KRULL–GABRIEL DIMENSION AND THE ZIEGLER SPECTRUM

ROSANNA LAKING

In this survey we describe the notion of the Krull–Gabriel dimension $\mathrm{KG}(R)$ of a ring R (or more generally of a small preadditive category) and how it relates to notions of purity and the Ziegler spectrum of a module category.

The dimension was first defined in this form by Geigle [?Gei] as a variation of the Krull dimension of an abelian category $\mathcal{A}$ defined by Gabriel in [?Gab] (this has subsequently been referred to as the Gabriel dimension of $\mathcal{A}$). When R is a finite-dimensional algebra, the Krull–Gabriel dimension of R is zero if and only if R is of finite representation type [?Aus]. In this context, it seems reasonable to consider the Krull–Gabriel dimension as a measure of how far R is from finite representation type, however the precise connection between representation type and Krull–Gabriel dimension is far from well-understood.

In order to define $\mathrm{KG}(R)$, we consider a transfinite filtration of the category $\mathcal{F} = \mathrm{F}_{\mathrm{fp}}(R)$ of finitely presented functors from mod-R to the category Ab of abelian groups. The initial part of the filtration is given by the sequence

$$0 = \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \cdots \subseteq \mathcal{F}_n \subseteq \cdots$$

of Serre subcategories $\mathcal{F}_n$ of $\mathcal{F}$ consisting of the objects that become finite length in $\mathcal{F}/\mathcal{F}_{n-1}$ for $n \geq 0$. By taking unions at limit ordinals, this process may be continued transfinitely in the obvious way. We define $\mathrm{KG}(R) := \min\{\alpha \mid \mathcal{F}_\alpha = \mathcal{F}\}$ if such an ordinal exists and $\mathrm{KG}(R) = \infty$ otherwise.

There is a close relationship between the Serre localisations of $\mathrm{F}_{\mathrm{fp}}(R)$ and the Ziegler spectrum $\mathrm{Zg}(R\text{-Mod})$ of R-Mod. The Ziegler spectrum is a topological space with its underlying set given by isomorphism classes of indecomposable pure-injective objects in R-Mod. Both the closed subsets of $\mathrm{Zg}(R\text{-Mod})$ and the Serre localisations of $\mathrm{F}_{\mathrm{fp}}(R)$ are parametrised by the hereditary torsion pairs of finite type in the category $\mathrm{F}(R)$ of all functors from mod-R to Ab. In particular, if the Krull–Gabriel dimension of R is defined then it coincides with the Cantor–Bendixson rank of $\mathrm{Zg}(R\text{-Mod})$.

Throughout this paper we will consider modules over a skeletally small preadditive category $\mathcal{R}$. One motivation for this level of generality is that it encompasses the theory of purity in a compactly generated triangulated category. See Example **??** for more details. References will often be given for the case where $\mathcal{R}$ is a ring as well as the more general case.

The paper is organised as follows. In the first section we outline notions of purity in categories of $\mathcal{R}$-modules. There are many equivalent ways of giving these definitions but we favour the definitions given in terms of the embedding of $\mathcal{R}$-Mod

2010 *Mathematics Subject Classification.* Primary 16G99, 18E15 ; Secondary 16G60, 18E30.
The author was supported by the DFG SFB / Transregio 45 during the period that this article was written.

into $\mathrm{F}(\mathcal{R})$. Similarly, we follow the definition of the Ziegler spectrum with an explanation of how the topology can be seen in terms of localisations of $\mathrm{F}(\mathcal{R})$. In the second section we define the Krull–Gabriel dimension of $\mathcal{R}$ and explain the connections between this dimension and the Cantor–Bendixson rank of $\mathrm{Zg}(\mathcal{R}\text{-Mod})$. Finally, in Section 3 we give some examples where the Krull–Gabriel dimension of $\mathcal{R}$ has been calculated. We first give examples where $\mathcal{R}$ is a ring: serial rings, Dedekind domains, tame hereditary algebras, string algebras and canonical algebras. We also give examples of compactly generated triangulated categories where the dimension has been calculated; these include derived and homotopy categories of derived-discrete algebras, and derived categories of hereditary algebras.

1. Purity in categories of modules

Throughout this survey, let $\mathcal{R}$ denote a skeletally small preadditive category. In this section we introduce pure-exact sequences in the category of $\mathcal{R}$-modules. We then discuss the theory surrounding the pure-injective $\mathcal{R}$-modules i.e. the injective objects relative to the pure-exact structure. In particular, we consider the Ziegler spectrum of the module category.

Definition 1.1. A **left $\mathcal{R}$-module** is a covariant (additive) functor $M\colon \mathcal{R} \to \mathrm{Ab}$ where Ab denotes the category of abelian groups. A left $\mathcal{R}$-module M is **finitely presented** if there exists an exact sequence

$$\bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathcal{R}}(r_i, -) \to \bigoplus_{j=1}^{m} \mathrm{Hom}_{\mathcal{R}}(s_i, -) \to M \to 0$$

for objects $r_i, s_j \in \mathcal{R}$. We will denote the category of left $\mathcal{R}$-modules by $\mathcal{R}$-Mod and the full subcategory of finitely presented left $\mathcal{R}$-modules by $\mathcal{R}$-mod.

A **(finitely presented) right $\mathcal{R}$-module** is a (finitely presented) left $\mathcal{R}^{\mathrm{op}}$-module. We will denote the category of right $\mathcal{R}$-modules by Mod-$\mathcal{R}$ and the full subcategory of finitely presented right $\mathcal{R}$-modules by mod-$\mathcal{R}$.

Note that if $\mathcal{R}$ has a single object r, then $\mathrm{End}_{\mathcal{R}}(r)$ is a unital ring and the above are equivalent to the usual definitions.

Notation 1.2. The category mod-$\mathcal{R}$ is a small preadditive category and we will often consider the category of left (mod-$\mathcal{R}$)-modules. To avoid cumbersome notation we will fix the following notation for these particular module categories:

$$\mathrm{F}(\mathcal{R}) := (\text{mod-}\mathcal{R})\text{-Mod} \qquad \mathrm{C}(\mathcal{R}) := \text{Mod-}(\mathcal{R}\text{-mod})$$

$$\mathrm{F}_{\mathrm{fp}}(\mathcal{R}) := (\text{mod-}\mathcal{R})\text{-mod} \qquad \mathrm{C}_{\mathrm{fp}}(\mathcal{R}) := \text{mod-}(\mathcal{R}\text{-mod})$$

Since it will always be clear from the context which category we are working in, we will denote representable functors $\mathrm{Hom}_{\mathcal{R}}(r, -)$ and $\mathrm{Hom}_{\mathcal{R}}(-, r)$ by $(r, -)$ and $(-, r)$ respectively.

Let M be an object in Mod-$\mathcal{R}$. We will also use the notation $(M, -)$ and $(-, M)$ for the representable functors $\mathrm{Hom}_{\text{Mod-}\mathcal{R}}(M, -)$ and $\mathrm{Hom}_{\text{Mod-}\mathcal{R}}(-, M)$ restricted to mod-$\mathcal{R}$.

1.1. **The pure-exact structure.** Next we introduce the pure-exact structure on $\mathcal{R}$-Mod. There are many equivalent definitions of a pure-exact sequence; we place an emphasis on those given by embedding $\mathcal{R}$-Mod into a category of functors since this is in line with the perspective taken in the subsequent sections.

Definition 1.3 (See, for example, [?JL, Lemma 1.2.13])**.** Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathcal{R}$-Mod. If the following equivalent statements are satisfied, then the sequence is called a **pure-exact sequence**, the morphism f is called a **pure monomorphism** and g is called a **pure epimorphism**.

