The measurement of segregation sensitive spatial income deprivation

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Abstract

We develop dominance criteria to assess the patterns of residential ethnic segregation and urban income deprivation across neighborhoods of a city. The results combine aggregate information on inequality and residential segregation within neighborhoods and disparities across neighborhoods in average incomes. We use this methodology to investigate the dynamic of these phenomena in the cities of Chicago and in New York from 1990 to 2012.

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1 Introduction

In the last decades inequality has been evolving in heterogeneous ways across major American cities (Moretti 2013, Chetty et al. 2014), with Gini indices reaching level 0.5 in New York City, or falling below 0.4 in other major cities such as Washington DC. However, not all places in large cities display the same inequality. While differences in income and poverty between neighborhoods are large and persistent (Iceland and Hernandez 2017, Wheeler and La Jeunesse 2008), inequality within the neighborhoods is fast growing in all large American cities (Andreoli and Peluso 2017).[1] Not only inequality and poverty are unevenly distributed geographically, but they could also be associated with different levels of spatial ethnic segregation across counties or districts. The geographical dispersion of incomes could then lead to more salient inequalities if they are perceived in more segregated area that exhibit the predominance of some ethnic groups and if the population living in these area is experiencing also lower incomes.

In this paper, we study the patterns of residential ethnic segregation and urban (relative) deprivation revealed by neighborhood-specific distributions of equivalized household income, and by the differences that these distributions display across neighborhoods of the same city. Our analysis focuses on two American Metropolitan Statistical Areas (MSA): Chicago, displaying historical patterns of inequalities and segregation and New York City, the largest MSA in the US. For both cities, we can retrieve information about the income distribution within neighborhoods. Neighborhoods define interesting partitions of the urban population. In fact, to a large extent they are chosen by residents, who reveal through the residential choice their distribution of reference. Within each neighborhood, we measure the degree of deprivation experienced by residents. Moreover, we extend the deprivation comparisons also outside the neighborhood by allowing comparisons with the

[1] Analogous spatial comparisons based on poverty evaluation are also illustrated in Andreoli et al. (2021).

individuals living in all the city. Because of information constraints these latter comparisons are assumed to be made considering only the average income in the different neighborhoods. The overall income deprivation felt by each individual is therefore made by two components, one within neighborhood and another computed making comparisons across the neighborhoods that cover all the city.

In the analysis, we use counties to delimit the geography of neighborhoods. US counties are well-defined spatial entities whose administrative and social borders affect residents economic, social and political life (for example through schooling catchment areas or voting districts). The boundaries of counties are fixed, implying that the spatial dimensions of neighborhoods is well defined across decades. Furthermore, the MSA that we analyze are composed by at least 14 counties, representing sufficient degree of heterogeneity in the distribution of people across urban space. Our analysis makes use of information about the distribution of incomes and on the ethnic composition within each county and in a MSA. These distributional measures could be computed at aggregate level for each county but it is not possible to link directly the income of an individual to his/her ethnic origin. From a normative perspective, we combine information about the average income and the ethnic segregation in each county to determine social weights that aggregate the deprivation evaluations of each county in a MSA measure. Average income quantifies the affluence in the neighborhoods. Ethnic segregation is an important factor for access to jobs, schools and social life in major American cities. The interaction between the average income and the degree of ethnic segregation can be hence informative of the extent to which income is evenly distributed across groups. Poorer counties always receive higher priority, or social weight, in our analysis. For low income areas priority increases with the degree of segregation, indicating that residents in these counties not only face, on average, a poor neighborhood, but also that poverty is concentrated among few ethnic groups that are over-represented in those neighborhoods. On the contrary, rich counties always receive low priority, that decreases with the degree of ethnic segregation therein. In fact, high-income counties where income is concentrated on few ethnic groups, over-represented with respect to the citywide ethnic composition receive least priority on the social order.

We are interested in comparing the dynamic of deprivation at MSA level, while considering that individuals choose neighborhoods as their main reference distributions, and that neighborhoods receive different degrees of priority on the social scale. Consistently with the methodology developed, we derive robust comparison criteria that take into account sets of possible normative aggregation weights. These criteria generate sequential dominance conditions (see Atkinson and Bourguignon 1987, Jenkins and Lambert 1993, Zoli and Lambert 2012, Apablaza et al. 2016). They will be used to study the evolution of deprivation and inequality in American cities in order to highlight patterns and trends over two decades from 1990 to 2012. The spatial income deprivation dominance conditions are shown to depend on the levels of inequality at county level measured by the *Gini index*, on the inequality in the distribution of income across counties measured by the *Schutz inequality index* and on the level of ethnic segregation in each county of a MSA measured by a dissimilarity index analogous to the *Duncan and Duncan index of segregation*.

2 Relative deprivation and spatial inequality

We consider the framework introduced in Hey and Lambert (1980) for applying the relative deprivation approach to the measurement of welfare and inequality. We extend the logic of this approach to the measurement of spatial relative income deprivation combining the information on income distribution with the one on ethnic segregation at county level in a MSA.

The approach is based on the seminal paper by Yitzhaki (1979) that provides a quantification of the concept of relative deprivation introduced by Runciman (1966). The perception of deprivation felt by each individual is represented through an "envy factor" that decreases proportionately the utility of each individual according to the gap between the income of wealthier subjects and his/her income. An individual perceives to be deprived if someone else that belongs to his/her reference group has access to resources which he/she considers useful while he/she does not have access to them.[2] In order to operationalize this notion we define the *reference groups* in a spatial context by considering two types of deprivation, the first taking place through comparisons with the population living in the county where the individual resides and the second considering the whole city population. Individuals make their comparisons in terms of income and compare themselves with those living in their county making use of the income distribution information in the county and with the whole population by using information on the average income of the counties.

2.1 Modelling deprivation

We consider a city partitioned into $n \geq 2$ counties where N is the set of all counties and N^i denotes the set of individuals living in county $i = 1, 2, ..., n$. These individuals are in number $m_i > 0$ and their income profile is given by the m_i-dimensional vector $\mathbf{y}^i = (y_1^i, y_2^i, ..., y_l^i, ..., y_{m_i}^i)$, where the income of individual ℓ is denoted $y_\ell^i > 0$. The average income in county i is denoted $\mu_i := \frac{1}{m_i} \sum_{\ell=1}^{m_i} y_\ell^i$. Considering that the whole city population is $m = \sum_{i=1}^n m_i$ where $q_i := m_i/m$ is the proportion of population in county i, the total city average income is $\mu := \sum_{i=1}^n q_i \mu_i = \frac{1}{m} \sum_{i=1}^n \sum_{\ell=1}^{m_i} y_\ell^i$. We make use of $\mathbf{y}$ to denote the m-dimensional vector of incomes of the entire population, that includes the incomes of

[2] Runciman's definition of relative deprivation states: "We can roughly say that (a person) is relatively deprived of X when (i) he does not have X, (ii) he sees some other person or persons, which may include himself at some previous or expected time, as having X (whether or not that is or will be in fact the case), (iii) he wants X, and (iv) he sees it as feasible that he should have it." (1966, p.10).

the individuals of all counties composing a city.

The index $d^i_\ell(\mathbf{y})$ represents the individual deprivation of person ℓ, belonging to county i, evaluated considering the city income distribution $\mathbf{y}$. Following Yitzhaki (1979) and Hey and Lambert (1980) we consider individual deprivation as the aggregation of the feeling of deprivation felt by each individual with respect to all the other individuals belonging to the reference group.

In our case we have *two components* for $d^i_\ell(\mathbf{y})$. An internal, within county, component and an overall city component of deprivation. The internal component $I^i_{\ell j}(\mathbf{y})$ is not affected by changes in the income distribution occurring outside the reference group identified by the people living in the same county as individual ℓ. We denote

$$I^i_{\ell j}(\mathbf{y}) := \begin{cases} y^i_j - y^i_\ell & \text{if } \; y^i_j > y^i_\ell \\ 0 & \text{if } \; y^i_j \leq y^i_\ell \end{cases} \tag{1}$$

the deprivation felt by individual ℓ with respect to individual j within the same reference county i. The city component instead is obtained by the comparison of the income of the individual with the one of all individuals in the city. For this component we assume that each individual identifies him/herself with his/her community and therefore the comparison is made in terms of the average incomes of each county. The comparison with individuals living in other counties lacks of detailed information on the income distribution and is approximated by the average income in each considered county.

