop morphism from $(X, C_u(X))$ to itself. Actually, by Corollary 3.8.9 in [27] the Bishop morphisms from $(X, C_u(X))$ to itself is the set $C_u(X, X)$ of uniformly continuous functions from X to itself. If $+$ is uniformly continuous, then for every $f \in C_u(X)$ we have that $f_+ \in C_u(X \times X)$, and by Theorem 4.2.8 in [27] we have that the Bishop topology $C_u(X \times X)$ is the product of the Bishop topologies $C_u(X) \times C_u(X)$, hence $f_+ \in C_u(X) \times C_u(X)$ and $+ \in \mathrm{Mor}\big((X \times X, C_u(X \times X)), (X, C_u(X))\big)$. Consequently, a compact metric group is a Bishop topological group.
(ii) If $-$ is a Bishop morphism from $(X, C_u(X))$ to itself, then by Bridges's backward uniform continuity theorem (see [9] and [27]-Corollary 3.8.9) we conclude that $-$ is uniformly continuous. If $+$ is a Bishop morphism from $\big(X \times X, C_u(X) \times C_u(X)\big)$ to $(X, C_u(X))$, and since $X \times X$ is compact and $C_u(X) \times C_u(X) = C_u(X \times X)$, then by Corollary 3.8.9 in [27] again we conclude that $+$ is uniformly continuous.
(iii) If $-$ is a Bishop morphism from $(X, \mathrm{Bic}(X))$ to itself, then by Bridges's Proposition 16 in [9] the map $-$ is in $\mathrm{Bic}(X, X)$. According to that Proposition, $-$ is what is called there B-continuous i.e., uniformly continuous near each compact image,

[5] The impredicativity in Bishop's notion of a locally compact metric space is avoided by Mandelkern in [21], and by us in [41], where a modulus of local compactness is added to the definition of a locally compact metric space.
[6] The notion of a compact metric group is a special case of a locally compact metric group. Clearly, if X is compact, then $\mathrm{Bic}(X) = C_u(X)$.

but as X is locally compact, a function on X is B-continuous if and only if it is uniformly continuous on every bounded subset of X (see [9] p. 107). Suppose next that $+$ is a Bishop morphism from $\big(X \times X, \mathrm{Bic}(X) \times \mathrm{Bic}(X)\big)$ to $(X, \mathrm{Bic}(X))$. Clearly, $X \times X$ is locally compact and by a straightforward inductive proof we get that the product Bishop topology $\mathrm{Bic}(X) \times \mathrm{Bic}(X)$ is included in $\mathrm{Bic}(X \times X)$. Hence, $+$ is a Bishop morphism from $\big(X \times X, \mathrm{Bic}(X \times X)\big)$ to $(X, \mathrm{Bic}(X))$. By Bridges's Proposition 16 in [9] again we conclude that $+$ is B-continuous, or, equivalently, $+ \in \mathrm{Bic}(X \times X, X)$. □

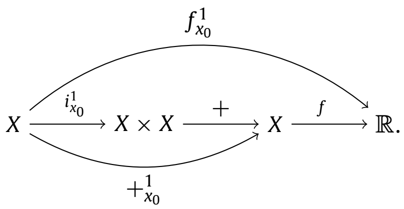
Suppose that X is a locally compact metric group i.e., $- \in \mathrm{Bic}(X, X)$ and $+ \in \mathrm{Bic}(X \times X, X)$. The fact that $-$ is a morphism from $(X, \mathrm{Bic}(X))$ to itself follows immediately from the fact that the composition of uniformly continuous functions on bounded subsets is uniformly continuous on bounded subsets. We still do not have a proof of $+ \in \mathrm{Bic}(X) \times \mathrm{Bic}(X)$.[7]

Unless otherwise stated, *from now on, X, Y are Bishop topological groups with F and G Bishop topologies on X and Y, respectively*. Next we show that the Bishop topology of a Bishop topological group contains the left (right) translates of its elements.

Proposition 4.5.

(i) *The function $- : X \to X$ is a Bishop isomorphism.*
(ii) *For every $x_0 \in X$ the functions $+^1_{x_0}, +^2_{x_0} : X \to X$, defined by $+^1_{x_0}(x) := x_0 + x$ and $+^2_{x_0}(x) := x + x_0$, for every $x \in X$, are Bishop morphisms.*
(iii) *If $f \in F$, the functions $f^1_{x_0}, f^2_{x_0} : X \to \mathbb{R}$, defined by $f^1_{x_0}(x) := f(x_0 + x)$ and $f^2_{x_0}(x) := f(x + x_0)$, for every $x \in X$, are in F.*
(iv) *For every $x_0 \in X$ the functions $+^1_{x_0}, +^2_{x_0} : X \to X$ are Bishop isomorphisms.*

Proof. (i) By definition $- \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$, and it is a bijection. It is also open i.e., $\forall_{f \in F} \exists_{g \in F}\big(f = g_-\big)$. If $f \in F$, we have that $f = (f_-)_-$ and $f_- \in F$.
(ii) and (iii) If $i^1_{x_0} : X \to X \times X$ is defined by $i^1_{x_0}(x) := (x_0, x)$, for every $x \in X$, then $i^1_{x_0} \in \mathrm{Mor}(\mathcal{F}, \mathcal{F} \times \mathcal{F})$ and $+^1_{x_0} := + \circ i^1_{x_0} \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$ as a composition of Bishop morphisms



$f^1_{x_0} = f \circ +^1_{x_0} \in F$, as a composition of Bishop morphisms. For $+^2_{x_0}$ we work similarly.
(iv) Clearly, $+^1_{x_0}$ is a bijection. It is also open, since for every $f \in F$ we have that $f = f^1_{-x_0} \circ +^1_{x_0}$, and by (iv) $f^1_{-x_0} \in F$. For $f^2_{x_0}$ we work similarly. □

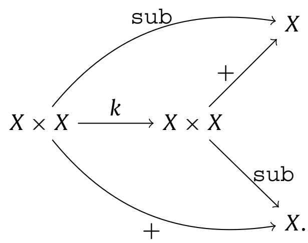
Proposition 4.6. *The function $k : X \times X \to X \times X$, defined by $k(x, y) := (x, -y)$, for every $(x, y) \in X \times X$, is a Bishop isomorphism.*