(1) The sequence
$$0 \to - \otimes_{\mathcal{R}} X \to - \otimes_{\mathcal{R}} Y \to - \otimes_{\mathcal{R}} Z \to 0$$
is exact in $\mathrm{F}(\mathcal{R})$.

(2) The sequence
$$0 \to (-, X) \to (-, Y) \to (-, Z) \to 0$$
is exact in $\mathrm{C}(\mathcal{R})$.

(3) The sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is isomorphic to a filtered colimit of split exact sequences.

The first two conditions in Definition **??** allude to the following fully faithful embeddings of $\mathcal{R}$-Mod into larger module categories.

The (restricted) tensor embedding: Consider the functor $\Phi\colon \mathcal{R}\text{-Mod} \to \mathrm{F}(\mathcal{R})$ defined by $M \mapsto - \otimes_{\mathcal{R}} M$ and $f \mapsto - \otimes_{\mathcal{R}} f$ for all $\mathcal{R}$-modules M and morphisms f. The functor Φ has the following properties.

- Φ induces an equivalence between $\mathcal{R}$-Mod and the full subcategory of **fp-injective objects** in $\mathrm{F}(\mathcal{R})$ i.e. those functors F such that $\mathrm{Ext}^1(X, F) = 0$ for all X in $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})$.
- Φ takes pure-exact sequences to exact sequences.
- A functor F in $\mathrm{F}(\mathcal{R})$ is right exact if and only if $F \cong \Phi(M)$ for some left $\mathcal{R}$-module M.

The proof of these assertions can be found in [?JL, Theorem B.16] for when $\mathcal{R}$ is a ring. Note that the argument given in [?CB, Theorem 3.3, Remark 3.3(2)] extends this result to the general case since there is a duality $\mathrm{F}_{\mathrm{fp}}(\mathcal{R}) \to \mathrm{F}_{\mathrm{fp}}(\mathcal{R}^{\mathrm{op}})$ for any $\mathcal{R}$, see [?PSL, Theorem 10.3.4].

The (restricted) Yoneda embedding: Consider the functor $\Psi\colon \mathcal{R}\text{-Mod} \to \mathrm{C}(\mathcal{R})$ defined by $M \to (-, M)$ and $f \to (-, f)$ for all $\mathcal{R}$-modules M and all morphisms f. Then Ψ has the following properties.

- Ψ induces an equivalence between $\mathcal{R}$-Mod and the full subcategory of **flat objects** in $\mathrm{C}(\mathcal{R})$ i.e. those F such that $F \otimes -$ is exact.
- Ψ takes pure-exact sequences to exact sequences.
- A functor F in $\mathrm{C}(\mathcal{R})$ is left exact if and only if $F \cong \Psi(M)$ for some left $\mathcal{R}$-module M.

The proof of these assertions can be found in [?JL, Theorem B.11] for when $\mathcal{R}$ is a ring. For the general case, see [?CB, Theorem 1.4(2)].

We can therefore see the pure-exact structure on $\mathcal{R}$-Mod as a reflection of the exact structure on the full subcategories of fp-injective objects in $\mathrm{F}(\mathcal{R})$ or equally the exact structure on the full subcategory of flat objects in $\mathrm{C}(\mathcal{R})$.

Definition 1.4. A module M in $\mathcal{R}$-Mod is called **pure-injective** if M is injective with respect to pure monomorphisms. That is, for every pure monomorphism $f\colon A \to B$ and morphism $g\colon A \to M$, there exists a morphism $h\colon B \to M$ such that the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f", "\text{pure}"'] \arrow[d, "g"'] & B \arrow[dl, dotted, "h"] \\ & M & \end{tikzcd}$$

The definition of a **pure-projective** module is given dually.

The following theorem was originally proved in [?GJ] for the case where $\mathcal{R}$ is a ring. The general case follows by combining [?CB, Lemma 3.5.1, Remark 3.3] with [?PSL, Theorem 10.3.4].

Theorem 1.5. *The functor* $\Phi\colon \mathcal{R}\text{-Mod} \to \mathrm{F}(\mathcal{R})$ *induces an equivalence*

$$\Phi\colon \mathrm{P.inj}(\mathcal{R}\text{-Mod}) \xrightarrow{\sim} \mathrm{Inj}(\mathrm{F}(\mathcal{R}))$$

where $\mathrm{P.inj}(\mathcal{R}\text{-Mod})$ *denotes the full subcategory of pure-injective modules in* $\mathcal{R}$-Mod *and* $\mathrm{Inj}(\mathrm{F}(\mathcal{R}))$ *denotes the full subcategory of injective functors in* $\mathrm{F}(\mathcal{R})$.

Similarly, the contravariant Yoneda embedding induces an equivalence between the pure-projective modules in $\mathcal{R}$-Mod and the projective functors in $\mathrm{C}(\mathcal{R})$ (see [?JL, Theorem B.11]).

1.2. **The Ziegler spectrum.** The set of isomorphism classes of indecomposable pure-injective modules carries with it a natural topology. We introduce this space here and describe how it connects to localisations of the functor category $\mathrm{F}(\mathcal{R})$.

Definition 1.6. Define the **Ziegler spectrum** $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ to be the topological space with points and basic open sets given by the following data:

Points: The isomorphism classes of indecomposable pure-injective modules in $\mathcal{R}$-Mod.

Topology: The following sets define a basis of open sets for the topology on $\mathrm{Zg}(\mathcal{R}\text{-Mod})$:

$$(F) := \{M \in \mathrm{Zg}(\mathcal{R}\text{-Mod}) \mid (F,\, - \otimes_{\mathcal{R}} M) \neq 0\}$$

where F ranges over functors in $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})$ and $(F, -\otimes_{\mathcal{R}} M)$ denotes the group of natural transformations from F to $- \otimes_{\mathcal{R}} M$.

The topology on $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ was originally defined in [?Z] in terms of pairs of pp-formulas (ϕ/ψ) and encodes much of the model theoretic information in $\mathcal{R}$-Mod. For an account of how the model theoretic definition of the Ziegler spectrum connects with the perspective given here see [?PSL].

Definition 1.7 ([?CB2, §2.3])**.** Let $\mathcal{X}$ be a full subcategory of R-Mod. Then $\mathcal{X}$ is called a **definable subcategory** if the following equivalent statements are satisfied.

(1) There exists a set $\mathcal{S}$ of functors in $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})$ such that

$$\mathcal{X} = \{M \in \mathcal{R}\text{-Mod} \mid (F,\, - \otimes_{\mathcal{R}} M) = 0 \text{ for all } F \in \mathcal{S}\}.$$

(2) The subcategory $\mathcal{X}$ is closed under direct products, direct limits and pure-submodules i.e. submodules where the canonical inclusion is a pure monomorphism.

There is a bijection between definable subcategories of $\mathcal{R}$-Mod and the closed subsets of $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ given by $\mathcal{X} \mapsto \mathcal{X} \cap \mathrm{Zg}(\mathcal{R}\text{-Mod})$ and $\mathbf{X} \mapsto \langle \mathbf{X} \rangle$ where $\langle \mathbf{X} \rangle$ is the smallest definable subcategory containing $\mathbf{X}$ [?Z].

1.3. **The Ziegler spectrum via localisations.** Since the closed subsets of $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ are parametrised by the Serre subcategories of $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})$, it is natural to consider localisations of this category. In fact, there is also a bijective correspondence between the Serre subcategories and hereditary torsion pairs of finite type in $\mathrm{F}(\mathcal{R})$. In this section we consider the Ziegler spectrum from the perspective of localisations of $\mathrm{F}(\mathcal{R})$ at hereditary torsion classes of finite type.