Moreover, differently from what assumed for the within county comparisons, we allow for the case where an individual belonging to a community with high average income could reduce his/her level of deprivation felt within the community when comparing his/her situation with individuals belonging to poorer communities. The city component of deprivation $C^{ii'}_{\ell j}(\mathbf{y})$ for the comparison made by individual ℓ living in county i with individual

j in county i' is then

$$C_{\ell j}^{ii'}(\mathbf{y}) := \mu_{i'} - \mu_i. \tag{2}$$

The individual ℓ deprivation $d_\ell^i(\mathbf{y})$ is computed as the average of across individuals deprivation comparisons $I_{\ell j}^i(\mathbf{y})$ made within the county population, that is $I_\ell^i(\mathbf{y}) := \frac{1}{m_i}\sum_{j\in N^i} I_{\ell j}^i(\mathbf{y})$, to which is added the city component that coincides with the average of $C_{\ell j}^{ii'}(\mathbf{y})$ across all individuals in the society, that is $C_\ell^i(\mathbf{y}) := \sum_{i'=1}^n q_{i'} C_{\ell j}^{ii'}(\mathbf{y})$ which leads to $C_\ell^i(\mathbf{y}) = \sum_{i'=1}^n q_{i'}\mu_{i'} - \sum_{i'=1}^n q_{i'}\mu_i = \mu - \mu_i$. Thus, $d_\ell^i(\mathbf{y}) = I_\ell^i(\mathbf{y}) + C_\ell^i(\mathbf{y})$.

The overall deprivation within the city is obtained by aggregating the average levels of deprivation experienced within each county. The average or representative level of deprivation for county i is denoted $\delta_i := \frac{1}{m_i}\sum_{\ell\in N^i} d_\ell^i(\mathbf{y}) = \frac{1}{m_i}\sum_{\ell\in N^i}[I_\ell^i(\mathbf{y}) + C_\ell^i(\mathbf{y})]$. Following Yitzhaki (1979) and Hey and Lambert (1980) we know that $\frac{1}{m_i}\sum_{\ell\in N^i} I_\ell^i(\mathbf{y}) = \mu_i G_i$ where G_i denotes the relative Gini inequality index for the income distribution $\mathbf{y}^i$ of county i. Moreover, we can note that $\frac{1}{m_i}\sum_{\ell\in N^i} C_\ell^i(\mathbf{y}) = \mu - \mu_i$. As a result we obtain that

$$\delta_i := \mu_i G_i + \mu - \mu_i = \mu - \mu_i(1 - G_i). \tag{3}$$

The formula makes explicit that the representative deprivation in the county is measured by the gap between the city average income and a representative monetary measure of welfare that is obtained as the product of the county average income times the Gini based income equality index. This latter measure $\mu_i(1-G_i)$ coincides with what Lambert (2001) calls "abbreviated social welfare function".

The simple additive aggregation rule applied to obtain the average deprivation of the individuals in a county is deprivation inequality neutral, that is any individual level of deprivation is treated in the same way irrespective of its extent. However, even if the total deprivation is insensitive with respect to the differences in the individual deprivation levels, it is inequality averse if we consider the income distribution, in fact, the average deprivation index coincides with the absolute Gini index of the income distribution within a county.[3]

To take a more general approach we may consider to combine the two components of deprivation by attributing a weight $\alpha \in (0, 1]$ to the city component $\mu - \mu_i$. In this case we obtain the representative deprivation

$$\delta_i^\alpha := \mu_i G_i + \alpha(\mu - \mu_i) = \alpha\mu - \mu_i(\alpha - G_i). \tag{4}$$

In our exposition we will use this general index and compare it with δ_i in (3) obtained for $\alpha = 1$. These measures can be computed using aggregate data at county and city level and should be aggregated across counties to derive an overall deprivation index.

2.2 The measurement of segregation sensitive relative deprivation

In order to compute a segregation sensitive evaluation of deprivation in metropolitan cities we use estimates of the average income μ_i and Gini relative index G_i by each county. We derive an aggregate relative deprivation measure $D(t)$ for a city for each year t. Each aggregate evaluation of deprivation will take into account both the distribution of the average deprivation in each county and the geographical ethnic segregation within each county.

In short we will denote by Y^t the set of variables that characterize the available information at time t for a given city. In line with the deprivation model introduced above we will consider the set Y^t composed by $\{G_i^t, \mu_i^t, q_i^t\}_{i=1,2,\ldots,n}$ and by the data on the ethnic composition at county and city level. Here the superscript denotes the year

[3]See Chakravarty (1990) for a survey of ethically based deprivation indices. For alternative approaches based on the decomposition of the Gini index see Ebert (2010), Lambert and Decoster (2005), and Mornet et al. (2013).

considered, it will be made explicit in the next exposition only when necessary.

Making use of this information we will derive a robust assessment of deprivation constructing segregation sensitive spatial deprivation curves. These curves cumulate across counties the relative level of individual deprivation perceived in each county (weighted according to the county population share). The order of cumulation will be derived according to the priority that is given to a county or a group of them considering the relative level of affluence of the population living there and their level of ethnic segregation. For any given city and year, the average deprivation δ_i in a county, or its general formulation δ_i^α, can be readily computed from available data on income, inequality, population shares and segregation at the county level. Deprivation curves represent a mapping between cumulated deprivation and the level of segregation and average income in the counties. Dominance can be verified by comparing the levels of the curves for counties that exhibit segregation and average income levels that belong to given sets of values.

The level of segregation in each county i is measured by comparing the data on the ethnic composition within the county and the one within the city. The city population is composed by $H \geq 2$ ethnic groups denoted with index $h = 1, 2, ..., H$. The proportion of individuals belonging to ethnic group h that live in the city is q^h, while the proportion computed within county i is q_i^h. The ethnic residential segregation index s_i computed for each county i that we propose is inspired by the Duncan and Duncan (1955) segregation index and is

$$s_i := \frac{1}{2} \sum_{h=1}^{H} \left| q_i^h - q^h \right| . \tag{5}$$

The value of the index $s_i \in [0, 1)$ measures how dissimilar is the ethnic composition of the population in each county compared to the entire city composition. A value of $s_i = 0$ is obtained if the proportional ethnic composition in a county fully reflects the composition in the entire city. A high value of the index represents a situation where an ethnic group is predominant in a county.[4]

For each county i we therefore have information on the population share of the county q_i, its average income μ_i, the Gini index G_i and the level of segregation s_i. Moreover, we have information on the city average income μ. Based on (4) we can construct an index D of deprivation within the city that aggregates each county deprivation $q_i \cdot \delta_i^\alpha$ using bounded and non-negative weights $v(\mu_i, \mu, s_i)$ that take into account both the level of relative affluence in the county and its ethnic segregation. Therefore the aggregate deprivation is:

$$D := \sum_{i=1}^{n} v(\mu_i, \mu, s_i) \cdot q_i \cdot [\alpha\mu - \mu_i(\alpha - G_i)] \tag{6}$$

for $\alpha \in (0, 1]$. Our aim is to derive *relative* deprivation evaluations that are *scale invariant* with respect to proportional changes in the income of all the individuals. In this case the notion of relativity of the evaluations refers both to the fact that the perception of deprivation is computed making use of relative comparisons of the situations of the individuals, and that these comparisons are based on relative values of the relevant variables that are not influenced by proportional changes affecting all the population. The obtained measure will depend on income distribution patterns within the city and the counties and on how these are related to the distribution of segregation across counties but not on the scale of the city income.

In view of this latter property the deprivation evaluation component for each county should satisfy the following condition, where by construction G_i is scale invariant.

Condition 1 ((DSI) Deprivation Scale Invariance) *For all $\lambda > 0$ and for all $i \in N$*

$$v(\lambda\mu_i, \lambda\mu, s_i) \cdot q_i \cdot [\alpha\lambda\mu - \lambda\mu_i(\alpha - G_i)] = v(\mu_i, \mu, s_i) \cdot q_i \cdot [\alpha\mu - \mu_i(\alpha - G_i)] \, .$$

[4] In the extreme case where a group, say group H, is the only located in the considered county, the index takes the value $s_i = \frac{1}{2}\sum_{h=1}^{H-1} \left|0 - q^h\right| + \frac{1}{2}\left|1 - q^H\right|$, that is $s_i = \frac{1}{2}(1 - q^H) + \frac{1}{2}(1 - q^H) = 1 - q^H$. If the proportion in the entire city population of individuals belonging to group H is small the index approaches the value of 1.

