COTILTING DUALITY FOR ARTINIAN RINGS

Abstract. A classical result due to Goro Azumaya establishes that any duality between the subcategories of finitely generated modules over two given arbitrary rings is representable by a unique bimodule which is a finitely generated injective cogenerator on both sides and, moreover, the two rings are necessarily artinian. We generalise this well-known result to the *cotilting dualities*, i.e. to the case of a cotilting bimodule in the derived categories of the given rings.

1. Introduction

The notion of cotilting module is dual to the notion of tilting module in a very large sense. Besides the definition itself of cotilting module, namely of a module which is injective w.r.t. the subcategory it cogenerates, any cotilting module yields a *Cotilting Theorem* [**?**, **?**, **?**] which in fact reflects the analogies and the pathologies arising when one dualises Morita's and the tilting equivalences, namely on the one hand the presence of a pair of dualities (as well as any tilting module induces a pair of equivalences), on the other hand the partial characterisation of the modules involved in such dualities (as well as a complete classification of the so-called *reflexive* module is unknown). Nevertheless, from the "classical" literature [**?**] some particular cases in which these pathologies are better understood are known:

1.1. Theorem. *Let R, S be two arbitrary rings and let $\operatorname{gen}{}_S S$ and $\operatorname{gen} R_R$ be the full subcategories of finitely generated left S- resp. right R-modules. A duality $F : \operatorname{gen}{}_S S \rightleftarrows \operatorname{gen} R_R : G$ is representable by a Morita bimodule ${}_S U_R$ which is finitely generated both over S and over R. In particular, S is left artinian and R is right artinian.*

1.2. Theorem. *Let R be a right artinian ring. Then R admits a Morita duality if and only if R admits a finitely generated injective cogenerator U_R. In this case, $S = \operatorname{End}_R(U)$ is a left artinian ring and ${}_S U$ is finitely generated.*

On the other hand, "less classical" homological tools seem to give new insights to this issue [**?**, **?**]. In order either to dualise the known results about adjoint pairs of derived (covariant) functors, and to extend the known results from module categories concerning the reflexive modules, in the paper [**?**] authors introduced the notion of $\mathcal{D}$-*reflexivity* among the derived categories of two abelian categories involved in an adjunction of contravariant functors. More precisely, given any such right adjoint pair $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$, [**?**, Lemma 13.6] establishes that whenever the total right derived functors $\mathbf{R}F, \mathbf{R}G$ of F and G respectively are defined in the derived categories $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$ of the underlying abelian categories, then

$$\mathbf{R}F : \mathcal{D}(\mathcal{A}) \rightleftarrows \mathcal{D}(\mathcal{B}) : \mathbf{R}G$$

is a right adjoint pair of functors. Thus, the complexes making the units of this adjunction into a natural isomorphism are said to be $\mathcal{D}$-*reflexive*. In particular, an object $A \in \mathcal{A}$ is said to be $\mathcal{D}$-*reflexive* if the corresponding stalk complex is $\mathcal{D}$-reflexive. This nomenclature is needed [**?**, Example 2.2] in order to distinguish the reflexivity in the underlying abelian categories from the reflexivity provided by the stalk complexes in the derived framework.

In this work, we will exploit the results concerning $\mathcal{D}$-reflexivity induced by a cotilting bimodule ${}_SU_R$, i.e. concerning a *cotilting duality*, in order to show that when at least one of the two rings is artinian, several of the good features of Morita bimodules and dualities hold true as well.

The paper is organised as follows. After a preliminary section where we recall some homological tools needed throughout the work, in particular the notion of $\mathcal{D}$-reflexivity specialised to the case of a cotilting bimodule [**?**, **?**], the last section contains the main results, concerning the duality —in the setting of the derived categories— induced by a cotilting bimodule defined at least over an artinian ring. Namely, Theorem 3.9 at once provides a strong finiteness condition for such a module, i.e. it is finitely generated and product complete over the artinian ring; then Theorem 4.3 and Theorem 4.5 form the announced generalisation of Azumaya's classical results, for the former establishes that a cotilting duality occurs between the finitely generated modules if and only the cotilting bimodule is defined over two artinian rings, while the latter provides the representability of a wide class of derived dualities precisely by such a kind of bimodules.