Let $\mathcal{B}$ be a class of objects in $\mathrm{F}(\mathcal{R})$, then we will use the notation $\mathcal{B}^{\perp}$ to denote the class $\{F \in \mathrm{F}(\mathcal{R}) \mid (G, F) = 0 \text{ for all } G \in \mathcal{B}\}$ and the notation ${}^{\perp}\mathcal{B}$ to denote the class $\{G \in \mathrm{F}(\mathcal{R}) \mid (G, F) = 0 \text{ for all } F \in \mathcal{B}\}$.

Definition 1.8. Let $(\mathcal{T}, \mathcal{F})$ be a pair of subclasses of the objects of $\mathrm{F}(\mathcal{R})$. Then we say that $(\mathcal{T}, \mathcal{F})$ is a **torsion pair** if the following equivalent conditions are satisfied.

(1) $\mathcal{F} = \mathcal{T}^{\perp}$ and $\mathcal{T}$ is closed under quotient objects, extensions and arbitrary direct sums.
(2) $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F}$ is closed under subobjects, extensions and arbitrary products.

We call $\mathcal{T}$ a **torsion class** and $\mathcal{F}$ a **torsion-free class**. A torsion pair is **hereditary** if $\mathcal{T}$ is closed under subobjects, or equivalently, if $\mathcal{F}$ is closed under injective envelopes. For any $F \in \mathrm{F}(\mathcal{R})$, let $t(F) := \sum\{G \in \mathcal{T} \mid G \text{ is a subobject of } F\}$. This induces a functor $t\colon \mathrm{F}(\mathcal{R}) \to \mathrm{F}(\mathcal{R})$ called the **torsion functor** of $(\mathcal{T}, \mathcal{F})$. If t commutes with direct limits, then $(\mathcal{T}, \mathcal{F})$ is said to be of **finite type**. Let $\mathcal{E}$ be a class of objects in $\mathrm{F}(\mathcal{R})$. Then $({}^{\perp}\mathcal{E}, ({}^{\perp}\mathcal{E})^{\perp})$ is a torsion pair and is **cogenerated by** $\mathcal{E}$.

If $\mathcal{T}$ is a torsion class of a hereditary torsion pair, then $\mathcal{T}$ is a **Serre subcategory** of $\mathrm{F}(\mathcal{R})$. That is, for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{F}(\mathcal{R})$, we have that $X, Z \in \mathcal{T}$ if and only if $Y \in \mathcal{T}$.

Theorem 1.9 ([?Kr1, Corollary 4.3])**.** *Let $\mathcal{T}$ be a class of objects in $\mathrm{F}(\mathcal{R})$. Then the following statements are equivalent.*

(1) The pair $(\mathcal{T}, \mathcal{T}^{\perp})$ is a hereditary torsion pair of finite type.
(2) There exists a closed subset $\mathbf{X}$ of $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ such that $(\mathcal{T}, \mathcal{T}^{\perp})$ is the torsion pair cogenerated by the set $\mathcal{E} := \{- \otimes_{\mathcal{R}} M \mid M \in \mathbf{X}\}$.

There is a bijective correspondence between hereditary torsion pairs of finite type in $\mathrm{F}(\mathcal{R})$ and Serre subcategories of $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})$ given by the following mutually inverse bijections:

$$(\mathcal{T}, \mathcal{F}) \mapsto \mathcal{T} \cap \mathrm{F}_{\mathrm{fp}}(\mathcal{R}) \quad \text{ and } \quad \mathcal{S} \mapsto (\varinjlim \mathcal{S}, (\varinjlim \mathcal{S})^{\perp})$$

where $\varinjlim(\mathcal{S})$ is the class of objects obtained by taking all direct limits of objects in $\mathcal{S}$. This can be found in [?Kr1, Corollary 2.10].

Definition 1.10. Let $\mathcal{A}$ be an abelian category. For any Serre subcategory $\mathcal{S}$ of $\mathcal{A}$, we define the **quotient category** $\mathcal{A}/\mathcal{S}$ to be the category with the same objects as $\mathcal{A}$ and morphisms given by

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(X, Y) := \varinjlim \mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$$

where X' and Y' range over subobjects of X and Y respectively such that $X/X', Y' \in \mathcal{S}$. For each quotient category, there is a **localisation functor** $q\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ defined on objects to be $q(X) = X$. For more details see, for example, [?pop].

Theorem 1.11 ([?Kr1, Theorem 2.6], [?Gab, III.4, Corolaire 2]). *Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair of finite type in $\mathrm{F}(\mathcal{R})$ and let $\mathcal{S} = \mathcal{T} \cap \mathrm{F}_{\mathrm{fp}}(\mathcal{R})$ be the corresponding Serre subcategory of $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})$. Then the following statements hold.*

(1) There is an equivalence of categories

$$\mathrm{F}_{\mathrm{fp}}(\mathcal{R})/\mathcal{S} \xrightarrow{\sim} \mathrm{fp}(\mathrm{F}(\mathcal{R})/\mathcal{T})$$

where $\mathrm{fp}(\mathrm{F}(\mathcal{R})/\mathcal{T})$ is the full subcategory of finitely presented objects in $\mathrm{F}(\mathcal{R})/\mathcal{T}$.

(2) The functor $q \colon \mathrm{F}(\mathcal{R}) \to \mathrm{F}(\mathcal{R})/\mathcal{T}$ has a right adjoint (i.e. $\mathcal{T}$ is a ***localising subcategory****) denoted $s \colon \mathrm{F}(\mathcal{R})/\mathcal{T} \to \mathrm{F}(\mathcal{R})$.*

(3) The (indecomposable) injective objects in $\mathrm{F}(\mathcal{R})/\mathcal{T}$ are exactly those that are isomorphic to $q(E)$ for some (indecomposable) injective object E in $\mathcal{F}$.

(4) For any injective object E in $\mathcal{F}$ we have $E \cong s \circ q(E)$.

Remark 1.12. Combining the previous theorem with Theorems **??** and **??**, we have that the Ziegler spectrum $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ is homeomorphic to the topological space with points given by the set $\mathcal{I}(\mathcal{R})$ isomorphism classes of indecomposable injective objects in $\mathrm{F}(\mathcal{R})$ and the closed sets given by the sets of indecomposable injective objects in $\mathrm{F}(\mathcal{R})/\mathcal{T}$ for hereditary torsion pairs $(\mathcal{T}, \mathcal{F})$ of finite type (or, equivalently by the sets $\mathcal{F} \cap \mathcal{I}(\mathcal{R})$).

Given a closed set $\mathbf{X}$ in $\mathrm{Zg}(\mathcal{R}\text{-Mod})$, the full subcategory of $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})$ with objects $\mathrm{ann}(\mathbf{X}) := \{F \in \mathrm{F}_{\mathrm{fp}}(\mathcal{R}) \mid (F, - \otimes_{\mathcal{R}} M) = 0 \text{ for all } M \in \mathbf{X}\}$ is a Serre subcategory. The compact open sets $(F) \cap \mathbf{X}$ of the relative topology on $\mathbf{X}$ are parametrised by the objects F in $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})/\mathrm{ann}(\mathbf{X})$.

Example 1.13. Finite-dimensional algebras. Let R be a finite-dimensional algebra. Then $\mathrm{Zg}(R\text{-Mod})$ is a compact topological space (in fact, this is the case whenever R is a ring). Every indecomposable module in R-mod is pure-injective and hence a point of $\mathrm{Zg}(R\text{-Mod})$ and they enjoy some particular nice properties. For example, every indecomposable module M in R-mod, the set $\{M\}$ is open and closed. Moreover, the open set $U := R\text{-mod} \cap \mathrm{Zg}(R\text{-Mod})$ contains all of the isolated points in $\mathrm{Zg}(R\text{-Mod})$ and is dense in $\mathrm{Zg}(R\text{-Mod})$. That is, for every non-empty open set V, the set $U \cap V$ is non-empty. It follows directly that $\mathrm{Zg}(R\text{-Mod})$ is finite if and only if R is of finite representation type.