Recalling that $\mu > 0$ and setting $\lambda = 1/\mu$, one can derive a necessary condition for (6) to satisfy DSI. That is, $v(\mu_i/\mu, 1, s_i)\frac{1}{\mu}\left[\alpha\mu - \mu_i(\alpha - G_i)\right] = v(\mu_i, \mu, s_i)\left[\alpha\mu - \mu_i(\alpha - G_i)\right].$ This condition is satisfied if and only if one sets $v(\mu_i, \mu, s_i) = v(\mu_i/\mu, 1, s_i)\frac{1}{\mu}$. Letting $r_i := \mu_i/\mu$ and denoting by $w(r_i, s_i) := v(\mu_i/\mu, 1, s_i)$, we can then substitute $w(r_i, s_i)\frac{1}{\mu}$ to $v(\mu_i, \mu, s_i)$ in (6) obtaining

$$D = \sum_{i=1}^{n} w(r_i, s_i) \cdot q_i \cdot \left[\alpha - r_i(\alpha - G_i)\right] = \sum_{i=1}^{n} w(r_i, s_i) \cdot q_i \cdot \left[\alpha(1 - r_i) + r_i G_i\right]. \qquad (7)$$

Note that the adopted specification of the weighting function is not only necessary for D to satisfy DSI but is also sufficient, given that the formulation in (7) is not affected by proportional changes in all the incomes. We can then write

$$D = \sum_{i=1}^{n} w(r_i, s_i) \cdot \left[\alpha\sigma_i + \gamma_i\right], \qquad (8)$$

where $\sigma_i := q_i \cdot (1 - r_i)$ denotes the proportion of average income that needs to be transferred to county i in order to reach the same level of average income of the city, while $\gamma_i := q_i r_i G_i$ denotes the proportion of the Gini index of county i that contributes to the overall measure of deprivation. Note that in this case $q_i r_i = \frac{q_i \mu_i}{\mu} = \frac{q_i \mu_i}{\sum_{i=1}^{n} q_i \mu_i}$ denotes the proportion β_i of the average/city income that is distributed in county i. As a result $\sum_{i=1}^{n} \sigma_i = 0$, and $\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \beta_i G_i$ with $\sum_{i=1}^{n} \beta_i = 1$. Most interestingly, if we compute the sum of all σ_i for counties where $r_i < 1$, that is, for those counties with average income below the city average income, then we obtain the Schutz (inequality) index σ that measures the proportion of total/average income that needs to be redistributed, in our case across counties, in order to reach perfect equality in terms of average income between the counties. If we denote by N_L the set of low income counties where $r_i < 1$ then $\sigma := \sum_{i \in N_L} \sigma_i$.

In line with (8) each county deprivation is weighted considering whether it is perceived in a low income county or in a richer one and whether the level of segregation is large in the county.

We assume that the deprivation felt in relatively poorer counties should receive an higher weight than the one perceived in richer counties and that the level of segregation is more relevant in poorer counties where high segregation means that the income is distributed mainly on few ethnic groups, while the concern is reversed if one considers a relatively rich county.

We partition the geographic areas in three disjoint groups of counties on the basis of the value of r_i. The set N_H of high income counties where $r_i > 1$, the set N_L of low income counties where $r_i < 1$, and we consider (as a threshold) also the set N_M of middle income counties with $r_i = 1$. We assume that the weight function $w(r_i, s_i)$ is the same for all counties within each (income based) group and depends only on s_i. We therefore obtain three sets of weights $w_H(s_i)$, $w_L(s_i)$ and $w_M(s_i)$. These are ordered giving more relevance to the counties with lower average income. We formalize this assumption through the next condition.

Condition 2 *((IP) Income based priority) For all* $s, s', s'' \in [0, 1)$ *we have that* $w_L(s) \geq w_M(s') \geq w_H(s'') \geq 0$.

Moreover, the segregation level is considered relevant both for low and high income counties. In particular we assume that for low income counties the priority is given to the deprivation felt in highly segregated counties, while for high income counties the priority is given to the deprivation felt in counties with low segregation. For medium income counties instead the county weight is assumed independent from the segregation level. This view is formalized in the following condition.

Condition 3 *((SP) Segregation based priority) For all* $s, s' \in [0, 1)$ *we have that* $w_L(s) \geq w_L(s')$ *while* $w_H(s) \leq w_H(s')$ *and* $w_M(s) = w_M(s') = w_M$ *for all* $s > s'$.

The logic behind the SP condition is that the deprivation felt in more segregated counties is affecting unevenly specific ethnic groups. For this reason for low income counties is normatively more important to consider this deprivation because it affects specific ethnic groups in neighbors that are less affluent. While for more affluent neighborhoods it is less relevant whether the deprivation affects specific ethnic groups because on average they are in a better economic condition.

We can therefore write the aggregate deprivation in (8) as:

$$D = \sum_{i \in N_L} w_L(s_i) \left[\alpha\sigma_i + \gamma_i\right] + \sum_{i \in N_H} w_H(s_i) \left[\alpha\sigma_i + \gamma_i\right] + \sum_{i \in N_M} w_M \left[\alpha\sigma_i + \gamma_i\right]. \tag{9}$$

We can now derive robust deprivation evaluations that are valid for all sets of weights $w(r_i, s_i)$ that satisfy conditions IP and SP. For this purpose we construct the following curves that generate a sequential dominance condition. Consider the evaluation made at time t and denote by $D_L^t(k)$ the curve associated with the deprivation in the set N_L of counties with $s_i \geq 1-k$. While $D_H^t(k)$ is computed considering the deprivation in the set N_H of counties with $s_i \leq k$. That is:

$$D_L^t(k) := \sum_{i \in N_L^t : s_i^t \geq 1-k} \alpha\sigma_i^t + \gamma_i^t \text{ and } D_H^t(k) := \sum_{i \in N_H^t : s_i^t \leq k} \alpha\sigma_i^t + \gamma_i^t. \tag{10}$$

Moreover, consider the value $D_M^t := \sum_{i \in N_M^t} \alpha\sigma_i^t + \gamma_i^t$. The sequential dominance between the distributions Y^t and $Y^{t'}$ in a given city that are derived at two different moments in time t and t' generates a partial order across distributions and is denoted by $Y^t \preceq_D Y^{t'}$ meaning that distribution Y^t is considered at most as deprived as distribution $Y^{t'}$. Formally we write:

Definition 1 *The evaluation made at time t exhibits no more deprivation as the one at time t', that is $Y^t \preceq_D Y^{t'}$, if and only if the following conditions hold:*

(i) $D_L^t(k) \leq D_L^{t'}(k)$ *for all* $k \in [0,1]$,

(ii) $D_L^t(1) + D_M^t \leq D_L^{t'}(1) + D_M^{t'}$, *and*

(iii) $D_L^t(1) + D_M^t + D_H^t(k) \leq D_L^{t'}(1) + D_M^{t'} + D_H^{t'}(k)$ *for all* $k \in [0,1]$.

Note that once the two sets of conditions (i) and (iii) are computed for $k = 1$, assuming that there are no counties with $r_i = 1$, a necessary condition for $Y^t \preceq_D Y^{t'}$ is obtained that requires:

$$\sum_{i \in N_L^t} (\alpha\sigma_i^t + \gamma_i^t) \leq \sum_{i \in N_L^{t'}} (\alpha\sigma_i^{t'} + \gamma_i^{t'}) \text{ and} \sum_{i \in N} (\alpha\sigma_i^t + \gamma_i^t) \leq \sum_{i \in N} (\alpha\sigma_i^{t'} + \gamma_i^{t'}). \tag{11}$$

Moreover, it is also required that the maximum level of segregation in the counties in N_L^t is not larger than the one in those in $N_L^{t'}$.