Proof. Clearly, k is a bijection. By the $\bigvee$-lifting of morphisms we have that $k \in \mathrm{Mor}(\mathcal{F} \times \mathcal{F}, \mathcal{F} \times \mathcal{F})$ if and only if $(f \circ \pi_1) \circ k \in F \times F$ and $(f \circ \pi_2) \circ k \in F \times F$, for every $f \in F$. Let $f \in F$ and $(x, y) \in X \times X$. Since $[(f \circ \pi_1) \circ k](x, y) := (f \circ \pi_1)(x, -y) := f(x) =: (f \circ \pi_1)(x, y)$, we get $(f \circ \pi_1) \circ k = f \circ \pi_1 \in F \times F$. Moreover, $[(f \circ \pi_2) \circ k](x, y) := (f \circ \pi_2)(x, -y) := f(-y) := f_-(y) := (f_- \circ \pi_2)(x, y)$, i.e., $(f \circ \pi_2) \circ k = (f_- \circ \pi_2) \in F \times F$, since $f_- \in F$. Since $(k \circ k)(x, y) := (x, -(-y)) = (x, y)$, k is its own inverse, hence k is a Bishop isomorphism. □

Proposition 4.7. *Let $\mathcal{X} := (X, +, 0, -)$ be a group, F a Bishop topology on X, and $\mathtt{sub} : X \times X \to X$ be defined by $\mathtt{sub}(x, y) := x - y$, for every $(x, y) \in X \times X$. Then $\boldsymbol{\mathcal{F}} := (X, +, 0, -; F)$ is a Bishop topological group if and only if $\mathtt{sub} \in \mathrm{Mor}(\mathcal{F} \times \mathcal{F}, \mathcal{F})$.*

Proof. If $\boldsymbol{\mathcal{F}}$ is a Bishop topological group, then $\mathtt{sub} = + \circ k \in \mathrm{Mor}(\mathcal{F} \times \mathcal{F}, \mathcal{F})$

[7] We also do not know yet if $\mathrm{Bic}(X \times X)$ is included in the product Bishop topology $\mathrm{Bic}(X) \times \mathrm{Bic}(X)$. Hence, we still do not know if a locally compact metric group is a Bishop topological group. See also the discussion in section 6.



For the converse, notice that $+ = \mathtt{sub} \circ k \in \mathrm{Mor}(\mathcal{F} \times \mathcal{F}, \mathcal{F})$. By Proposition 2.5(iii) we get $- = \mathtt{sub}_0 \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$, where $\mathtt{sub}_0(y) := \mathtt{sub}(0, y) := -y$, for every $y \in X$. $\square$

Proposition 4.8. *Let $\boldsymbol{\mathcal{F}} := (X, +, 0, -; F)$ be a Bishop topological group and $\mathcal{G} := (Y, G)$ a Bishop space. Let the functions $+^{\rightarrow} : \mathrm{Mor}(\mathcal{G}, \mathcal{F}) \times \mathrm{Mor}(\mathcal{G}, \mathcal{F}) \to \mathrm{Mor}(\mathcal{G}, \mathcal{F})$, $-^{\rightarrow} : \mathrm{Mor}(\mathcal{G}, \mathcal{F}) \to \mathrm{Mor}(\mathcal{G}, \mathcal{F})$, and $0^{\rightarrow} : Y \to X$ be defined by $(h_1 +^{\rightarrow} h_2)(y) := h_1(y) + h_2(y)$, $(-^{\rightarrow} h)(y) := -h(y)$, and $0^{\rightarrow}(y) := 0$, for every $y \in Y$, and $h_1, h_2, h \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$. Then $\big(\mathrm{Mor}(\mathcal{G}, \mathcal{F}), +^{\rightarrow}, 0^{\rightarrow}, -^{\rightarrow}\big)$ is a group.*

One can show that the group $\big(\mathrm{Mor}(\mathcal{G}, \mathcal{F}), +^{\rightarrow}, 0^{\rightarrow}, -^{\rightarrow}\big)$, equipped with the pointwise exponential Bishop topology (see [27], section 4.3), is a Bishop topological group. Notice that if Y has also a group structure compatible with G, and if h_1, h_2, h are group homomorphisms, then $h_1 +^{\rightarrow} h_2$, $-^{\rightarrow} h$ and $0^{\rightarrow}$ are also group homomorphisms.

Definition 4.9. Let $\boldsymbol{\mathcal{F}} := (X, +^X, 0^X, -^X; F)$ and $\boldsymbol{\mathcal{G}} := (Y, +^Y, 0^Y, -^Y; G)$ be Bishop topological groups. If $h \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$ such that h is a $(\mathcal{X}, \mathcal{Y})$-group homomorphism, then we call h a *Bishop group homomorphism*, or simpler, a *Bishop homomorphism*. We denote by $\mathbf{Mor}(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{G}})$ the set of all Bishop homomorphisms from $\boldsymbol{\mathcal{F}}$ to $\boldsymbol{\mathcal{G}}$. Let **BTopGrp** be the category of Bishop topological groups with Bishop group homomorphisms.

Proposition 4.10. *Let $a \in \mathbb{R}$ and let $h_a : \mathbb{R} \to \mathbb{R}$ be defined by $h_a(x) = ax$, for every $x \in \mathbb{R}$. Then $h_a \in \mathbf{Mor}(\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{R}})$. Conversely, if $h \in \mathbf{Mor}(\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{R}})$, there is $a \in \mathbb{R}$ such that $h = h_a$.*

Proof. Clearly, h_a is a group homomorphism, and by the $\bigvee$-lifting of morphisms we have that $h_a \in \mathrm{Mor}(\mathbb{R}, \mathbb{R}) \Leftrightarrow \mathrm{id}_{\mathbb{R}} \circ h_a := h_a \in \mathrm{Bic}(\mathbb{R})$, which holds, since $h_a := a \cdot \mathrm{id}_{\mathbb{R}} \in \mathrm{Bic}(\mathbb{R})$. If $h \in \mathbf{Mor}(\boldsymbol{\mathcal{R}}, \boldsymbol{\mathcal{R}})$, let its restriction $h_{|\mathbb{Q}}$, where $h_{|\mathbb{Q}} : \mathbb{Q} \to \mathbb{Q}$ is a group homomorphism defined by $\big(h_{|\mathbb{Q}}\big)(q) = h(1)q$, for every $q \in \mathbb{Q}$. Since $\mathbb{Q}$ is metrically dense in $\mathbb{R}$, by Proposition 4.7.15 in [27] we have that

$$\mathrm{Bic}(\mathbb{Q}) = \mathrm{Bic}(\mathbb{R})_{|\mathbb{Q}} = \{\phi_{|\mathbb{Q}} \mid \phi \in \mathrm{Bic}(\mathbb{R})\}.$$