For some classes of finite-dimensional algebras, the Ziegler spectrum has been explicitly described. Examples include, tame hereditary algebras [?Pr1, ?Ri1], domestic string algebras [?PP, ?LPP], and canonical algebras of tubular type (the topology has been described but the algebraic structure of some of the points is not known) [?HP], [?LG].

Example 1.14. Compactly generated triangulated categories. Let $\mathcal{C}$ be a compactly generated triangulated category and let $\mathcal{C}^{\mathrm{c}}$ be the full subcategory of compact objects in $\mathcal{C}$. For definitions and more details on such categories see [?Nee].

The theory of purity and the Ziegler spectrum of compactly generated triangulated categories is well-developed (see [?Bel, ?GP, ?Kr2, ?Kr3]). Let $\mathrm{Zg}(\mathcal{C})$ denote the Ziegler spectrum of $\mathcal{C}$ as it is defined in [?Kr2]. Then this topological space is homeomorphic to a closed subset of $\mathrm{Zg}(\mathrm{Mod}\text{-}\mathcal{C}^{\mathrm{c}})$ and all the definitions surrounding the theory of purity in $\mathcal{C}$ coincide with those given in the preceding sections. This is explicitly proved in [?ALPP]; we give a brief account here.

Recall that a right $\mathcal{C}^c$-module N is fp-injective if $\mathrm{Ext}^1(M,N) = 0$ for all finitely presented right $\mathcal{C}^c$-modules M. The full subcategory $\mathcal{D}$ of fp-injective modules in Mod-$\mathcal{C}^c$ is a definable subcategory and so $\mathbf{Z} := \mathcal{D} \cap \mathrm{Zg}(\text{Mod-}\mathcal{C}^c)$ is a closed subset of the $\mathrm{Zg}(\text{Mod-}\mathcal{C}^c)$. In fact, $\mathbf{Z} = \{(-,N) \in \text{Mod-}\mathcal{C}^c \mid N \in \mathrm{Zg}(\mathcal{C})\}$ and these are exactly the indecomposable injective objects of Mod-$\mathcal{C}^c$ [?Kr3, Corollary 1.9]. Although the following result is not explicitly stated, much of the preliminary work needed for this result is contained in [?Kr3].

Theorem 1.15 ([?ALPP, Theorem 1.9])**.** *The closed subset* $\mathbf{Z}$ *of* $\mathrm{Zg}(\text{Mod-}\mathcal{C}^c)$ *with the relative topology is homeomorphic to* $\mathrm{Zg}(\mathcal{C})$*. Moreover there is an equivalence*

$$\mathrm{F_{fp}}(\mathcal{C}^c)/\mathrm{ann}(\mathbf{Z}) \xrightarrow{\sim} (\text{mod-}\mathcal{C}^c)^{\mathrm{op}}$$

yielding the following description of the compact open sets of $\mathbf{Z}$*:*

$$(M) := \{(-,N) \in \mathcal{Z} \mid (M,(-,N)) \neq 0\}$$

where M *is a module in* mod-$\mathcal{C}^c$*.*

There are some examples of compactly generated triangulated categories where the Ziegler spectrum is known. For example, the homotopy category $\mathrm{K}(\mathrm{Proj}\Lambda)$ of projective modules over a derived discrete algebra [?ALPP]; the derived category $\mathrm{D}(R\text{-Mod})$ where R is a right hereditary ring [?GP, Theorem 8.1]; and the derived category $\mathrm{D}(R\text{-Mod})$ where R is Von Neumann regular [?GP, Theorem 8.5].

2. The Krull–Gabriel dimension of $\mathcal{R}$

In this section we define the Krull–Gabriel dimension of $\mathcal{R}$. We begin by defining the more general notion of the Krull–Gabriel dimension of an abelian category $\mathcal{A}$ and then consider the special case where $\mathcal{A} = \mathrm{F_{fp}}(\mathcal{R})$; we refer to this dimension as the Krull–Gabriel dimension of $\mathcal{R}$. We then discuss the connections between the Krull–Gabriel analysis of $\mathrm{F_{fp}}(\mathcal{R})$ and the Ziegler spectrum of $\mathcal{R}$-Mod.

2.1. **The Krull–Gabriel dimension of an abelian category.** Let $\mathcal{A}$ be an abelian category. The following filtration of $\mathcal{A}$ was introduced by Gabriel in [?Gab, IV.1]. Let $\mathcal{A}_0$ be the Serre subcategory of $\mathcal{A}$ generated (as a Serre subcategory) by the simple objects in $\mathcal{A}$. For each ordinal α, we define a Serre subcategory $\mathcal{A}_\alpha$ via the following transfinite induction. For $\beta < \alpha$, let $q_\beta \colon \mathcal{A} \to \mathcal{A}/\mathcal{A}_\beta$ denote the corresponding localisation functor.

- If $\alpha = \beta + 1$, then define $\mathcal{A}_\alpha$ to be the Serre subcategory of $\mathcal{A}$ generated by objects X such that $q_\beta(X)$ is zero or simple in $\mathcal{A}/\mathcal{A}_\beta$.
- If α is a limit ordinal, then define $\mathcal{A}_\alpha := \bigcup_{\beta<\alpha} \mathcal{A}_\beta$.

Definition 2.1. Let $\mathcal{A}$ be an abelian category. If $\mathcal{A} = \mathcal{A}_\alpha$ for some ordinal α, then we define the **Krull–Gabriel dimension** $\mathrm{KG}(\mathcal{A})$ to be α. If no such ordinal exists then the **Krull–Gabriel dimension of $\mathcal{A}$ is undefined** and this is denoted $\mathrm{KG}(\mathcal{A}) = \infty$.

Note that Gabriel and many subsequent authors refers to this as the Krull dimension of $\mathcal{A}$. The name Krull–Gabriel dimension was adopted later.

Example 2.2. In [?baer, Theorem 3.9, Theorem 4.3], Baer shows that if $\mathcal{P}$ (respectively $\mathcal{R}$) is the preprojective (respectively regular) component of the Auslander-Reiten quiver of R-mod where R is a hereditary connected Artin algebra of infinite representation type, then $\mathrm{KG}(\mathcal{P}\text{-mod})$ is undefined if and only if $\mathrm{KG}(\mathcal{R}\text{-mod})$ is undefined if and only if R is wild.

Definition 2.3. Let a be an object in an abelian category $\mathcal{A}$. The **Krull–Gabriel dimension** $\mathrm{KG}(a)$ **of** a is defined to be the greatest ordinal such that $q_\alpha(a) \neq 0$. If no such ordinal exists then the Krull–Gabriel dimension of a is undefined and this is denoted $\mathrm{KG}(a) = \infty$.

Let $L(a)$ denote the modular lattice consisting of subobjects of a. Let $\sim$ denote the congruence relation on $L(a)$ generated by the relations: $b \sim c$ whenever $b < c$ and, for all $d \in L(a)$ such that $b < d < c$, either $b = d$ or $c = d$. Let $L'(a)$ denote the modular lattice consisting of the $\sim$-congruence classes in $L(a)$.

Then let $L_0(a) := L'(a)$ and for each ordinal α we define a modular lattice $L_\alpha(a)$ as follows.

- If $\alpha = \beta + 1$, then define $L_\alpha(a) := L'_\beta(a)$.
- If α is a limit ordinal, then define $L_\alpha(a) := \varinjlim_{\beta<\alpha} L_\beta(a)$.

The following lemma follows directly from the definitions.