The following proposition holds for the deprivation evaluations in (8) where $D(t) := \sum_{i=1}^n w(r_i^t, s_i^t) \cdot (\alpha\sigma_i^t + \gamma_i^t)$ and $D(t') := \sum_{i=1}^n w(r_i^{t'}, s_i^{t'}) \cdot (\alpha\sigma_i^{t'} + \gamma_i^{t'})$ denote respectively the evaluation based on Y^t and $Y^{t'}$.

Proposition 1 *$D(t) \leq D(t')$ for all weights $w(r,s)$ satisfying conditions IP and SP if and only if $Y^t \preceq_D Y^{t'}$.*

We can now further specify the set of conditions IP and SP by adding a requirement that simplifies the dominance criterion.

We split the set of realizations in terms of segregation by assuming that there is a segregation level $s^* \in (0,1)$ such that the weights take the same value for segregation levels that are on the same side with respect to s^*, more formally we set the following condition.

Condition 4 *((SI) Segregation Indifference) For all $s \in [0,1)$ and for $s^* \in (0,1)$ we have that* *(a)* $w_L(s) = w_L^+$ *if* $s \geq s^*$ *and* $w_L(s) = w_L^-$ *if* $s < s^*$;

(b) $w_H(s) = w_H^+$ *if* $s \leq s^*$ *and* $w_H(s) = w_H^-$ *if* $s > s^*$.

The combined effect of conditions IP, SP and SI leads to the following restrictions on the set of weights: $w_L^+ \geq w_L^- \geq w_M \geq w_H^+ \geq w_H^- \geq 0$.

As a result the set of comparisons in the sequential algorithm are restricted. We define the related dominance condition as follows:

Definition 2 *The evaluation made at time* t *exhibits no more [restricted] deprivation as the one at time* t'*, that is* $Y^t \preceq_{D^*} Y^{t'}$ *for* $s^* \in (0,1)$ *if and only if the following conditions hold:*

(i) $D_L^t(k) \leq D_L^{t'}(k)$ *for* $k \in \{1-s^*, 1\}$,

(ii) $D_L^t(1) + D_M^t \leq D_L^{t'}(1) + D_M^{t'}$, *and*

(iii) $D_L^t(1) + D_M^t + D_H^t(k) \leq D_L^{t'}(1) + D_M^{t'} + D_H^{t'}(k)$ *for all* $k \in \{s^*, 1\}$.

For each configuration one has to compute five values, and the two necessary conditions above in (11), if $r_i \neq 1$ for all i, become also sufficient if supplemented by the conditions computed at s^*, that is by

$$\sum_{i \in N_L^t : s_i^t \geq 1-s^*} (\alpha\sigma_i^t + \gamma_i^t) \leq \sum_{i \in N_L^{t'} : s_i^{t'} \geq 1-s^*} (\alpha\sigma_i^{t'} + \gamma_i^{t'}) \text{ and } \sum_{i \in N_H^t : s_i^t \leq s^*} (\alpha\sigma_i^t + \gamma_i^t) \leq \sum_{i \in N_H^{t'} : s_i^{t'} \leq s^*} (\alpha\sigma_i^{t'} + \gamma_i^{t'}). \tag{12}$$

We can now formalize next result.

Proposition 2 $D(t) \leq D(t')$ *for all weights* $w(r,s)$ *satisfying conditions IP, SP and SI for* $s^* \in (0,1)$ *if and only if* $Y^t \preceq_{D^*} Y^{t'}$.

Benchmark cases. We illustrate the dominance condition $Y^t \preceq_D Y^{t'}$ in Proposition 1 by referring to benchmark cases derived under simplifying assumptions. We consider first the necessary conditions in (11). By recalling that $\sum_{i=1}^{n} \sigma_i = 0$ and $\sigma := \sum_{i \in N_L} \sigma_i$ they can be rewritten as

$$\alpha\sigma^t + \sum_{i \in N_L^t} \gamma_i^t \leq \alpha\sigma^{t'} + \sum_{i \in N_L^{t'}} \gamma_i^{t'} \text{ and } \sum_{i \in N} \gamma_i^t \leq \sum_{i \in N} \gamma_i^{t'}. \tag{13}$$

If the inequality is the same in each county, that is, if $G_i^t = G^t$ for all $i \in N$ at time t, then the conditions become

$$\alpha\sigma^t + G^t \cdot I^t \leq \alpha\sigma^{t'} + G^{t'} \cdot I^{t'} \text{ and } G^t \leq G^{t'}. \tag{14}$$

where $I^t := \sum_{i \in N_L^t} \beta_i^t$ denotes the proportion of total income distributed in the low income counties in N_L^t. The first condition is a weighted combination of the Schutz index and a condition that is reminiscent of the Gini based poverty measure in Sen (1976). The second condition simply requires a lower Gini index in all the counties at time t with respect to time t'.

If there is no inequality in each county then the conditions boil down to $\sigma^t \leq \sigma^{t'}$. In this case the sources of disparities between the counties are only the differences in segregation and σ_i. Deprivation then becomes $D = \alpha \sum_{i=1}^{n} w(r_i, s_i) \cdot \sigma_i$.

In general, if segregation decreases in the low income counties, all things kept equal, then the less segregated configuration dominates the initial one.

If there is no segregation in each county, the necessary conditions in (13) turn out also to be sufficient.

If all the counties exhibit the same average income, that is if $\mu_i^t = \mu^t$ and therefore $r_i^t = 1$ for all i, then the dominance conditions in Proposition 1 boil down to $D_M^t \leq D_M^{t'}$ that also coincides with $\sum_{i \in N} \alpha\sigma_i^t + \gamma_i^t \leq \sum_{i \in N} \alpha\sigma_i^{t'} + \gamma_i^{t'}$. Then noticing that in this case $\beta_i^t = q_i^t$ and $\sigma_i^t = 0$, the condition simply requires that $\sum_{i \in N} \gamma_i^t = \sum_{i=1}^{n} q_i^t G_i^t \leq \sum_{i \in N} \gamma_i^{t'} =$

$\sum_{i=1}^{n} q_i^{t'} G_i^{t'}$. With no disparities on average between counties the only relevant component for deprivation is an average measure of the inequalities within the counties.

These type of results can be extended if one considers a more general deprivation model where the definition of the middle income counties in N_M is not only restricted to the special case where $r_i = 1$, but considers an interval of values for r_i around this point.

3 Deprivation in American cities

In this section we illustrate the pattern of deprivation in two selected American MSAs: Chicago-Naperville-Elgin (IL-IN-WI) and New York-Newark-Jersey (NY-NJ-PA) MSAs, which we label *Chicago* and *New York* for simplicity. The geographic extension of each MSA is given by the Census Bureau definition of American MSAs based on the Census 2000 boundaries (the midpoint in our analysis). Using such definition, we identify the urban counties making up the territory of each of the MSAs. The demographic extension of the cities is hence stable over time and span over multiple counties and potentially across multiple states. We hence end up with a variable number of counties across municipalities which define our data points: 14 counties for Chicago and 25 counties for New York.

The purpose of this section is to illustrate the applicability and usefulness of the deprivation rankings discussed above. We hence limit our analysis to two cities and two periods, 1990 and 2012. This allows to study the evolution of deprivation over time and across the two cities. A broader set of results, as well as robustness checks, to which these are an extract, can be found in Andreoli, Prete and Zoli (2023).