Hence, $h_{|\mathbb{Q}} \in \mathrm{Bic}(\mathbb{Q})$. By Lemma 4.7.13. in [27] there is a unique (up to equality) extension of $h_{|\mathbb{Q}}$ in $\mathrm{Bic}(\mathbb{R})$. Consequently, $h = h_{h(1)}$. $\square$

Proposition 4.11. *Let $\boldsymbol{\mathcal{F}} := (X, +^X, 0^X, -^X; F)$ and $\boldsymbol{\mathcal{G}} := (Y, +^Y, 0^Y, -^Y; G)$ be Bishop topological groups. Let the functions $+^{X \times Y} : (X \times Y) \times (X \times Y) \to X \times Y$, $-^{X \times Y} : X \times Y \to X \times Y$, and $0^{X \times Y} \in X \times Y$ be defined by $(x, y) +^{X \times Y} (x', y') := \big(x +^X x', y +^Y y'\big)$, $-^{X \times Y}(x, y) := \big(-^X x, -^Y y\big)$, and $0^{X \times Y} := \big(0^X, 0^Y\big)$. Then $\boldsymbol{\mathcal{F}} \times \boldsymbol{\mathcal{G}} := (X \times Y, +^{X \times Y}, 0^{X \times Y}, -^{X \times Y}; F \times G)$ is a Bishop topological group.*

Proof. The proof that $X \times Y$ is a group is omitted as trivial. By the $\bigvee$-lifting of morphisms $+^{X \times Y} \in \mathrm{Mor}\big([\mathcal{F} \times \mathcal{G}] \times [\mathcal{F} \times \mathcal{G}], \mathcal{F} \times \mathcal{G}\big)$ if and only if

$$\forall_{f \in F} \forall_{g \in G} \Big(\big(f \circ \pi^X\big) \circ +^{X \times Y} \in [F \times G] \times [F \times G] \,\&\, \big(g \circ \pi^Y\big) \circ +^{X \times Y} \in [F \times G] \times [F \times G] \Big),$$

where by the $\bigvee$-lifting of the product Bishop topology

$$[F \times G] \times [F \times G] = \bigvee_{f \in F}^{g \in G} \big(f \circ \pi^X\big) \circ \pi_1^{X \times Y}, \big(g \circ \pi^Y\big) \circ \pi_1^{X \times Y}, \big(f \circ \pi^X\big) \circ \pi_2^{X \times Y}, \big(g \circ \pi^Y\big) \circ \pi_2^{X \times Y}.$$

If $f \in F$, $x, x' \in X$, and $y, y' \in Y$, and if $z := \big((x, y), (x', y')\big)$, then

$$\begin{aligned} \big[\big(f \circ \pi^X\big) \circ +^{X \times Y}\big](z) &:= \big(f \circ \pi^X\big)\big(x +^X x', y +^Y y'\big) \\ &:= f(x +^X x') \\ &:= (f \circ +^X)(x, x') \end{aligned}$$

$$
\begin{aligned}
&:= (f \circ +^X)\big(\pi^X(x,y), \pi^X(x',y')\big)\\
&:= (f \circ +^X)\Big(\big[\pi^X \circ \pi_1^{X\times Y}\big](z), \big[\pi^X \circ \pi_2^{X\times Y}\big](z)\Big)\\
&:= \big[(f \circ +^X) \circ h\big)\big](z),
\end{aligned}
$$

where by Corollary 2.4 the function $h : [(X \times Y) \times (X \times Y)] \to X \times X$, where $h := \big[\pi^X \circ \pi_1^{X\times Y}\big] \times \big[\pi^X \circ \pi_2^{X\times Y}\big]$ is a Bishop morphism. Since $\big(f \circ \pi^X\big) \circ +^{X\times Y} = (f \circ +^X) \circ h$

$$
(X \times Y) \times (X \times Y) \xrightarrow{\ h\ } X \times X \xrightarrow{\ +^X\ } X \xrightarrow{\ f\ } \mathbb{R}, \qquad (f\circ\pi^X)\circ +^{X\times Y} : (X \times Y) \times (X \times Y) \to \mathbb{R}
$$

we get $\big(f \circ \pi^X\big) \circ +^{X\times Y} \in \mathrm{Mor}\big([\mathcal{F} \times \mathcal{G}] \times [\mathcal{F} \times \mathcal{G}], \mathcal{R}\big)$ as a composition of Bishop morphisms, hence $\big(f \circ \pi^X\big) \circ +^{X\times Y} \in [F \times G] \times [F \times G]$. Working similarly, we get $\big(g \circ \pi^Y\big) \circ +^{X\times Y} \in [F \times G] \times [F \times G]$. By the $\bigvee$-lifting of morphisms we also have that

$$
-^{X\times Y} \in \mathrm{Mor}\big(\mathcal{F} \times \mathcal{G}, \mathcal{F} \times \mathcal{G}\big) \Leftrightarrow \forall_{f\in F}\forall_{g\in G}\Big(\big(f \circ \pi^X\big) \circ -^{X\times Y} \in F \times G \;\&\; \big(g \circ \pi^Y\big) \circ -^{X\times Y} \in F \times G\Big).
$$

If $f \in F$, $x \in X$, and $y \in Y$, then

$$
\big[(\big(f \circ \pi^X\big) \circ -^{X\times Y}\big](x,y) := (\big(f \circ \pi^X\big)\big(-^X x, -^Y y\big) := f(-^X x) := f_-(x) := \big(f_- \circ \pi^X\big)(x,y)
$$

i.e., $(\big(f \circ \pi^X\big) \circ -^{X\times Y} = f_- \circ \pi^X \in F \times G$, since $f_- \in F$. Similarly, $\big(g \circ \pi^Y\big) \circ -^{X\times Y} \in F \times G$. □

Since the projections π^X, π^Y are homomorphisms, they are Bishop homomorphisms. By the universal property of the product Bishop topology, $\boldsymbol{\mathcal{F} \times \mathcal{G}}$ is the product in **BTopGrp**.

5. Closed subsets

In this section we prove fundamental properties of the closed subsets in Bishop topological groups. For all algebraic notions within BISH used here without further explanation, we refer to [22].