Lemma 2.4. *Let a be an object in an abelian category $\mathcal{A}$. Then, for every ordinal α, we have $L_\alpha(a) \cong L(q_\alpha(a))$.*

This allows us to understand an important distinction between abelian categories $\mathcal{A}$ with defined Krull–Gabriel dimension and those with undefined Krull–Gabriel dimension.

Definition 2.5. A **dense chain** in a lattice L is a non-empty sublattice C such that, for all ordered pairs $x < y$ with $x, y \in C$, there exists some $z \in C$ such that $x < z < y$.

Proposition 2.6 (See, for example, [?Kr4, Lemma B.8])**.** *For an abelian category $\mathcal{A}$, the following conditions are equivalent.*

(1) There exists an object a in $\mathcal{A}$ such that $L(a)$ contains a dense chain.
(2) There exists an object a such that $\mathrm{KG}(a) = \infty$ if and only if $\mathrm{KG}(\mathcal{A}) = \infty$.

2.2. **The Krull–Gabriel dimension of $\mathcal{R}$.** The Krull–Gabriel dimension of $\mathcal{R}$, denoted $\mathrm{KG}(\mathcal{R})$, is defined to be the Krull–Gabriel dimension of the category $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})$. The dimension was first studied in [?Gei]. Let F be a functor in $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})$. By definition there exists an exact sequence of functors

$$(B, -) \xrightarrow{(f,-)} (A, -) \longrightarrow F \longrightarrow 0$$

where $f \colon A \to B$ is a morphism in mod-$\mathcal{R}$. That is, we have $F \cong (A, -)/\mathrm{im}(f, -)$. Moreover, it is well-known that the finitely presented subfunctors of $(A, -)$ are of the form $\mathrm{im}(g, -)$ for some $g \colon A \to C$ (see, for example, [?PSL, Lemma 10.2.2]). It follows that subfunctors of F are of the form $\mathrm{im}(g, -)/\mathrm{im}(f, -)$ where $f = hg$ for some $h \colon C \to B$. For a careful treatment of this approach see, for example, [?Pr-Sch].

With this description of subfunctors of $F \in \mathrm{F}_{\mathrm{fp}}(\mathcal{R})$ in mind, it is clear that the next result is a direct application of Proposition **??**.

Theorem 2.7 ([?Pr2])**.** *Let $\mathcal{R}$ be a skeletally small preadditive category. The Krull–Gabriel dimension of $\mathcal{R}$ is undefined if and only if there exist $n \geq 1$ and a collection of finitely presented n-pointed modules $\{(M_i, \underline{m}^{(i)}) \mid i \in \mathbb{Q}_{\geq 0},\ \underline{m}^{(i)} = (m_1^{(i)}, ..., m_n^{(i)}) \in M_i^n\}$ together with a collection of morphisms $\{f_{ij} \colon M_j \to M_i \mid i < j \in \mathbb{Q}_{\geq 0}\}$ such that the following conditions are satisfied for each $i < j$ and $1 \leq k \leq n$.*

(1) The morphism $f_{ij}\colon M_j \to M_i$ is such that $f_{ij}(m_k^{(j)}) = m_k^{(i)}$.
(2) There is no $g\colon M_i \to M_j$ such that $g(m_k^{(i)}) = m_k^{(j)}$.

2.3. **Krull–Gabriel dimension and the Ziegler spectrum.** Since every simple object in $\mathrm{F}(\mathcal{R})$ has an indecomposable injective hull, we have a direct connection between the first stage of the Krull–Gabriel filtration and $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ via the equivalence induced by the tensor embedding (see Theorem **??**). The following proposition tells us that this connection also exists for the latter stages.

Proposition 2.8. *Let $\mathcal{R}$ be a small preadditive category. Consider a closed subset $\mathbf{X}$ in $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ with $\mathcal{T}_{\mathbf{X}}$ the torsion class in $\mathrm{F}(\mathcal{R})$ associated to $\mathbf{X}$ under the correspondence in Theorem **??**. Let S be a functor in $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})$ such that $q_{\mathbf{X}}(S)$ is simple in $\mathrm{F}(\mathcal{R})/\mathcal{T}_X$ where $q_{\mathbf{X}}\colon \mathrm{F}(\mathcal{R}) \to \mathrm{F}(\mathcal{R})/\mathcal{T}_{\mathbf{X}}$ is the corresponding localisation functor. Then the following statements hold.*

(1) The injective envelope $E(q_{\mathbf{X}}(S))$ of $q_{\mathbf{X}}(S)$ in $\mathrm{F}(\mathcal{R})/\mathcal{T}_{\mathbf{X}}$ is indecomposable.
(2) There is a pure-injective $\mathcal{R}$-module $N \in \mathrm{Zg}(\mathcal{R}\text{-Mod})$ such that $E(q_{\mathbf{X}}(S))$ is isomorphic to $q_{\mathbf{X}}(- \otimes_{\mathcal{R}} N)$ and $(S) \cap \mathbf{X} = \{N\}$.

Proof. It is clear that $E(q_{\mathbf{X}}(S))$ is indecomposable because $q_{\mathbf{X}}(S)$ is uniform i.e. any pair of non-zero subobjects of $q_{\mathbf{X}}(S)$ have non-zero intersection (see [?Sten, Proposition 2.8]). By the uniqueness of injective envelopes, as well as Theorem **??** and the last part of Theorem **??**, there is a unique $N \in \mathrm{Zg}(\mathcal{R}\text{-Mod})$ such that $E(q_{\mathbf{X}}(S)) = q_{\mathbf{X}}(- \otimes_{\mathcal{R}} N)$. By Remark **??**, N is isolated in $\mathbf{X}$ by the open set corresponding to S. □

Definition 2.9. If all isolated points in a closed set $\mathbf{X}$ of $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ arise as injective envelopes of finitely presented simple functors in $\mathrm{F}(\mathcal{R})/\mathcal{T}_{\mathbf{X}}$, then we say that **X satisfies the isolation condition**.

Note that Jensen and Lenzing prove the following theorem in the case where $\mathcal{R}$ is a ring but the argument works equally well in the general case.

Theorem 2.10 ([?JL, Proposition 8.52, Theorem 8.53])**.** *If the Krull–Gabriel dimension of $\mathcal{R}$ is defined, then every pure-injective module has an indecomposable direct summand and, for every ordinal α, the closed subset $\mathbf{X}_\alpha$ of $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ corresponding to $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})/\mathrm{F}_{\mathrm{fp}}(\mathcal{R})_\alpha$ satisfies the isolation condition.*

Remark 2.11. If the isolation condition holds for $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ (e.g. if $\mathrm{KG}(\mathcal{R})$ is defined), then the Krull–Gabriel dimension of $\mathcal{R}$ coincides with the Cantor–Bendixson rank of $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ which is defined as follows.

For a topological space $\mathbf{Z}$ we define $\mathbf{Z}'$ to be the closed subset of $\mathbf{Z}$ consisting of non-isolated points. Setting $\mathbf{Z}_0 = \mathbf{Z}$, we define $\mathbf{Z}_{\alpha+1} := \mathbf{Z}'_\alpha$ for each ordinal α and let $\mathbf{Z}_\lambda := \bigcap_{\alpha<\lambda} \mathbf{Z}_\alpha$ for each limit ordinal λ. The **Cantor–Bendixson rank** of **Z** is defined to be the least α such that $\mathbf{Z}_{\alpha+1} = \emptyset$. If no such ordinal exists then the Cantor–Bendixson rank of $\mathbf{Z}$ is said to be undefined.