As shown in Figure 1, the period 1990-2012 has been a decade of economic progress for all cites and across all counties, as measured by the average equivalent household income, even after accounting for MSA-specific changes in cost of living. As shown in Andreoli et al. (2023) the data in 2012 are affected by the consequences of the Great

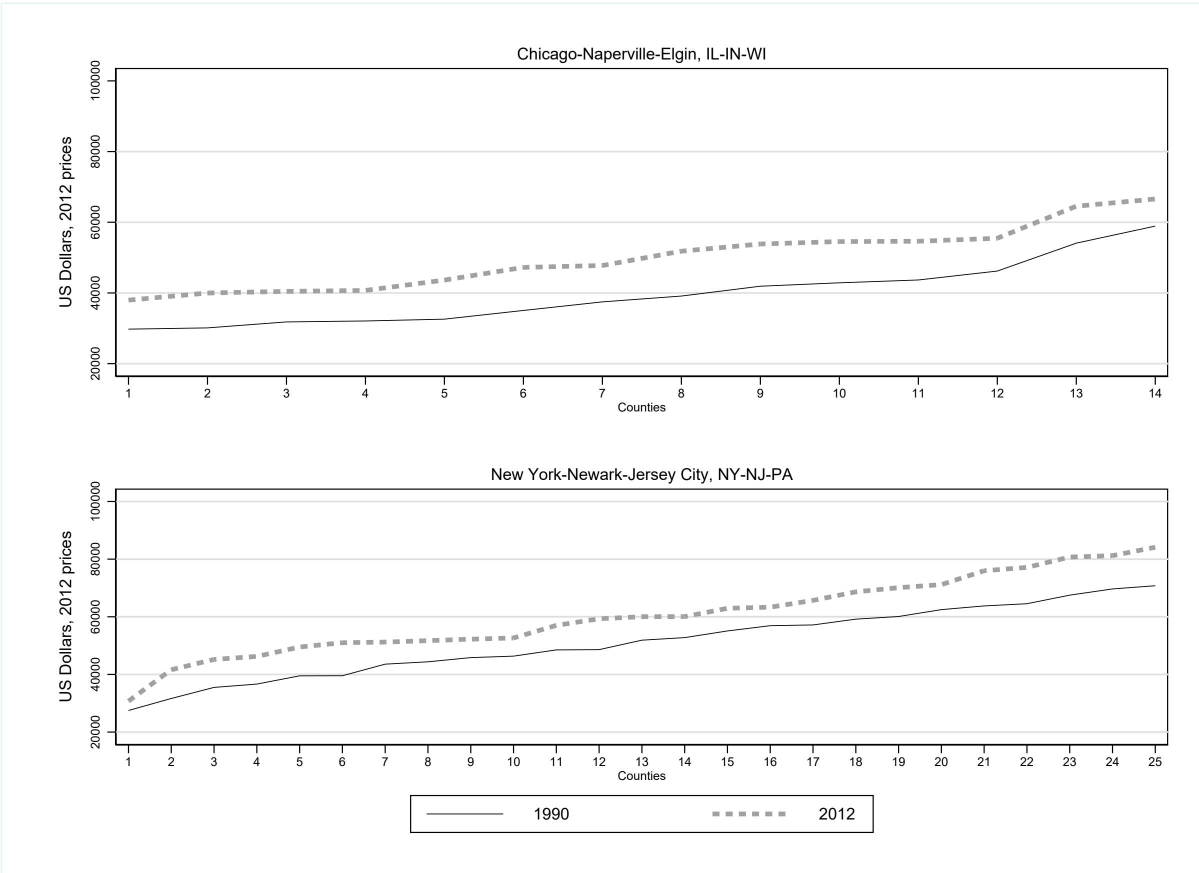

Figure 1: Mean equivalent household income in 2012 dollars, by county-city-year.
Note: Data: 1990 US Census and 2010-2014 ACS 5-years estimates. CPI (all urban consumers) at MSA level is from the US BLS.

Recession: average incomes in 2012 are not dissimilar from average income registered in early 2000, the effect being particularly strong for counties where average income is closer or above the respective city average. Despite large progress, the Great Recession may have considerably affected other distributional features of the income distribution.

In this section we investigate the extent at which such changes have impacted urban deprivation as assessed by the criterion $\preceq_D$ characterized in Proposition 1.

3.1 Data

Data are from the 1990 American census as well as from the 2010-2014 5-years module of the American Community Survey (ACS). Data from ACS are assumed to be representative for the distribution of income in American cities for the target year 2012.[5]

Census and ACS data come in the form of tables: for different spatial partitions of the American territory (states, counties, census tracts up to block groups), the data report information about the income distribution therein (average income, Gini index and income quantiles) and the demographic composition of the territories (total population and population by ethnic group).

Following Andreoli and Peluso (2017), we construct first distributions of household income, equivalized by the square root scale rule, that are representative at the county level. The so-obtained income distribution is representative at the equivalent individual level. We hence interpreted aggregate income statistics at the county level (mean income and the Gini inequality index) as representative for this individual in each county. Aggregations across counties are performed by weighting county estimates by each county relative demographic size, which deliver estimates at the city level.[6] All income measures are in constant 2012 dollars, which we have obtained by scaling income statistics by the relative MSA-specific CPI (all urban consumers) series available on the Bureau of Labour Statistics website. Figure 1 displays county average incomes. In the figure, counties are ordered from the poorest to the richer in each of the three years (so that position 1 always identifies the poorest county in every year).

We use population counts to determine counties demographic weights (q_i in the model).

[5]Differently from the decennial census, ACS reports estimates drawn from a collection of yearly surveys representative of very fine geographic partitions of the US. Pooling multiple consecutive yearly waves of these surveys allows to produce reliable estimates of counties income distributions.

[6]All estimates are representative and valid conditional on the choice of the equivalence scale transformation. Considering alternative transformations would lead to a more sophisticated set of dominance relations in the spirit of Atkinson (1992). We abstract from this issue in this paper.

The two cities display very different patterns in terms of demographic composition. Counties in New York are of comparable demographic size (albeit heterogeneous), with larger counties being also among the poorest in this city. Conversely, Chicago is dominated by a large county making up to more than 60% of the urban population, whereas the rest of the counties score to less than 10% of the residents. In Chicago, deprivation in a large county is likely to drive deprivation in the city. The effect however, is not easy to predict, as the dominating county scores (as expected) relatively close to the city average in terms of household income.

The census and ACS also report population counts for each ethnic group at the county level, which we use to construct county-specific measures of ethnic segregation. We abide to the census definition of ethnic groups: White, Black, Hispanic, Asian, natives and other groups. We then compute the share of population in any given county of an MSA that belongs to each ethnic group (q_i^h in the model) and we make use of population counts by ethnic groups to measure ethnic segregation at the county level, using the dissimilarity index as a benchmark (denoted s_i in the model). Our data show that both in New York and Chicago, segregation is usually above 0.1 across counties and years and reaches 0.4 for some counties. We do not detect any relevant sign of correlation between segregation and county size or county income in these cities. Overall, measured segregation does not vary significantly across time in both cities.[7]

3.2 Results: Deprivation and average incomes

We first describe the segregation-sensitive deprivation curve $D(k)$ obtained after setting $\alpha = 1$ and $\gamma_i = 0$ for all counties i. Such a curve measures deprivation in the city by cumulating deprivation experienced by the representative individual in each county,

[7]Data and codes are available in the replication package of this paper. We refer to Figures 2 and 3 in Andreoli et al. (2023) for a visual description of the data on population density and segregation by county-year for a broader number of cities and periods.