Proposition 5.1. *Let $C \subseteq X$ and $x_0 \in X$.*

(i) *If C is closed, then $-C := \{-c \mid c \in C\}$ is closed.*
(ii) $\overline{-C} = -\overline{C}$.
(iii) *If C is closed, then $x_0 + C := \{x_0 + c \mid c \in C\}$ is closed.*
(iv) $\overline{x_0 + C} = x_0 + \overline{C}$.

Proof. (i) We suppose that $u \in \overline{-C}$ i.e., if $f(u) > 0$, there is $w \in -C$ such that $f(w) > 0$, for every $f \in F$, and we show that $u \in -C$ i.e., $-u \in C$. Since C is closed, it suffices to show that $-u \in \overline{C}$. Let $f \in F$ such that $f(-u) > 0 :\Leftrightarrow f_-(u) > 0$. By our hypothesis on u there is $w \in -C$ such that $f_-(w) := f(-w) > 0$, and $-w \in C$.
(ii) Since $C \subseteq \overline{C}$, we get $-C \subseteq -\overline{C}$. Since $\overline{C}$ is closed, by (i) $-\overline{C}$ is also closed, hence $\overline{-C} \subseteq \overline{-\overline{C}} = -\overline{C}$. To show the converse inclusion $-\overline{C} \subseteq \overline{-C}$, let $x \in -\overline{C}$, hence $-x \in \overline{C}$ i.e., if $f(-x) > 0$, there is $u \in C$ with $f(c) > 0$, for every $f \in F$. We show that $x \in \overline{-C}$. Let $f \in F$ with $f(x) = f(-(-x)) = f_-(-x) > 0$. By our hypothesis on $-x$, there exists $u \in C$ with $f_-(u) := f(-u) > 0$, and $-u \in -C$.
(iii) We suppose that $y \in \overline{x_0 + C}$ i.e., if $f(y) > 0$, there is $c \in C$ such that $f(x_0 + c) > 0$, for every $f \in F$, and we show that $y \in x_0 + C$ by showing that $-x_0 + y \in C$. As C is closed, it suffices to show that $-x_0 + y \in \overline{C}$. Let $f \in F$ such that $f(-x_0 + y) > 0 \Leftrightarrow f^1_{-x_0}(y) > 0$. We show that there is $c \in C$ such that $f(c) > 0$. By our hypothesis on y, there is $c \in C$ such that $f^1_{-x_0}(x_0 + c) := f(-x_0 + x_0 + c) = f(c) > 0$.
(iv) Since $\overline{C}$ is closed, and $x_0 + C \subseteq x_0 + \overline{C}$, we get $\overline{x_0 + C} \subseteq \overline{x_0 + \overline{C}} = x_0 + \overline{C}$. For the converse inclusion, let $x := x_0 + y$ with $y \in \overline{C}$. We show that $x \in \overline{x_0 + C}$. Let $f \in F$ such that $f(x_0 + y) > 0 \Leftrightarrow f^1_{x_0}(y) > 0$. We find $c \in C$ such that $f(x_0 + c) > 0$. By our hypothesis on y though, there is $c \in C$ such that $f^1_{x_0}(c) := f(x_0 + c) > 0$. □

Corollary 5.2. *F separates the points of X if and only if $\{0^X\}$ is closed.*

Proof. If $\{0^X\}$ is closed, then by Proposition 5.1(iii) $\{x\} = x + \{0^X\}$ is closed, for every $x \in X$. By Proposition 3.5 F is separating. The converse follows immediately from Proposition 3.5. $\square$

Proposition 5.3. *If C is an open subgroup of X, then C is closed in X*

Proof. Let $x \in \overline{C}$ i.e., if $f(x) > 0$, there is $u \in C$ such that $f(u) > 0$, for every $f \in F$. We show that $x \in C$. Since C is a subgroup of X, we have that $0 \in C$. Since C is open in X, there is $g \in F$ such that $g(0) > 0$ and $U(g) \subseteq C$. Since $g^1_{-x} \in F$ and

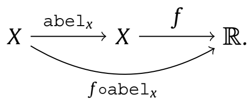
$$g^1_{-x}(x) := g(-x + x) = g(0) > 0,$$

by our hypothesis on x there is $u \in C$ such that $g^1_{-x}(u) := g(-x + u) > 0$. Since $U(g) \subseteq C$, we get $-x + u \in C$, and since C is a subgroup of X, we get $x \in C$. $\square$

Classically, C is closed, since its complement in X is the open set $\bigcup\{x + C \mid x \notin C\}$, where $x + C$ is open, for every $x \in X$, as C is open (this holds also constructively). The double use of negation in the classical proof is replaced here by the clear algorithm of the previous proof.

Lemma 5.4. *The commutator map $\mathtt{abel} : X \times X \to X$ is defined by $\mathtt{abel}(x, y) := x + y - x - y$, for every $(x, y) \in X \times X$.*

(i) $\mathtt{abel} \in \mathrm{Mor}(\mathcal{F} \times \mathcal{F}, \mathcal{F})$.
(ii) *If $x, y \in X$, then $x + y =_X y + x \Leftrightarrow \mathtt{abel}(x, y) = 0^X$.*
(iii) *If $x \in X$, the mapping $\mathtt{abel}_x : X \to X$, where $\mathtt{abel}_x(y) := \mathtt{abel}(x, y)$, for every $y \in X$, is in $\mathrm{Mor}(\mathcal{F}, \mathcal{F})$, and for every $f \in F$ the composition $f \circ \mathtt{abel}_x \in F$*

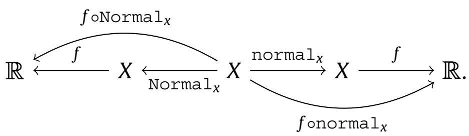


(iv) *If $x, y \in X$, then $\mathtt{abel}_x(y) = -\mathtt{abel}_y(x)$.*
(v) *If $x, y \in X$, then $\mathtt{abel}_x(y) = 0^X \Leftrightarrow \mathtt{abel}_y(x) = 0^X$.*

Proof. (i) By the definition of the product Bishop topology $\pi_1, \pi_2 \in \mathrm{Mor}(\mathcal{F} \times \mathcal{F}, \mathcal{F})$. Since $\mathtt{abel} := \pi_1 + \pi_2 - \pi_1 - \pi_2$, by Proposition 4.8 we get $\mathtt{abel} \in \mathrm{Mor}(\mathcal{F} \times \mathcal{F}, \mathcal{F})$.
(ii) and (iii) The proof for (ii) is immediate. By (i) and Proposition 2.5(iii) $\mathtt{abel}_x \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$, hence by the definition of a Bishop morphism $f \circ \mathtt{abel}_x \in F$, for every $f \in F$.
(iv) is trivial and (v) follows immediately from (iv). $\square$