Thus a description of the open neighbourhoods in $\mathrm{Zg}(\mathcal{R}\text{-Mod})$ gives rise to a description of the functors in $\mathrm{F}_{\mathrm{fp}}(\mathcal{R})/\mathrm{F}_{\mathrm{fp}}(\mathcal{R})_\alpha$ for each ordinal α. This was the strategy employed by the authors of [?LPP]; for a description of the simple functors see [?L].

There are no known examples of rings R where the isolation condition does not hold for $\mathrm{Zg}(R\text{-Mod})$. In other words, it is an open question as to whether the Krull–Gabriel dimension of R and the Cantor–Bendixson rank of $\mathrm{Zg}(R\text{-Mod})$ always coincide.

Remark 2.12. Let R be a ring. In [?Z], Ziegler shows that if every pure-injective module in R-Mod has an indecomposable direct summand or if R is countable then $\mathrm{Zg}(R\text{-Mod})$ satisfies the isolation condition.

Remark 2.13. As Jensen and Lenzing point out in [?JL, Remark 8.54], if the Krull–Gabriel dimension of $\mathcal{R}$ is defined, then the classification of indecomposable pure-injective modules amounts to the classification of simple functors in $\mathrm{F}(\mathcal{R})_{\alpha+1}/\mathrm{F}(\mathcal{R})_{\alpha}$ for each ordinal $\alpha < \mathrm{KG}(\mathcal{R})$.

This procedure is demonstrated in [?JL] for Dedekind domains and the Kronecker algebra. This is also the method employed by the authors of [?ALPP] in order to classify the indecomposable pure-injective objects in $\mathrm{K}(\mathrm{Proj}\Lambda)$ where Λ is a derived-discrete algebra.

3. Examples

In this final section we give some examples where the Krull–Gabriel dimension of $\mathcal{R}$ has been calculated.

3.1. **Serial rings.** Let R be a unital ring. Then R is a **serial ring** if it is serial when considered as both a left and a right R-module. Equivalently there exists a collection $e_1, \dots e_n$ of pairwise orthogonal idempotents in R such that $e_1 + \dots e_n = 1$ and the modules e_iR and Re_i are **uniserial** for each $1 \leq i \leq n$, i.e. the lattice of submodules for each module is a chain.

In [?Pun], Puninski proves the following about the Krull–Gabriel dimension of R when R is a serial ring. For the definition of Krull dimension see, for example, [?McC-Rob, Chapter 6].

Theorem 3.1. *Let R be a serial ring. Then the following statements hold.*

(1) *The Krull–Gabriel dimension $\mathrm{KG}(R)$ is defined if and only if the Krull dimension of R is defined.*
(2) *If α is the Krull dimension of R, then $\mathrm{KG}(R) \leq \alpha \oplus \alpha$ where $\oplus$ denotes the Cantor sum.*
(3) *The Krull–Gabriel dimension $\mathrm{KG}(R)$ is not equal to 1.*

Puninski also shows that the isolation condition holds for such rings. There are classes of serial rings where the Cantor–Bendixson rank has been calculated and so, by Remark **??**, the Krull–Gabriel dimension can be given exactly in terms of the Krull dimension of R. A ring R is called **Krull–Schmidt** if every finitely presented R-module is a finite direct sum of indecomposable modules with local endomorphism ring.

Corollary 3.2 ([?Pun, Proposition 5.8, Corollary 5.9], [?Pun2], [?Rey])**.** *Let R be either:*

- *a Krull–Schmidt uniserial ring;*
- *a Krull–Schmidt serial ring with finite Krull dimension; or*
- *a commutative valuation domain.*

If the Krull dimension of R is α, then the Krull–Gabriel dimension of R is $\alpha \oplus \alpha$.

3.2. **Dedekind domains.** Let R be an integral domain, then R is a **Dedekind domain** if every non-zero proper ideal can be written as a product of prime ideals. Examples include $\mathbb{Z}$ or $\mathrm{k}[T]$ for a field k.

Theorem 3.3. *Let R be a Dedekind domain that is not a field. Then the Cantor–Bendixson rank of* $\mathrm{Zg}(R\text{-Mod})$ *is equal to the Krull–Gabriel dimension of R is equal to* 2.

Proof. It follows from Ziegler's description of $\mathrm{Zg}(R\text{-Mod})$ [?Z] that the isolation condition is satisfied so, by Remark **??**, the Krull–Gabriel dimension of R is equal to the Cantor–Bendixson rank of $\mathrm{Zg}(R\text{-Mod})$. Thus, by [?Z], both dimensions are equal to 2. □

Remark 3.4. There is an explicit description of the Krull–Gabriel filtration of $\mathrm{F}_{\mathrm{fp}}(R)$ in [?JL, Chapter 8] for R a Dedekind domain that is not a field. Following the notation in [?JL], for any R-module M, let $\overline{M}$ denote the unique functor that vanishes on all finite length modules and satisfies $\overline{M}(R) = 0$.

Let $\mathfrak{m}$ denote a maximal ideal in R. Then the functors $(R/\mathfrak{m}^n, -)/\mathrm{rad}(R/\mathfrak{m}^n, -)$ are shown to be simple in $\mathrm{F}_{\mathrm{fp}}(R)$ and the images of the functors $(R/\mathfrak{m}, -)$ are simple in $\mathrm{F}_{\mathrm{fp}}(R)/\mathrm{F}_{\mathrm{fp}}(R)_0$. One should note that the images of functors $\overline{R/\mathfrak{m}}$ are actually simple in $\mathrm{F}_{\mathrm{fp}}(R)/\mathrm{F}_{\mathrm{fp}}(R)_0$. Moreover the image of the functor $\overline{R}$ is simple in $\mathrm{F}_{\mathrm{fp}}(R)/\mathrm{F}_{\mathrm{fp}}(R)_1$. To see why this is the case, consider, for example, that there is a non-zero, non-surjective morphism

$$(- \otimes_R R/\mathfrak{m}^n) \to \overline{R/\mathfrak{m}}$$

and that $- \otimes_R R/\mathfrak{m}^n$ is a finitely presented functor for each $n \geq 1$. Moreover, the following is a chain of proper embeddings.

$$0 \to - \otimes_R R/\mathfrak{m} \to - \otimes_R R/\mathfrak{m}^2 \to - \otimes_R R/\mathfrak{m}^3 \to \dots.$$

3.3. **Finite-dimensional algebras.** For a finite-dimensional algebra R, the only known values of $\mathrm{KG}(R)$ are finite or undefined. We give examples of both cases and state some open questions and conjectures concerning the connections between Krull–Gabriel dimension and other notions of complexity in a module category.

The first examples we will consider are those with finite Krull–Gabriel dimension.

Theorem 3.5 ([?Aus, Corollary 3.14])**.** *Let R be a finite-dimensional algebra. Then* $\mathrm{KG}(R) = 0$ *if and only if R is of finite representation type.*

Theorem 3.6 ([?Kr5, Corollary 11.4],[?Herz2, Theorem 3.6])**.** *Let R be an Artin algebra. Then* $\mathrm{KG}(R) \neq 1$.

The first example of Krull–Gabriel dimension being explicitly calculated in the functor category $\mathrm{F}_{\mathrm{fp}}(R)$ was carried out by Geigle in [?Gei] for R a tame hereditary algebra of infinite representation type. He described both the simple functors at each level of the filtration and the factor categories $\mathcal{A}_{\alpha+1}/\mathcal{A}_{\alpha}$.

Theorem 3.7 ([?Gei])**.** *Let R be a tame hereditary algebra of infinite representation type. Then* $\mathrm{KG}(R) = 2$.

Since the Krull–Gabriel dimension of R is defined, we have that R satisfies the isolation condition. Thus, by Remark **??** the above result for tame hereditary algebras is implicit in the descriptions of the Ziegler spectrum given in [?Pr1] and [?Ri1].