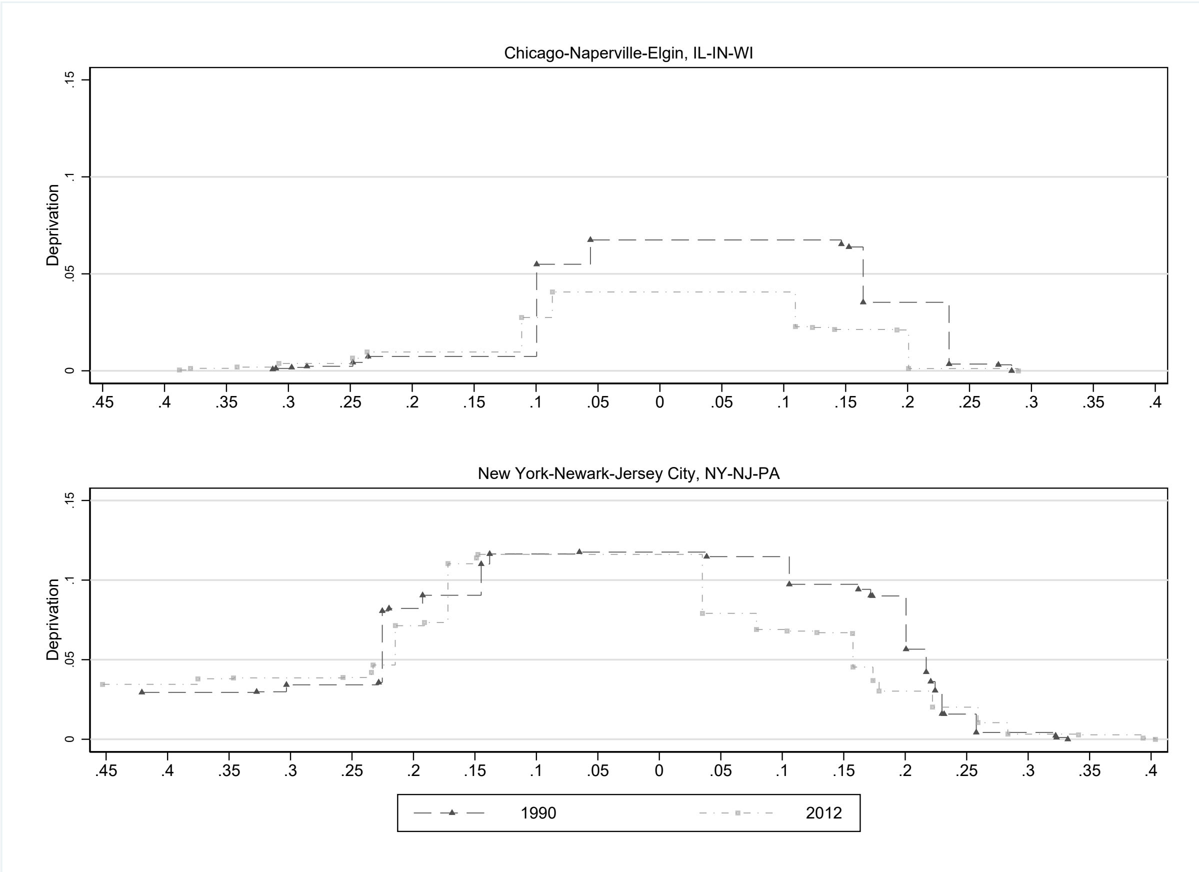

Figure 2: Deprivation curve $D(k)$ with $\alpha = 1$ and $\gamma_i = 0$ for all counties i, by city and year.

were counties are weighted according to the segregation they display. Deprivation in each county is the gap between the city average income and the county average income. Cumulative deprivation is displayed in Figure 2. In the figure, the deprivation curve coordinates are reported on the vertical axis. The horizontal axis reports instead the level of segregation observed across counties. The segregation scale is symmetric with respect to the point marked with "0". Counties with average income below the city average (that is, where $r_i < 1$) are on the left hand-side of this threshold. These counties are ranked in decreasing order of segregation. Then, their deprivation scores are cumulated according to this ranking and generate the intercept of the component $D_L(k)$ of the deprivation curve. Any county contributes to the curve D_L according to the income gap measured therein and its demographic size. The curve is hence stepwise increasing below the city average. As expected, there are no counties exactly at the average, implying that $D_M = 0$. The remaining counties are those more affluent where $r_i > 1$. They are ranked in increasing order according to the degree of segregation that they display, which is why the scale of the horizontal axis is increasing in values on the right hand-side of the "0" threshold. On this side of the graph, the deprivation curve is $D_L(1) + D_H(k)$ and counties contribute negatively to its level (given that their income gap $1 - r_i$ is negative). This explains the inverted-U shape of the curves.

We compute deprivation curves for both Chicago and New York in 1990 and in 2012. The scale of the graphs is fixed in Figure 2, so that visual inspection of the curves allows to conclude on robust changes in deprivation across years and cities.

Across years, we do not detect any sign of dominance in deprivation. Despite the steep improvement in average incomes from 1990 to 2012, there has been little dynamics in deprivation for the group of counties with incomes below the respective city averages. Average income in these counties has remained relatively stable over the Great Recession. Overall, the patters of the deprivation curves are indistinguishable in the group of low-income counties.

Deprivation patterns become more clear-cut when looking at the right-hand side of the graphs. The direction of changes are specific to each city. In Chicago, deprivation has substantially reduced since 1990 when considering the contribution of most affluent counties. New York displays a comparable pattern, with deprivation patterns being reduced substantially in 2012. In both cases, this pattern can be explained by the fact that income has grown faster in high-income counties in the period where the whole city has experienced income growth whereas income in these counties has reduced by effect of the Great Recession, thus reducing inequalities across counties.

Comparing curves across cities reveals a clear ranking of deprivation if we do not consider inequality within the counties: $Chicago \preceq_D New\ York$, which is stable over time. There is, nonetheless, evidence of convergence over time across these cities.

3.3 Results: Deprivation, average incomes and inequality

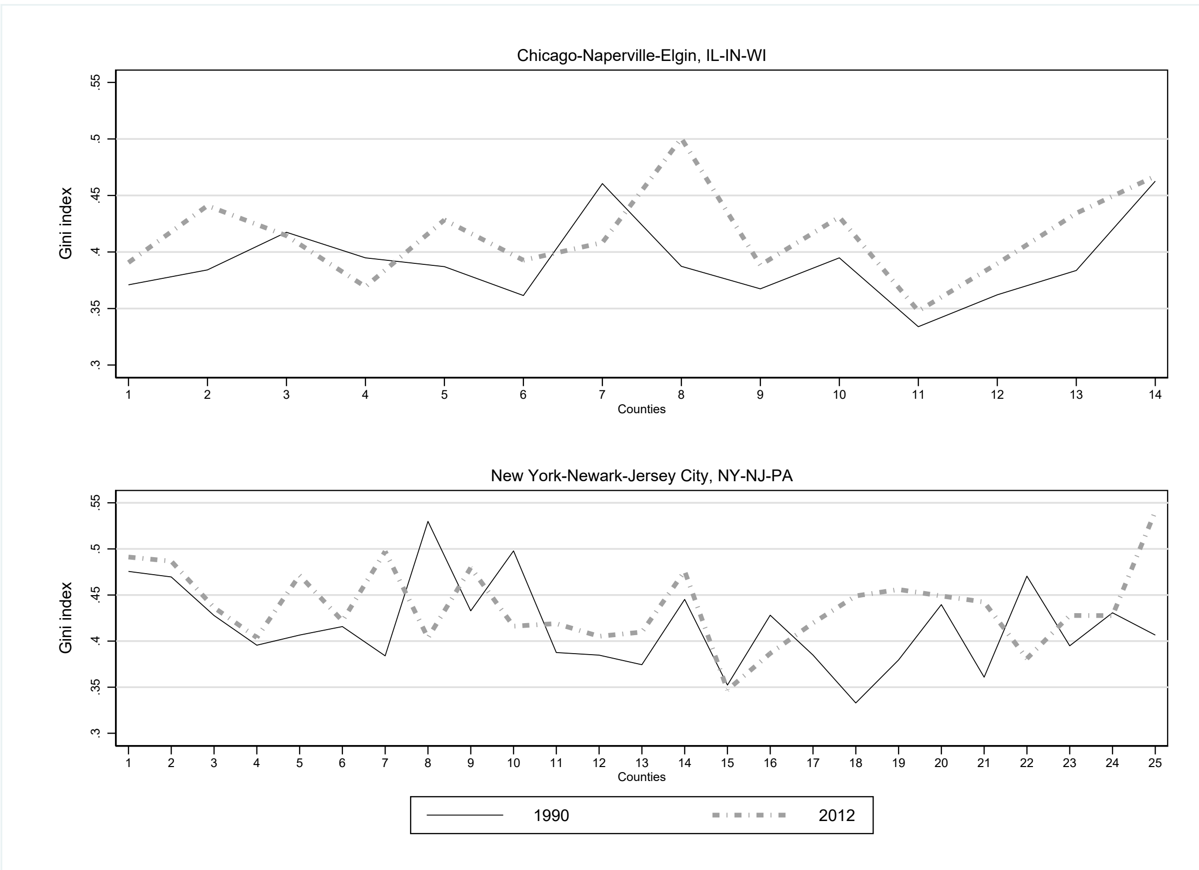

Figure 3: Inequality (Gini index G_i), by county-city-year.

The deprivation evaluations based on Figure 2 do not consider the role of income heterogeneity *within* counties. More unequal counties may suffer from an additional burden of deprivation, insofar deprivation is more likely to concern a large number of poor households in high-inequality counties. According to our model we use the Gini index to measure inequality in equivalent household income at the county-year level. Estimates of the Gini indices are in Figure 3. We do not detect a clear association between affluence, size and segregation across counties in Chicago and in New York. Including inequality considerations into the deprivation analysis has nonetheless consequences for the rankings of periods and cities discussed so far.