2. Preliminaries

All the rings considered will be associative with $1 \neq 0$. Given a ring R, Mod-R and R-Mod will denote the categories of right resp. left modules over R, while mod-R and R-mod are the full subcategories of the finitely presented right resp. left R-modules. The derived category of Mod-R will be denoted by $\mathcal{D}(R)$, while $\mathcal{D}^b(R)$ (resp. $\mathcal{D}^b_{\text{mod-}R}(R)$) denotes the bounded derived category of R (with finitely presented cohomologies). The injective dimension of $U \in$ Mod-R will be denoted by $\text{i.d.}_R(M)$. Let us recall some useful subcategories of Mod-R associated with U:

- $\operatorname{Gen} U$ (resp. $\operatorname{gen} U$) is the class of the homomorphic images of a direct sum of (finitely many) copies of U. Dually, $\operatorname{Cogen} U$ (resp. $\operatorname{cogen} U$) is the class of modules which embeds into a direct product of (finitely many) copies of U.
- $\operatorname{Add} U$ (resp. $\operatorname{add} U$) is the class of the direct summands of a direct sum of (finitely many) copies of U. Dually, $\operatorname{Prod} U$ is the class of direct summands of products of copies of U.

A module $U \in$ Mod-R is called *cotilting* in case $\operatorname{Ker}\operatorname{Ext}^1_R(-,U) = \operatorname{Cogen} U$; that is, if it is injective in the class it cogenerates. A well-known characterisation ensures that U_R is a cotilting modules if and only if

(i) $\text{i.d.}_R(U) \leq 1$;
(ii) $\operatorname{Ext}^1_R(U^\alpha, U) = 0$ for every cardinal α;

(iii) $\operatorname{Ker}\operatorname{Hom}_R(-,U) \cap \operatorname{Ker}\operatorname{Ext}^1_R(-,U) = 0$.

A module U_R satisfying conditions (i) and (ii) is said to be *partial cotilting*. This is equivalent to ask (i) and that $\operatorname{Cogen} U \subseteq \operatorname{Ker}\operatorname{Ext}^1_R(-,U)$.

A faithfully balanced bimodule ${}_SU_R$ which is cotilting on both sides will be called a *cotilting bimodule*.

2.1. **Approximation theory.** A *torsion pair* in Mod-R is a pair $(\mathcal{T},\mathcal{F})$ of classes of modules which are mutually orthogonal with respect to the Hom_R functors, namely

$$\mathcal{T} = \{X \in \text{Mod-}R \mid \operatorname{Hom}_R(X,Y) = 0 \text{ for all } Y \in \mathcal{F}\}$$
$$\mathcal{F} = \{Y \in \text{Mod-}R \mid \operatorname{Hom}_R(X,Y) = 0 \text{ for all } X \in \mathcal{T}\},$$

and maximal w.r.t. these orthogonality conditions. The first member $\mathcal{T}$ of the pair is called the *torsion class*, and its objects are the *torsion modules*, while $\mathcal{F}$ is the *torsionfree class* of the torsion pair, and its objects are the *torsionfree modules*. These classes may be both characterised in terms of closure properties: $\mathcal{T}$ is a torsion class iff its closed under homomorphic images, extensions and direct sums, whereas $\mathcal{F}$ is a torsionfree class iff it is closed under submodules, extensions and direct products. Yet, any torsion pair $(\mathcal{T},\mathcal{F})$ yields a functorial way to approximate the objects of Mod-R; that is, there exists an idempotent radical functor $r\colon \text{Mod-}R \to \mathcal{T}$ such that every module M_R fits in the short exact sequence $0 \to r(M) \to M \to M/r(M) \to 0$, where $M/r(M) \in \mathcal{F}$.