The Krull–Gabriel dimension of many other algebras closely related to tame hereditary has been shown to equal 2. For example, [?Gei2] covers the tame tilted algebras, algebras which are stably equivalent to tame hereditary algebras and domestic one relation algebras. Other examples of $\mathrm{KG}(R) = 2$ can be found in [?FS], [?W], [?Mal], [?Skow].

It has been known for some time that, for every $n \in \mathbb{N}$, there exists a domestic string algebra R of Krull–Gabriel dimension $n \geq 2$.

For example, the algebra with the quiver

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "x_1"] \arrow[r, bend right, "y_1"'] & \bullet \arrow[r, "z_1"] & \bullet \arrow[r, bend left, "x_2"] \arrow[r, bend right, "y_2"'] & \bullet \arrow[r, "z_2"] & \bullet \cdots \bullet \arrow[r, "z_n"] & \bullet \arrow[r, bend left, "x_{n+1}"] \arrow[r, bend right, "y_{n+1}"'] & \bullet
\end{tikzcd}$$

and relations $z_i x_i = y_{i+1} z_i = 0$ has Krull–Gabriel dimension $n+2$.

This was shown in [?PreBur] by describing the Ziegler spectrum of these algebras and then proving explicitly that the isolation condition holds.

This result was proved simultaneously in [?Sch] by making use of connections to the transfinite radical of mod-R.

Definition 3.8. Let $\mathcal{I}$ be an ideal in mod-R. Then the finite powers of $\mathcal{I}$ are defined in the usual way:

$$\mathcal{I}^n := \left\{ \sum_{i=1}^{m} f_{in} \dots f_{i1} \mid f_{ij} \in \mathcal{I} \text{ and } m < \omega \right\}.$$

Then for any limit ordinal λ let $\mathcal{I}^\lambda = \bigcap_{\beta<\lambda} \mathcal{I}^\beta$ and for any ordinal $\alpha := \lambda + n$ where λ is a limit ordinal and $n < \omega$ let $\mathcal{I}^\alpha := (\mathcal{I}^\lambda)^{n+1}$. We refer to the ideals $\mathcal{I}^\alpha$ for ordinals α as the **transfinite powers of** $\mathcal{I}$. Also let $\mathcal{I}^\infty := \bigcap_\alpha \mathcal{I}^\alpha$.

For an Artin algebra R, we will denote the Jacobson radical of mod-R by rad_R. That is, rad_R is the ideal of mod-R generated by the non-isomorphisms between indecomposable modules.

Proposition 3.9. *Let R be an Artin algebra and let $\mathbf{X}_\alpha$ denote the closed subset of* $\mathrm{Zg}(R\text{-Mod})$ *containing points with Cantor–Bendixson rank greater than or equal to α.*

If R has Krull–Gabriel dimension β, then $\mathrm{rad}^{\omega\beta+n} = 0$ *for some $n \in \mathbb{N}$. Moreover, for each successor ordinal α, every element of* $\mathrm{rad}_R^{\omega\alpha}$ *factors through a finite direct sum of modules in $\mathbf{X}_\alpha$.*

Proof. The first statement is [?Kr4, Corollary 8.14]. Since the Krull–Gabriel dimension is defined, the isolation condition holds and the Cantor–Bendixson rank is therefore defined. The latter result then follows immediately from [?PreRad] since, in any topological space where the Cantor–Bendixson rank is defined, the isolated points in the space are dense (see, for example, [?PSL, Lem. 5.3.57]). □

Open question 3.10 ([?PSL, Question 9.1.13])**.** If $\mathrm{rad}_R^{\omega\alpha+n} = 0$ for some ordinal α, is the Krull–Gabriel dimension of R less than or equal to α?

Conjecture 3.11 ([?Sch2])**.** *Let $n \geq 2$ and let R be a finite-dimensional algebra. Then the Krull–Gabriel dimension of R is equal to n if and only if* $\mathrm{rad}_R^{\omega(n-1)} \neq 0$ *and* $\mathrm{rad}_R^{\omega n} = 0$.

Theorem 3.12. *Let R be a domestic string algebra over an algebraically closed field. Then the following statements hold.*

(1) The Krull–Gabriel dimension of R is finite.
(2) Conjecture **??** *holds.*

Proof. The Krull–Gabriel dimension was shown to equal the length of the longest path in a finite graph known as the bridge quiver in [?LPP]. In [?Sch] Schröer has previously shown that this is also the value of n such that $\mathrm{rad}_R^{\omega(n-1)} \neq 0$ and $\mathrm{rad}_R^{\omega n} = 0$. □

There are also large classes of finite-dimensional algebras with undefined Krull–Gabriel dimension. The dividing line between algebras with defined and undefined Krull–Gabriel dimension is still unclear but it is conjectured to be connected with the representation type of the algebra.

Theorem 3.13 ([?Kr4, Proposition 8.15])**.** *Let R be a finite-dimensional algebra. If R is of wild representation type, then* $\mathrm{KG}(R) = \infty$*.*

One might guess, given the above, that the Krull–Gabriel dimension of an algebra being undefined is equivalent to being of wild representation type. However there are examples of tame algebras with undefined Krull–Gabriel dimension. Examples include non-domestic string algebras [?Sch] and non-domestic canonical algebras [?Gei2]. A different conjecture is therefore natural.

Conjecture 3.14 ([?Pr-Sch, Conjecture 1.5])**.** *Let R be a finite-dimensional algebra. Then the following are equivalent.*

(1) R is of tame domestic representation type.
(2) The Krull–Gabriel dimension of R is defined.
(3) The Krull–Gabriel dimension of R is finite.

3.4. **Compactly generated triangulated categories.** In Example **??** we saw that when considering a compactly generated triangulated category $\mathcal{C}$ with compact generators $\mathcal{C}^{\mathrm{c}}$ we should analyse the category $(\text{mod-}\mathcal{C}^{\mathrm{c}})^{\mathrm{op}}$.

A Krull–Gabriel filtration of this category yields the same connections with the morphisms in $\mathcal{C}^{\mathrm{c}}$ (as in Section **??**) and the Cantor–Bendixson rank of $\mathrm{Zg}(\mathcal{C})$ (as in Remark **??**).

The first two results in the following theorem follow from [?Bob]; the latter is explicitly contained in [?ALPP]. The third result follows from [?GP, Theorem 8.1] combined with the corresponding results for $\mathrm{KG}(R)$.

Theorem 3.15. *Let* $\mathcal{A} = (\text{mod-}\mathcal{C}^{\mathrm{c}})^{\mathrm{op}}$*.*

(1) Let R be a derived-discrete algebra and let $\mathcal{C} = \mathrm{D}(R\text{-Mod})$ *with compact generators* $\mathcal{C}^{\mathrm{c}} \cong \mathrm{K}^{\mathrm{b}}(R\text{-proj})$*. Then*

$$\mathrm{KG}(\mathcal{A}) = \begin{cases} 1 & \textit{if } gl.dim(R) = \infty \\ 2 & \textit{if } gl.dim(R) < \infty \end{cases}$$

(2) Let R be a derived-discrete algebra and let $\mathcal{C} = \mathrm{K}(R\text{-Proj})$ *with compact generators* $\mathcal{C}^{\mathrm{c}} \cong \mathrm{D}^{\mathrm{b}}(R\text{-mod})$*. Then* $\mathrm{KG}(\mathcal{A}) = 2$*.*
(3) Let R be a finite-dimensional hereditary algebra over an algebraically closed field and let $\mathcal{C} = \mathrm{D}(R\text{-Mod})$ *with compact generators* $\mathcal{C}^{\mathrm{c}} \cong \mathrm{K}^{\mathrm{b}}(R\text{-proj})$*.*

Then

$$\mathrm{KG}(\mathcal{A}) = \begin{cases} 0 & \text{if } R \text{ is of finite representation type} \\ 2 & \text{if } R \text{ is of tame representation type} \\ \infty & \text{if } R \text{ is of wild representation type} \end{cases}$$

References

[1] Kristin Arnesen, Rosanna Laking, David Pauksztello, and Mike Prest, *The Ziegler spectrum for derived-discrete algebras*, available at `arXiv:1603.00775`.