Lemma 5.5. *Let $x \in X$ and $H \subseteq X$. The maps $\mathtt{normal}_x : X \to X$ and $\mathtt{Normal}_x : X \to X$ are defined, for every $x \in X$, respectively, by $\mathtt{normal}_x(y) := x + y - x$ and $\mathtt{Normal}_x(y) := y + x - y$. Let $\mathtt{normal}^H_x, \mathtt{Normal}^H_x : H \to X$ be the restrictions of $\mathtt{normal}_x$ and $\mathtt{Normal}_x$ to H, respectively.*

(i) *If $x, y \in X$, then $\mathtt{normal}_x(y) = \mathtt{Normal}_y(x)$.*
(ii) *$\mathtt{normal}_x \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$ and $\mathtt{Normal}_x \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$.*
(iii) *If H is a subgroup of X ($H \leq X$), then H is normal if and only if $\mathtt{normal}^H_x : H \to H$, for every $x \in X$.*
(iv) *If $f \in F$, the compositions $f \circ \mathtt{normal}_x \in F$ and $f \circ \mathtt{Normal}_x \in F$.*



(v) *If H is normal, then $\mathtt{normal}^H_x \in \mathrm{Mor}(\mathcal{F}_{|H}, \mathcal{F}_{|H})$.*
(vi) *If $\mathtt{Normal}^H_x : H \to H$, then $\mathtt{Normal}^H_x \in \mathrm{Mor}(\mathcal{F}_{|H}, \mathcal{F}_{|H})$.*

Proof. (i) The proof is immediate. For the proof of (ii), the function $c_x : X \to X$, defined by $c_x(y) := x$, for every $y \in X$, is in $\mathrm{Mor}(\mathcal{F}, \mathcal{F})$; if $f \in F$, then $(f \circ c_x)(y) := f(x)$, for every $y \in X$, hence $f \circ c_x = \overline{f(x)}^X \in F$. The identity map id_X on X is also in $\mathrm{Mor}(\mathcal{F}, \mathcal{F})$. Since $\mathtt{normal}_x := c_x + \mathrm{id}_X - c_x$, by Proposition 4.8 $\mathtt{normal}_x \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$. Since $\mathtt{Normal}_x := \mathrm{id}_X + c_x - \mathrm{id}_X$, we get $\mathtt{Normal}_x \in \mathrm{Mor}(\mathcal{F}, \mathcal{F})$. (iii) and (iv) are immediate to show. For the proof of (v), by the $\bigvee$-lifting of morphisms we have that $\mathtt{normal}_x^H \in \mathrm{Mor}(\mathcal{F}_{|H}, \mathcal{F}_{|H})$ if and only if $\forall_{f \in F}(f_{|H} \circ \mathtt{normal}_x^H \in F_{|H})$. If $f \in F$ and $v \in H$, then $(f_{|H} \circ \mathtt{normal}_x^H)(v) := f_{|H}(x + v - x) := f(x + v - x) := (f \circ \mathtt{normal}_x)(v) := (f \circ \mathtt{normal}_x)_{|H}(v)$ i.e., $f_{|H} \circ \mathtt{normal}_x^H = (f \circ \mathtt{normal}_x)_{|H} \in F_{|H}$, since $f \circ \mathtt{normal}_x \in F$. For the proof of (vi), we proceed as in the proof of (v). □

Theorem 5.6. *Let H be a subgroup of X.*

(i) *The closure $\overline{H}$ of H is also a subgroup of X.*
(ii) *If F is a separating Bishop topology and H is abelian, then $\overline{H}$ is abelian.*
(iii) *If H is normal, then $\overline{H}$ is normal.*

Proof. (i) Since $H \subseteq \overline{H}$ and $0 \in H$, we get $0 \in \overline{H}$. Let $x, y \in \overline{H}$, where by definition

$$x \in \overline{H} :\Leftrightarrow \forall_{f \in F}\big(f(x) > 0 \Rightarrow \exists_{v \in H}(f(v) > 0)\big),$$
$$y \in \overline{H} :\Leftrightarrow \forall_{f \in F}\big(f(y) > 0 \Rightarrow \exists_{v \in H}(f(v) > 0)\big).$$

We show that $x + y \in \overline{H}$ i.e., $\forall_{f \in F}\big(f(x + y) > 0 \Rightarrow \exists_{v \in H}(f(v) > 0)\big)$. Let $f \in F$ such that $f(x + y) > 0$. By Proposition 4.5(iii) the map $f_y^2 : X \to \mathbb{R} \in F$, where $f_y^2(u) := f(u + y)$, for every $u \in X$. By hypothesis, $f_y^2(x) > 0$. Since $x \in \overline{H}$, there is $z \in H$ such that

$$f_y^2(z) > 0 :\Leftrightarrow f(z + y) > 0 :\Leftrightarrow f_z^1(y) > 0,$$

where by Proposition 4.5(iii) the map $f_z^1 : X \to \mathbb{R} \in F$, where $f_z^1(u) := f(z + u)$, for every $u \in X$, is in F. Since $y \in \overline{H}$, there is $w \in H$ such that $f_z^1(w) > 0 :\Leftrightarrow f(z + w) > 0$. Since $H \leq X$, we get $z + w \in H$, which is what we need to show. Next we show that $-x \in \overline{H}$ i.e.,

$$\forall_{f \in F}\big(f(-x) > 0 \Rightarrow \exists_{v \in H}(f(v) > 0)\big).$$