The orthogonality conditions display above are usually expressed with a more compact notation. Let $\mathcal{M}$ be a class of right R-modules, and define:

$$\begin{aligned} {}^{\perp_0}\mathcal{M} &= \operatorname{Ker}\operatorname{Hom}_R(-,\mathcal{M}) \\ {}^{\perp_1}\mathcal{M} &= \operatorname{Ker}\operatorname{Ext}^1_R(-,\mathcal{M}) \\ &\vdots \\ {}^{\perp}\mathcal{M} &= \bigcap_{n\in\mathbb{N}} {}^{\perp_n}\mathcal{M}, \end{aligned}$$

and dually $\mathcal{M}^{\perp_0}, \mathcal{M}^{\perp_1}, \dots, \mathcal{M}^{\perp}$. Therefore, $(\mathcal{T},\mathcal{F})$ is a torsion pair in Mod-R iff $\mathcal{T} = {}^{\perp_0}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp_0}$. More generally, given $\mathcal{M}$ as above, then

$$({}^{\perp_0}(\mathcal{M}^{\perp_0}), \mathcal{M}^{\perp_0}) \qquad \text{and} \qquad ({}^{\perp_0}\mathcal{M}, ({}^{\perp_0}\mathcal{M})^{\perp_0})$$

are called the torsion pairs *generated* resp. *cogenerated* by $\mathcal{M}$.

Any cotilting module U_R cogenerates the torsion pair $({}^{\perp_0}U, {}^{\perp_1}U)$, whose torsion radical is the so called reject in U, defined by $\operatorname{rej}_U(M) = \bigcap\{\operatorname{Ker} f \mid f \in \operatorname{Hom}_R(M,U)\}$. In particular, the two torsion pairs cogenerated by a cotilting bimodule ${}_SU_R$ are strictly related to the adjoint pair induced by this latter, for the reject actually are the kernel functor of the units ω's of the adjunction

$$\operatorname{Hom}_S(-,U) : S\text{-Mod} \rightleftarrows \text{Mod-}R : \operatorname{Hom}_R(-,U);$$

for instance, one unit ω is the natural transformation defined by the family of R-linear evaluation maps

$$\begin{aligned} \omega_M\colon M &\longrightarrow \operatorname{Hom}_S(\operatorname{Hom}_R(M,U),U) \\ x &\longmapsto \tilde{x}\colon (M_R \xrightarrow{f} U_R) \mapsto f(x)\ . \end{aligned}$$

Two cotilting modules $U, U' \in$ Mod-R are said to be *equivalent* if they cogenerate the same torsion pair.

We will use the forthcoming version of Hill's Lemma (a general form is [**?**, Theorem 7.10]). Let $M \in$ Mod-R and σ be an ordinal. An increasing chain of submodules $(M_\alpha \mid \alpha \leq \sigma)$ is called a *filtration* of M provided that $M_0 = 0$, $M_\alpha = \sum_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha \leq \sigma$, and $M_\sigma = M$. Given a class of right R-modules $\mathcal{M}$, a $\mathcal{M}$*-filtration* is a filtration $(M_\alpha \mid \alpha \leq \sigma)$ of M such that for anyall $\alpha < \sigma$ each factor module $M_{\alpha+1}/M_\alpha$ is isomorphic to some object of $\mathcal{M}$.

2.1. Theorem [**?**, Theorem 2.1]**.** *Let R be any ring and let $\mathcal{M} \subseteq$ mod-R be a set of finitely presented modules. Assume that the module M_R admits an $\mathcal{M}$-filtration. Then there exists a family $\mathcal{E}$ of submodules of M_R such that:*

(i) *$\mathcal{E}$ contains the $\mathcal{M}$-filtration of M_R;*
(ii) *$\mathcal{E}$ is closed under arbitrary sums and intersections;*
(iii) *For any $X' \leq X$ in $\mathcal{E}$, the factor X/X' admits an $\mathcal{M}$-filtration;*
(iv) *For any finitely generated submodule $K \leq M_R$, there exists $X \in \mathcal{E}$ such that $K \leq X$ and X admits an $\mathcal{M}$-filtration. In particular, X is finitely presented.*