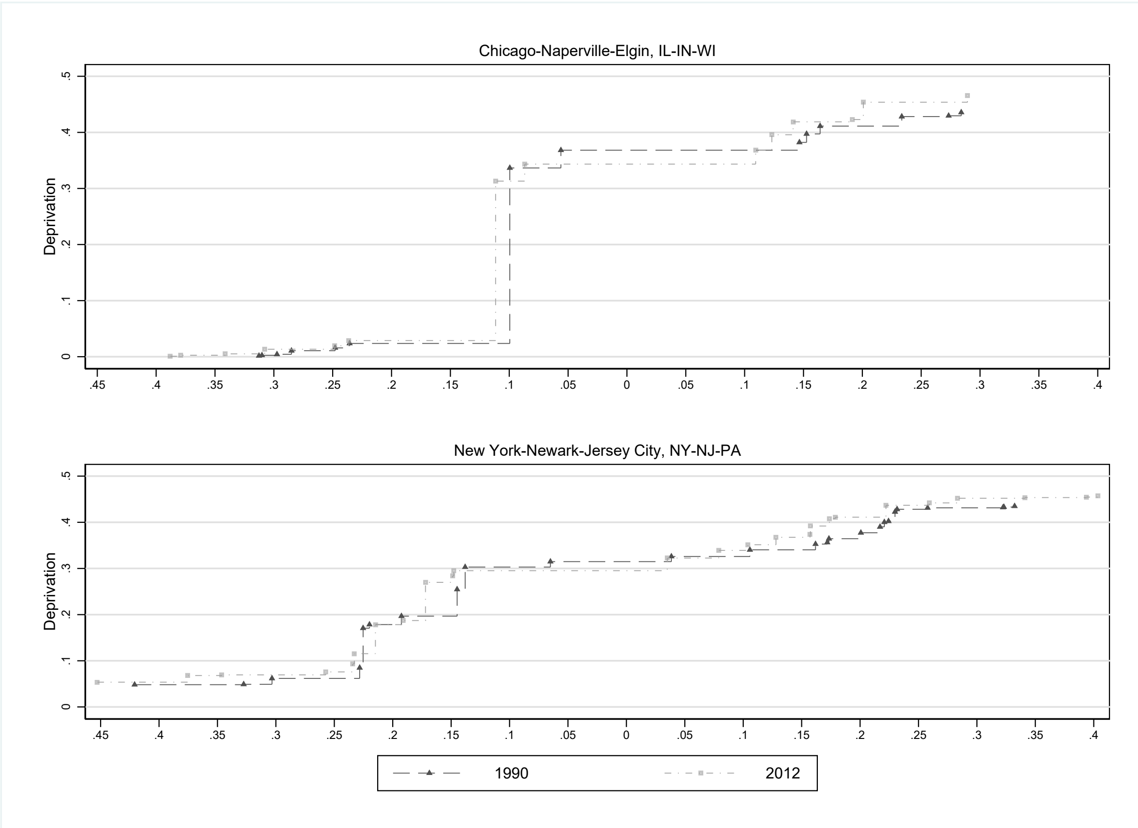

Figure 4: Deprivation curve D with $\alpha = 1$, by city and year

We analyze deprivation by mean of the segregation and inequality sensitive deprivation curves, which is based on county-specific measures of deprivation δ^{α} weighting the county income gap $1 - r_i$ by the county level of inequality G_i. We consider the benchmark case in which $\alpha = 1$.[8]

The deprivation curves of interest are in Figure 4. The curves are constructed and interpreted as in Figure 2, except that county-level deprivation is based on the deprivation indicator δ, which combines information on income gaps and a penalty due to inequality

[8]Conclusions are qualitatively robust to the choice of α, as shown in Andreoli et al. (2023).

in the county. The curve need not be decreasing in the right hand-side of the mean income threshold if the penalty is large enough to amplify cross-counties inequalities in average incomes. This is the case for both Chicago and New York.

Interestingly enough, we do not detect any sign of progress in terms of segregation-sensitive deprivation in both cities. In fact, we reject any form of dominance over time, thus challenging results previously obtained. When within-county inequality is put into perspective, we find that much of the progress observed with average incomes vanishes. Conversely, even the reductions in deprivation that were highlighted by the deprivation curves in Figure 2 have been wiped away. When within-county inequality is put into perspective, we rather find evidence that deprivation has increased over time, in a similar manner both in New York and Chicago. Not only deprivation in 2012 is larger than in 1990 in most-segregated low-income counties, as in the previous Figure, but deprivation has also increased slightly in high-income, high-segregated counties in both cities. While overall the curves do not allow to conclude on robust changes in deprivation (due to multiple intersections occurring in average-income counties), there is evidence of rising deprivation, driven by an increase in income inequality among high-income counties.

Based on Figure 2, it is difficult to draw conclusions about dominance in deprivation *across* cities, since curves would cross at multiple points across panels of the figure. Low-income counties in Chicago display less deprivation than the corresponding counties in New York, irrespectively of their priority (measured on the segregation scale). However, the curves of the two cities cross in correspondence of the largest county in Chicago, and the order of the curves reverts, thus making it difficult any conclusive assessment. The weaker dominance condition characterized in Proposition 2, as well as the deprivation index D, may be useful to draw more conclusive inference about deprivation over time and cities.

In Table 1 we report relevant statistics that can be used to perform the dominance test

Year	$D(t)$ dominance							
	$D^t_L(k)$			$D^t_L(1)+D^t_M$	$D^t_L(1)+D^t_M+D^t_H(k)$			D^t
	$s^*=0.3$	$s^*=0.2$	$s^*=0.1$		$s^*=0.1$	$s^*=0.2$	$s^*=0.3$	
Chicago								
1990	0.002	0.023	0.023	**0.368**	0.368	0.411	0.435	**0.435**
2012	0.013	0.028	0.313	**0.343**	0.343	0.422	0.465	**0.465**
New York								
1990	0.061	0.178	0.302	**0.314**	0.325	0.364	0.431	**0.435**
2012	0.069	0.178	0.295	**0.295**	0.339	0.411	0.452	**0.457**

Table 1: The deprivation criterion in Proposition 2 for multiple alternative choices of s^* and the deprivation index D ($\alpha = 1$).

in Proposition 2. The test can be performed on the basis of four coordinates: a measure of deprivation for the low-income counties corresponding to benchmark segregation s^* (the statistic $D^t_L(k)$), the deprivation level corresponding to the middle-income class ($D^t_L(1) + D^t_M$, remember that in our data there is no county located exactly at the average income in the city, implying that the class M is empty), the deprivation for the high-income counties corresponding to benchmark segregation s^* (the statistics $D^t_L(1) + D^t_M + D^t_H(k)$) and the deprivation in the city at a certain year (D^t). The table reports information on deprivation for Chicago and New York in separate panels and for different years. The deprivation statistic corresponds to the intercepts of the city-year deprivation curves obtained in correspondence of different segregation levels. We report three potential alternative choices of s^* for robustness checks: $s^* = 0.1$, $s^* = 0.2$ and $s^* = 0.3$. While comparisons should be based on one of the three thresholds, the table leaves however some degrees of flexibility in selecting the relevant threshold. The numbers in bold-faced character in the table are instead fixed. Focusing on these sets of numbers, intertemporal dominance in deprivation cannot be established in any of the two cities. In fact, in both cities the deprivation among the low and middle income counties is larger in 1990 than it was in 2012, whereas the ranking reverses when the focus is on deprivation among all counties in each city. This reversal reflects the fact that the high-income counties have substantially contributed rising deprivation in 2012 compared to 1990 in both Chicago and New York, more so among counties with a moderate to high level of segregation (i.e., when $s^* = 0.2$ or $s^* = 0.3$). The table shows some evidence that deprivation has also increased among low-income counties with a moderate to high level of segregation. Overall, there is evidence that deprivation has increased in 2012 in both Chicago and New York, although a clear patterns of dominance cannot be established because of the contribution of changes in deprivation in low to middle-income counties (i.e. counties whose income is close to the city average) that display sufficiently low levels (below 0.2) of ethnic segregation.