Let $f \in F$ such that $f(-x) > 0 \Leftrightarrow f_-(x) > 0$. Since $f_- \in F$ and $x \in \overline{H}$, there is $v \in H$ such that $f_-(v) := f(-v) > 0$. Since $H \leq X$, we get $-v \in H$, and our proof is completed.
(ii) **Case I**: $x \in H$ and $y \in \overline{H}$.
We show that $\mathtt{abel}_x(y) =_X 0^X$. Suppose that $\mathtt{abel}_x(y) \neq_F 0^X$. By Proposition 5.2.5 in [27] there is $f \in F$ such that $f(\mathtt{abel}_x(y)) = 1$ and $f(0^X) = 0$. Since $y \in \overline{H}$, and $(f \circ \mathtt{abel}_x)(y) = 1 > 0$, and $f \circ \mathtt{abel}_x \in F$, there is $v \in H$ such that $(f \circ \mathtt{abel}_x)(v) > 0$. Since $x, v \in H$ and H is abelian, we have that $\mathtt{abel}_x(v) = 0^X$, hence $0 = f(0^X) = (f \circ \mathtt{abel}_x)(v) > 0$, which is a contradiction. Since F is separating, the canonical apartness relation $\neq_F$ of F is tight (see Proposition 5.1.3 in [27]), hence the negation of $\mathtt{abel}_x(y) \neq_F 0^X$ implies that $\mathtt{abel}_x(y) =_X 0^X$.
Case II: $x \in \overline{H}$ and $y \in \overline{H}$.
We show that $\mathtt{abel}_x(y) =_X 0^X$. Suppose that $\mathtt{abel}_x(y) \neq_F 0^X$. As in the previous case, there is $f \in F$ such that $f(\mathtt{abel}_x(y)) = 1$ and $f(0^X) = 0$. Since $y \in \overline{H}$, and $(f \circ \mathtt{abel}_x)(y) = 1 > 0$, there is $v \in H$ such that $(f \circ \mathtt{abel}_x)(v) > 0$. By case I we have that $\mathtt{abel}_v(x) = 0^X$, hence by Lemma 5.4(v) we get $\mathtt{abel}_x(v) = 0^X$, hence $0 = f(0^X) = (f \circ \mathtt{abel}_x)(v) > 0$, which is a contradiction. Since F is separating, we conclude, similarly to Case I, that $\mathtt{abel}_x(y) =_X 0^X$.
(iii) By Lemma 5.5(ii) it suffices to show that $\mathtt{normal}_x^{\overline{H}} : \overline{H} \to \overline{H}$, for every $x \in X$. If $x \in X$ and $y \in \overline{H}$, we show that

$$\mathtt{normal}_x^{\overline{H}}(y) := x + y - x \in \overline{H}.$$

Let $f \in F$ such that $f(x + y - x) > 0 \Leftrightarrow (f \circ \mathtt{normal}_x)(y) > 0$. We need to find $u \in H$ such that $f(u) > 0$. Since $y \in \overline{H}$ and $f \circ \mathtt{normal}_x \in F$, there is $v \in H$ with

$$(f \circ \mathtt{normal}_x)(v) := f(x + v - x) > 0.$$

Since H is normal and $v \in H$, we have that $u := x + v - x \in H$ and $f(u) > 0$. □

As we explain in [27], section 5.8, by the Stone-Čech theorem for Bishop spaces we have that the separating hypothesis on F in case (ii) of the previous proposition is not a serious restriction on F. Note that classically, if X is a Hausdorff topological group, then if H is an abelian subgroup, then $\overline{H}$ is abelian. The F-version of a Hausdorff topology is the following (see [27], section 5.2): if $\neq$ is a given apartness relation on X, then F is $\neq$-Hausdorff, if $\neq \subseteq \neq_F$. In Proposition 5.2.3. of [27] we show that this is equivalent to the positive $\neq$-version of the induced neighborhood structure being Hausdorff. If F is $\neq$-Hausdorff, n-many pairwise $\neq$-apart points of X are mapped to given n-many real numbers. This is essential to the proof we gave above. In this way, we capture computationally the requirement of a Hausdorff topology for $\overline{H}$ being abelian. By Proposition 3.5 the tightness of $\neq_F$ in a Bishop topological group implies that the induced topology is Hausdorff in the classical sense (classically, a topological group is Hausdorff if and only if there is a closed singleton). In the next proposition the subset H of X is extensional i.e., H is closed under the given equality $=_X$ on X, so that the defining property of $\mathtt{Normal}_X(H)$ in the use of the separation scheme is also extensional.

Theorem 5.7. *Let H be an extensional subset of X. The normalizer $\mathtt{Normal}_X(H)$ of H in X and the center $\mathtt{Center}_X(H)$ of H in X, are defined by*

$$\mathtt{Normal}_X(H) := \big\{x \in X \mid \mathtt{Normal}_x^H : H \to H\big\},$$
$$\mathtt{Center}_X(H) := \big\{x \in X \mid \forall_{v \in H}\big(\mathtt{abel}_x(v) = 0^X\big)\big\}.$$

(i) *If H is closed, then $\mathtt{Normal}_X(H)$ is closed.*
(ii) *If F is separating, then $\mathtt{Center}_X(H)$ is closed.*

Proof. (i) We suppose that H is closed i.e.,

$$(\mathtt{Hyp}_1) \qquad \forall_{x \in X}\Big(\forall_{f \in F}\big(f(x) > 0 \Rightarrow \exists_{v \in H}\big(f(v) > 0\big)\big) \Rightarrow x \in H\Big),$$

and we show that $\mathtt{Normal}_X(H)$ is closed i.e.,

$$(\mathtt{Goal}_1) \quad \forall_{x \in X}\Big(\forall_{f \in F}\big(f(x) > 0 \Rightarrow \exists_{u \in \mathtt{Normal}_X(H)}\big(f(u) > 0\big)\big) \Rightarrow x \in \mathtt{Normal}_X(H)\Big).$$

For that we fix some $x \in X$, we suppose that

$$(\mathtt{Hyp}_2) \qquad \forall_{f \in F}\big(f(x) > 0 \Rightarrow \exists_{u \in \mathtt{Normal}_X(H)}\big(f(u) > 0\big)\big),$$

and we show $(\mathtt{Goal}_2)$ $x \in \mathtt{Normal}_X(H) :\Leftrightarrow \mathtt{Normal}_x^H : H \to H$. If we fix $v \in H$, we show next $(\mathtt{Goal}_3)$ $\mathtt{Normal}_x^H(v) := v + x - v \in H$. By $\mathtt{Hyp}_1$ it suffices to show the following

$$(\mathtt{Goal}_4) \qquad \forall_{f \in F}\Big(f\big(\mathtt{Normal}_x^H(v)\big) > 0 \Rightarrow \exists_{w \in H}\big(f(w) > 0\big)\Big).$$