[2] Maurice Auslander, *A functorial approach to representation theory*, Representations of algebras (Puebla, 1980), Lecture Notes in Math., vol. 944, Springer, Berlin-New York, 1982, pp. 105–179.

[3] Dagmar Baer, *Homological properties of wild hereditary Artin algebras*, Representation theory, I (Ottawa, Ont., 1984), Lecture Notes in Math., vol. 1177, Springer, Berlin, 1986, pp. 1–12.

[4] Apostolos Beligiannis, *Relative homological algebra and purity in triangulated categories*, J. Algebra **227** (2000), no. 1, 268–361.

[5] Grzegorz Bobiński and Henning Krause, *The Krull–Gabriel dimension of discrete derived categories*, Bull. Sci. Math. **139** (2015), no. 3, 269–282.

[6] Kevin Burke and Mike Prest, *The Ziegler and Zariski spectra of some domestic string algebras*, Algebr. Represent. Theory **5** (2002), no. 3, 211–234.

[7] William Crawley-Boevey, *Locally finitely presented additive categories*, Comm. Algebra **22** (1994), no. 5, 1641–1674.

[8] ———, *Infinite-dimensional modules in the representation theory of finite-dimensional algebras*, Algebras and modules, I (Trondheim, 1996), CMS Conf. Proc., vol. 23, Amer. Math. Soc., Providence, RI, 1998, pp. 29–54.

[9] Rolf Farnsteiner and Andrzej Skowroński, *Classification of restricted Lie algebras with tame principal block*, J. Reine Angew. Math. **546** (2002), 1–45.

[10] Pierre Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448 (French).

[11] Grigory Garkusha and Mike Prest, *Triangulated categories and the Ziegler spectrum*, Algebr. Represent. Theory **8** (2005), no. 4, 499–523.

[12] Werner Geigle, *The Krull–Gabriel dimension of the representation theory of a tame hereditary Artin algebra and applications to the structure of exact sequences*, Manuscripta Math. **54** (1985), no. 1-2, 83–106.

[13] ———, *Krull dimension and Artin algebras*, Representation theory, I (Ottawa, Ont., 1984), Lecture Notes in Math., vol. 1177, Springer, Berlin, 1986, pp. 135–155.

[14] Lorna Gregory, *Decidability of theories of modules over canonical algebras of tubular type*, available at `arXiv:1603.03284`.

[15] Laurent Gruson and Christian U. Jensen, *Modules algébriquement compacts et foncteurs* $\varprojlim^{(i)}$, C. R. Acad. Sci. Paris Sér. A-B **276** (1973), A1651–A1653.

[16] Richard Harland and Mike Prest, *Modules with irrational slope over tubular algebras*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 3, 695–720.

[17] Ivo Herzog, *The endomorphism ring of a localized coherent functor*, J. Algebra **191** (1997), no. 1, 416–426.

[18] Christian U. Jensen and Helmut Lenzing, *Model-theoretic algebra with particular emphasis on fields, rings, modules*, Algebra, Logic and Applications, vol. 2, Gordon and Breach Science Publishers, New York, 1989.

[19] Henning Krause, *The spectrum of a locally coherent category*, J. Pure Appl. Algebra **114** (1997), no. 3, 259–271.

[20] ———, *Generic modules over Artin algebras*, Proc. London Math. Soc. (3) **76** (1998), no. 2, 276–306.

[21] ———, *Coherent functors in stable homotopy theory*, Fund. Math. **173** (2002), no. 1, 33–56.

[22] ———, *Smashing subcategories and the telescope conjecture—an algebraic approach*, Invent. Math. **139** (2000), no. 1, 99–133.

[23] ———, *The spectrum of a module category*, Mem. Amer. Math. Soc. **149** (2001), no. 707, x+125.

[24] Rosanna Laking, *String algebras in representation theory*, University of Manchester, 2016.

[25] Rosanna Laking, Mike Prest, and Gena Puninski, *Krull–Gabriel dimension of domestic string algebras*, Trans. Amer. Math. Soc., to appear.

[26] Piotr Malicki, *Krull dimension of tame generalized multicoil algebras*, Algebr. Represent. Theory **18** (2015), 881–894.

[27] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Revised edition, Graduate Studies in Mathematics, vol. 30, American Mathematical Society, Providence, RI, 2001. With the cooperation of L. W. Small.

[28] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.

[29] N. Popescu, *Abelian categories with applications to rings and modules*, Academic Press, London-New York, 1973. London Mathematical Society Monographs, No. 3.

[30] Mike Prest, *Ziegler spectra of tame hereditary algebras*, J. Algebra **207** (1998), no. 1, 146–164.

[31] ———, *Morphisms between finitely presented modules and infinite-dimensional representations*, Algebras and modules, II (Geiranger, 1996), CMS Conf. Proc., vol. 24, Amer. Math. Soc., Providence, RI, 1998, pp. 447–455.

[32] ———, *Ideals in mod-R and the ω-radical*, J. London Math. Soc. (2) **71** (2005), no. 2, 321–334.

[33] ———, *Purity, spectra and localisation*, Encyclopedia of Mathematics and its Applications, vol. 121, Cambridge University Press, Cambridge, 2009.

[34] Mike Prest and Jan Schröer, *Serial functors, Jacobson radical and representation type*, J. Pure Appl. Algebra **170** (2002), no. 2-3, 295–307.

[35] Gennadi Puninski, *Cantor–Bendixson rank of the Ziegler spectrum over a commutative valuation domain*, J. Symbolic Logic **64** (1999), no. 4, 1512–1518.

[36] Gena Puninski, *The Krull–Gabriel dimension of a serial ring*, Comm. Algebra **31** (2003), no. 12, 5977–5993.

[37] Gena Puninski and Mike Prest, *Ringel's conjecture for domestic string algebras*, Math. Z. **282** (2016), no. 1-2, 61–77.

[38] Geert Reynders, *Ziegler spectra of serial rings with Krull dimension*, Comm. Algebra **27** (1999), no. 6, 2583–2611.

[39] Claus Michael Ringel, *The Ziegler spectrum of a tame hereditary algebra*, Colloq. Math. **76** (1998), no. 1, 105–115.

[40] Jan Schröer, *On the Krull–Gabriel dimension of an algebra*, Math. Z. **233** (2000), no. 2, 287–303.

[41] ———, *The Krull–Gabriel dimension of an algebra—open problems and conjectures*, Infinite length modules (Bielefeld, 1998), Trends Math., Birkhäuser, Basel, 2000, pp. 419–424.

[42] Andrzej Skowroński, *The Krull–Gabriel dimension of cycle-finite Artin algebras*, Algebr. Represent. Theory **19** (2016), no. 1, 215–233.

[43] Bo Stenström, *Rings and modules of quotients*, Lecture Notes in Mathematics, Vol. 237, Springer-Verlag, Berlin-New York, 1971.

[44] Michał Wenderlich, *Krull dimension of strongly simply connected algebras*, Bull. Polish Acad. Sci. Math. **44** (1996), no. 4, 473–480.

[45] Martin Ziegler, *Model theory of modules*, Ann. Pure Appl. Logic **26** (1984), no. 2, 149–213.

Mathematisches Institut, Universität Bonn, Endenicher Allee 60 53115 Bonn, Germany

Email address: `laking@math.uni-bonn.de`