The data in Table 1 are also useful to compare Chicago and New York in the same year. In 1990, there is no clear pattern of dominance between the two cities, irrespectively of the choice of s^*. More so, we also find that the two cities display the same citywide level of deprivation, equal to 0.435. In 2012, we find robust evidence of dominance: $New - York \preceq_{D^*} Chicago$ if priority is given to neighborhoods with high-level of segregation (that is if the chosen threshold satisfies $s^* \leq 0.2$). Overall, deprivation in Chicago is found to be $D^{2012} = 0.465$ whereas in New-York is $D^{2012} = 0.457$. Such a ranking (consistent with the dominance criterion in Proposition 2), reverses the robust ranking identified in Figure 2, thus highlighting the important role of within-county inequality for deprivation analysis.

4 Conclusions

Differences in incomes and inequalities between neighborhoods are relevant to assess deprivation measured within a city. In this paper we take the view that the geographical dispersion of incomes could lead to more salient inequalities if they are perceived in more segregated area that exhibit the predominance of some ethnic groups and if the population living in these area is experiencing also lower incomes. In accordance with this view we characterize a measurement model that allows to assess the combined effect of segregation and inequalities at city level making use of available aggregate data at county level. The robust dominance conditions derived are applied for the analysis of the dynamics of relative deprivation for MSAs between 1990 and 2012. Even though the changes occurred in the relevant variables across the two decades do not allow to reach a robust assessment, our investigation highlights a trend in the increase of segregation sensitive spatial income deprivation in the more recent decade driven by the higher income counties.

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Appendix

Proof of Propositions 1 and 2.

We illustrate here the proof of the results in Propositions 1 and 2.

For *Propositions 1*, consider the deprivation index in (9). All weights $w(r,s)$ satisfying conditions IP and SP are such that $w_L(s) \geq w_M \geq w_H(s) \geq 0$ for all $s \in [0,1)$ with $w_L(s)$ non-decreasing in s and $w_H(s)$ non-increasing in s.

Recall that any monotonic continuous function, is the limit of a sequence of step functions that combine indicator functions (see Ch. 1 in Asplund and Bungart 1966). In our case we can consider non-decreasing functions, as $w_L(s)$, that could exhibit a countable number of discontinuities and adopt as "bases" the indicator functions $\omega_z(s)$ such that, for $z \in [0,1)$,

$$\omega_z(s) := \begin{cases} 0 & if\ 0 \leq s < z \\ 1 & if\ 1 > s \geq z \end{cases} . \tag{15}$$

It is then possible to write any admissible weighting function $w_L(s)$ as the limit of $w_M + \sum_{z\in\mathcal{Z}} a_z\omega_z(s)$ with $a_z > 0$ for an appropriate countable set $\mathcal{Z} \subseteq [0,1]$ leading to a finite value for $\sum_{z\in\mathcal{Z}} \alpha_z$. For each county i let $x_i(s_i) := \alpha\sigma_i + \gamma_i$ denote the variable that maps the county level of segregation s_i into the average county deprivation $\alpha\sigma_i + \gamma_i$. We can then write $\sum_{i\in N_L} w_L(s_i)\,[\alpha\sigma_i + \gamma_i]$ as $\sum_{i\in N_L}\left[w_M + \sum_{z\in\mathcal{Z}} a_z\omega_z(s)\right]\cdot x_i(s_i)$ that is, $w_M\sum_{i\in N_L} x_i(s_i) + \sum_{z\in\mathcal{Z}} a_z \sum_{i\in N;s_i\geq z} x_i(s_i)$. Similarly we can write the non-increasing functions $w_H(s)$ considering the indicator function $\omega'_{z'}(s) = 1$ if $0 \leq s \leq z'$ that takes value 0 if $1 > s \geq z'$. We obtain $w_H(s) = \sum_{z'\in\mathcal{Z}'} b_{z'}\omega_{z'}(s)$, with $b_{z'} > 0$ for an appropriate countable set $\mathcal{Z}' \subseteq [0,1]$ leading to a finite value for $\sum_{z'\in\mathcal{Z}'} b_{z'} \leq w_M$. We can then write $\sum_{i\in N_H} w_H(s_i)\,[\alpha\sigma_i + \gamma_i]$ as $\sum_{z'\in\mathcal{Z}'} b_{z'} \sum_{i\in N_H;s_i\leq z'} x_i(s_i)$.

Combining these results into the definition in (9) and recalling that $D_L(1-z) = \sum_{i\in N;s_i\geq z} x_i(s_i)$, $D_H(z') = \sum_{i\in N_H;s_i\leq z'} x_i(s_i)$, $D_M^t := \sum_{i\in N_M} x_i(s_i)$ and that by definition

$\sum_{i \in N_L} x_i(s_i) = D_L(1)$, we obtain that

$$D = w_M D_L(1) + \sum_{z \in \mathcal{Z}} a_z D_L(1-z) + \sum_{z' \in \mathcal{Z}'} b_{z'} D_H(z') + w_M D_M. \tag{16}$$

The level D can be specified as a combination of the following values obtained for specific values of $a_z, b_{z'}$ and w_M. Setting $w_M = 0$ and $b_{z'} \to 0$ for all $z' \in [0,1]$, $a_{z_0} > 0$ and $a_z \to 0$ for all $z \in [0,1]$ s.t. $z \neq z_0$ one obtains $D = a_{z_0} D_L(1-z_0)$ for $z_0 \in [0,1]$. Setting $w_M > 0$, $b_{z'} \to 0$ for all $z' \in [0,1]$, and $a_z \to 0$ for all $z \in [0,1]$ we obtain $D = w_M[D_L(1) + D_M]$. Setting $w_M > 0$, $b_{z'_0} = w_M$ and $b_{z'} \to 0$ for all $z' \in [0,1]$, s.t. $z' \neq z'_0$, and $a_z \to 0$ for all $z \in [0,1]$ we obtain $D = w_M[D_L(1) + D_M + D_H(z'_0)]$ for $z'_0 \in [0,1]$. All the other specifications of D can be obtained combining these three sets of values. If one compares $D(t)$ with $D(t')$ and sets $z_0 = 1-k$ and $z'_0 = k$, one obtains the conditions in Proposition 1. These conditions are necessary for $D(t) \leq D(t')$ because are derived as special cases of the deprivation indices, but are also sufficient because all the deprivation indices considered can be written as a combination of the associated special indices derived from (16).

For *Proposition 2* the set of deprivation indices that satisfy in addition condition SI are obtained from (16) by setting $a_0 = w_L^- - w_M$, $a_{s^*} = w_L^+ - w_L^-$, $a_z \to 0$ for all $z \in (0,1], z \neq s^*$, and $b_1 = w_H^-$, $b_{s^*} = w_H^+ - w_H^-$, $b_{z'} \to 0$ for all $z' \in [0,1)$, $z' \neq s^*$, where $w_L^+ \geq w_L^- \geq w_M \geq w_H^+ \geq w_H^- \geq 0$. As a result

$$\begin{aligned} D \;=\; & \left(w_L^+ - w_L^-\right) D_L(1-s^*) + \left(w_L^- - w_M\right) D_L(1) + w_M \left[D_L(1) + D_M\right] \\ & + \left(w_H^+ - w_H^-\right) D_H(s^*) + w_H^- D_H(1), \end{aligned} \tag{17}$$

where all coefficient of the elements in the sum are non-negative. By setting either $w_L^+ - w_L^- > 0$ or $w_L^- - w_M > 0$ and all the other coefficients equal to 0, we obtain the conditions in (i) of Proposition 2. If all coefficients are set equal to 0 except for $w_M > 0$, we obtain the condition in (ii). If $w_L^+ - w_L^- = w_L^- - w_M = w_H^- = 0$ and $w_M = w_H^+ - w_H^- > 0$

we obtain the first condition in (iii), and if $w_L^+ - w_L^- = w_L^- - w_M = w_H^+ - w_H^- = 0$ and $w_M = w_H^- > 0$ we obtain the second condition in (iii). These necessary conditions are also sufficient because the index in (17) could be obtained combining with non-negative weights the specifications above that are associated with the necessary conditions.