If we fix $f \in F$, we suppose that

$$(\mathtt{Hyp}_3) \qquad f\big(\mathtt{Normal}_x^H(v)\big) > 0 \Leftrightarrow f\big(\mathtt{normal}_v(x)\big) > 0 \Leftrightarrow (f \circ \mathtt{normal}_v))(x) > 0,$$

and we show $(\mathtt{Goal}_5)$ $\exists_{w \in H}\big(f(w) > 0\big)$. Since $f \circ \mathtt{normal}_v \in F$, by $(\mathtt{Hyp}_2)$ there is $u \in \mathtt{Normal}_X(H)$ with $(f \circ \mathtt{normal}_v)(u) := f(v + u - v) > 0$. Since $u \in \mathtt{Normal}_X(H)$, $\mathtt{Normal}_u^H : H \to H$, and since $v \in H$, we get $\mathtt{Normal}_u^H(v) := v + u - v \in H$. Hence, $w := v + u - v \in H$ and $f(w) > 0$.
(ii) We fix $x \in X$, we suppose that

$$(\mathtt{Hyp}_1^*) \qquad \forall_{f \in F}\big(f(x) > 0 \Rightarrow \exists_{u \in \mathtt{Center}_X(H)}\big(f(u) > 0\big)\big),$$

and we show $(\mathtt{Goal}_1^*)$ $x \in \mathtt{Center}_X(H) :\Leftrightarrow \forall_{v \in H}\big(\mathtt{abel}_x(v) = 0^X\big)$. Let $v \in H$ be fixed. Since F is a separating Bishop topology, it suffices to prove $\neg\big(\mathtt{abel}_x(v) \neq_F 0^X\big)$. If we suppose that $\mathtt{abel}_x(v) \neq_F 0^X$, there is $f \in F$ such that

$$f\big(\mathtt{abel}_x(v)\big) = 1 > 0 \ \ \& \ \ f\big(0^X\big) = 0.$$

By Lemma 5.4(iv) we get

$$1 = f\big(\mathtt{abel}_x(v)\big) = f\big(-\mathtt{abel}_v(x)\big) := f_-\big(\mathtt{abel}_v(x)\big) \ \ \& \ \ 0 = f\big(0^X\big) = f_-\big(0^X\big).$$

Since $f \in F$, we have that $f_- \circ \mathtt{abel}_v \in F$ and $(f_- \circ \mathtt{abel}_v)(x) > 0$. By $(\mathtt{Hyp}_1^*)$ there is $u \in \mathtt{Center}_X(H)$ such that $(f_- \circ \mathtt{abel}_v)(u) > 0$. Since $u \in \mathtt{Center}_X(H)$, we get $\forall_{w \in H}\big(\mathtt{abel}_u(w) = 0^X\big)$. Moreover, $(f_- \circ \mathtt{abel}_v)(u) := f_-(v + u - v - u) > 0$. Since $v \in H$, we get

$$v + u - v - u := \mathtt{abel}_v(u) = -\mathtt{abel}_u(v) = -0^X = 0^X,$$

hence $f_-\big(0^X\big) > 0$, which contradicts the previously established equality $f_-\big(0^X\big) = 0^X$. □

Proposition 5.8. *If G is a separating Bishop topology on Y and $h \in \mathbf{Mor}(\mathcal{F}, \mathcal{G})$, the kernel $\mathtt{Ker}(h) := \big\{x \in X \mid h(x) =_Y 0^Y\big\}$ of h is a closed set in $\mathcal{F}$.*

Proof. Let $x \in X$ such that $\forall_{f \in F}\big(f(x) > 0 \Rightarrow \exists_{v \in \mathtt{Ker}(h)}\big(f(v) > 0\big)\big)$. Since $\neq_G$ is tight, it suffices to show that $\neg\big(h(x) \neq_G 0^Y\big)$. If $h(x) \neq_G 0^Y$, there is $g \in G$ such that $g(h(x)) = 1 > 0$ and $g\big(0^Y\big) = 0$. Since $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, we have that $g \circ h \in F$. As $(g \circ h)(x) > 0$, there is $v \in \mathtt{Ker}(h)$ such that $0 = g\big(0^Y\big) = g(h(v)) := (g \circ h)(v) > 0$, which is a contradiction. □

Theorem 5.9 *(Characterization of a closed (open) subgroup). Let C be a subgroup of X.*

(i) *C is closed if and only if there is an open set O in X such that $O \cap C$ is inhabited and closed in O.*
(ii) *C is open if and only if there is an inhabited, open set O in X such that $O \subseteq C$.*

Proof. (i) Let O be open in X such that $O \cap C$ is inhabited and closed in O. We show that C is closed. Suppose that $x \in \overline{C}$ i.e., if $f(x) > 0$, there is $u \in C$ such that $f(u) > 0$, for every $f \in F$. We prove that $x \in C$. Let $c_0 \in O \cap C$. Since O is open, there is $g \in F$ such that $g(c_0) > 0$ and $U(g) \subseteq O$. Since $g^1_{-x+c_0} \in F$ and

$$g^1_{-x+c_0}(x) := g(x - x + c_0) = g(c_0) > 0,$$

by our hypothesis on x there is $c \in C$ such that

$$g^1_{-x+c_0}(c) := g(c - x + c_0) > 0.$$

As $U(g) \subseteq O$, we get $c - x + c_0 \in O$. The hypothesis "$O \cap C$ is closed in O" means

$$\forall_{z \in O} \big(z \in \overline{O \cap C} \Rightarrow z \in O \cap C\big).$$

Let $z_0 := c - x + c_0 \in O$. We show that $z_0 \in \overline{O \cap C}$, hence $z_0 \in O \cap C$. Since C is a subgroup, and since then $z_0 \in C$, we get the required membership $x \in C$. To show that $z_0 \in \overline{O \cap C}$, let $f \in F$ such that $f(z_0) > 0$. We find $w \in O \cap C$ such that $f(w) > 0$. Since $f(z_0) > 0$ and $g(z_0) > 0$, we have that $(f \wedge g)(z_0) > 0$ (see [7], p. 57). By Theorem 5.6(i) $\overline{C}$ is a subgroup of X. Since $C \subseteq \overline{C}$, and $c, c_0, -x \in \overline{C}$, we get $z_0 \in \overline{C}$. Since $f \wedge g \in F$ and $(f \wedge g)(z_0) > 0$, there is $w \in C$ such that

$$(f \wedge g)(w) > 0.$$

Since $g(w) \geq (f \wedge g)(w) > 0$, we get $w \in O$, hence $w \in O \cap C$. Since $f(w) \geq (f \wedge g)(w) > 0$, we conclude that $f(w) > 0$, as required. For the converse, if C is closed, then X is open, $C = C \cap X$ is inhabited by 0 and it is trivially closed in X.
(ii) Let O be open in X such that $O \subseteq C$, and let $c_0 \in O$. Suppose that $g \in F$ with $g(c_0) > 0$ and $U(g) \subseteq O$. Since also $U(g) \subseteq C$, we get $c_0 \in C$. Let $c \in C$. The function $g^1_{-c+c_0} \in F$ and $g^1_{-c+c_0}(c) = g(c_0) > 0$. We show that $U(g^1_{-c+c_0}) \subseteq C$, hence, since c is an arbitrary element of C, we conclude that C is open. Let $u \in X$ such that $g^1_{-c+c_0}(u) := g(u - c + c_0) > 0$. As $U(g) \subseteq O \subseteq C$, we get $u - c + c_0 \in C$. As $c, c_0 \in C$ and C is a subgroup of X, we have that $u \in C$. For the converse, if C is open, then C is an inhabited open set included in C. □

The classical proof of Theorem 5.9(i) is based on multiple use of negation (see [19]). In accordance to the standard practice of constructive mathematics, we replaced the "non-empty intersection" of O and C in Theorem 5.9(i) with the positive notion $O \between C$ of inhabited intersection, and the "non-emptyness" of O in Theorem 5.9(ii) with the stronger inhabitedness of O. Although, in general, it is not possible to show that the uniform F-complement $X \setminus^u_F C$ of a closed set in a Bishop space X is equal to $X \setminus C$, there is a number of cases in the theory of Bishop topological groups where this is possible.

Corollary 5.10. *If C is closed in X, such that $X \setminus C$ is a subgroup of X and $X \setminus^u_F C$ is inhabited, then $X \setminus C = X \setminus^u_F C$ and $X \setminus C$ is clopen.*

Proof. By Theorem 3.4 we have that $X \setminus^u_F C$ is open in X and $X \setminus^u_F C \subseteq X \setminus C$. Since $X \setminus^u_F C$ is inhabited and $X \setminus C$ is a subgroup of X, by Theorem 5.9(ii) we have that $X \setminus C$ is open. As $X \setminus C \subseteq (X \setminus C)^\circ$, by Theorem 3.4 we get $X \setminus C \subseteq X \setminus^u_F C$, hence $X \setminus C = X \setminus^u_F C$. As $X \setminus C$ is an open subgroup, by Proposition 5.3 we have that $X \setminus C$ is also closed. □

Notice that classically a subgroup C of a topological group is either clopen or has empty interior. By replacing the hypothesis of "non-empty interior of C" with the positive "existence of an inhabited open subset of C", constructively we prove the following corollary.

Corollary 5.11. *Let C be a subgroup of X. If O is an inhabited open set in X such that $O \subseteq C$, then C is clopen.*

Proof. By Theorem 5.9(ii) C is open, and by Proposition 5.3 C is also closed. □

6. Concluding remarks

We have presented some very first, fundamental results in the theory of Bishop topological groups and their closed subsets. Clearly, we can study similarly other algebraic structures, like rings and modules, equipped with a compatible Bishop topology. For example, it is important to investigate how a theory of Bishop topological rings relates to the constructive theory of Zariski spectrum in [20].

There is a plethora of open questions related to Bishop topological groups. The study of "compact" subsets of Bishop topological groups is a natural extension of this work, which depends though, on the notion of compactness considered within the theory of Bishop spaces. The notion of 2-compactness developed in [27,29] seems hard to work with, while the development in [39] of a constructive version of Comfort-compactness [11] is more promising.

The question whether

$$\mathrm{Bic}(X \times Y) \subseteq \mathrm{Bic}(X) \times \mathrm{Bic}(Y),$$

which implies that a locally compact metric group, in the sense of Bishop and Bridges, is a Bishop topological group (in the canonical sense), is an important open problem of independent interest. Categorically speaking, the assignment

$$X \mapsto \big(X, C_u(X)\big)$$

is a full and faithful functor from the category of compact metric spaces into 2-compact Bishop spaces (see Theorem 3.10, Proposition 3.11, and Corollary 3.14 in [28]). The assignment

$$X \mapsto \big(X, \mathrm{Bic}(X)\big)$$

should be compared with Palmgren's full and faithful embedding (that preserves products) of the category of locally compact metric spaces into the category of locally compact formal topologies, which is given in [25].

By Theorem 4.4(i)-(ii) compact metric groups, in the sense of Bishop and Bridges, are identified with the compact metric groups equipped with the canonical Bishop topology of uniformly continuous functions. By Theorem 4.4(iii) locally compact metric spaces that are also groups and they are equipped with the canonical Bishop topology of uniformly continuous functions on bounded subsets are locally compact metric groups in the sense of Bishop and Bridges. Consequently, the Haar measure theory of locally compact (abelian) metric groups, developed in [5], Chapter 8, applies to these Bishop topological groups.

The generalization of the Haar measure theory of locally compact metric spaces to a Haar measure theory of locally compact Bishop topological groups, for an appropriate notion of a *locally compact* Bishop space, is an important research project. In our work under construction [38] we study the integration theory of Bishop spaces. Using ideas from the "forgotten" Bishop measure theory in [3], we define the notion of an algebra of test functions T included in a Bishop topology of functions F in an abstract way. An integral on T is extended then to an integral on F. The key idea is to discover function theoretic translations of the principal steps taken in the integration theory of locally compact metric spaces and incorporate them into the integration theory of Bishop spaces, such that the integration theory of locally compact metric spaces will be a special case of the integration theory of Bishop spaces. In the integration theory of locally compact metric spaces the set of test functions T is the set of continuous functions with compact support and the Bishop topology $F = \mathrm{Bic}(X)$. The next step is to apply this method to the Haar measure theory of locally compact groups. That is, one has to find abstract, function theoretic characterizations of the key-lemmas in the proof of the existence of the Haar measure of a locally compact metric group and use these characterizations to define the notion of a locally compact Bishop space.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

An early version of this work was presented in "Conference Algebra and Algorithms" that took place in Djerba, Tunisia, in February 2020. I would like to thank the organizers, Peter Schuster and Ihsen Yengui, for inviting me. I would also like to thank the anonymous reviewers for their very useful comments and suggestions.

This research was partially supported by LMUexcellent, funded by the Federal Ministry of Education and Research (BMBF) and the Free State of Bavaria under the Excellence Strategy of the Federal Government and the Länder.

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