2.2. **$\mathcal{D}$-reflexivity.** Assume that $F : S\text{-Mod} \rightleftarrows \text{Mod-}R : G$ is a right adjoint pair of contravariant functors. Since the additive categories S-Mod and Mod-R have enough projectives, then the total right derived functors $\mathbf{R}F$ and $\mathbf{R}G$ exist and so, by Lemma [],

$$\mathbf{R}F : \mathcal{D}(S\text{-Mod}) \rightleftarrows \mathcal{D}(\text{Mod-}R) : \mathbf{R}G$$

is a right adjoint pair. In other words, any right adjoint pair of contravariant functor between module categories extends to a right adjoint pair between their derived categories. Moreover, if the cohomological codimension of F and G is finite, that is.... then, by [], the adjunction restricts to the bounded derived categories

$$\mathbf{R}F : \mathcal{D}^b(S\text{-Mod}) \rightleftarrows \mathcal{D}^b(\text{Mod-}R) : \mathbf{R}G.$$

Notice that the covariant functors GF and FG admit total left derived functors, since the base categories have enough projectives. Thus it is reasonable to ask for $\mathbf{L}GF = \mathbf{R}G\mathbf{R}F$ and $\mathbf{L}FG = \mathbf{R}F\mathbf{R}G$ in order to ensure that the adjunction on module categories extends properly to the adjunction on derived categories. As we will see in the next lemma this is equivalent to ask that F maps projectives in G-acyclics , and G maps projectives in F-acyclics. (recall what acyclic means)

2.2 Lemma. *Let $F : S\text{-Mod} \rightleftarrows \text{Mod-}R : G$ be a right adjoint pair of contravariant functors of finite cohomological dimension. Then F maps projective in G-acyclics and G maps projective in F-acyclics if and only if $\mathbf{L}GF = \mathbf{R}G\mathbf{R}F$ and $\mathbf{L}FG = \mathbf{R}F\mathbf{R}G$.*

Proof. Assume F maps projective modules in G-acyclics. Let $M^\bullet$ be a complex in $\mathcal{D}^b(S\text{-Mod})$ and $P^\bullet$ a complex of projectives quasi-isomorphic to $M^\bullet$. Then $\mathbf{R}F(M^\bullet) = F(P^\bullet)$ is a complex of G-acyclic modules in $\mathcal{D}^b(\text{Mod-}R)$ and so we can compute $\mathbf{R}G(\mathbf{R}F(M^\bullet)$ as $GF(P^\bullet)$. But $GF(P^\bullet) = \mathbf{L}GF(M^\bullet)$, since GF is an additive covariant functor. So we conclude $\mathbf{L}GF = \mathbf{R}G\mathbf{R}F$. Similar for the other composition. Conversely, assume $\mathbf{L}GF = \mathbf{R}G\mathbf{R}F$. Let P be a projective left S module. Then P is acyclic with respect to any additive covariant functor, so $\mathbf{L}GF(P) = P$ (where here P is seen as a *stalk complex*, i.e. a complex with cohomology concentrated in zero degree). Moreover $\mathbf{R}F(P) = F(P)$. Since $\mathbf{R}G\mathbf{R}F(P) = GF(P)$ is a stalk complex, we get that the module $F(P)$ is G-acyclic. □

From now we will always assume that (F, G) satisfies the assumption of Lemma 2.2.

Following [] a complex is called $\mathcal{D}$-reflexive if the corresponding unit of the adjoint pair $(\mathbf{R}F, \mathbf{R}G)$ is a quasi-isomorphism. A module is called $\mathcal{D}$-reflexive if it is $\mathcal{D}$-reflexive as a stalk complex. Then we get:

2.3. Theorem [**?**, Theorem 1.6]**.** *A complex $M^\bullet$ is $\mathcal{D}$-reflexive if and only if every stalk complex $H^n(M^\bullet)$ is $\mathcal{D}$-reflexive.*

2.4. Theorem. *The subcategory of $\mathcal{D}$-reflexive modules is exact.*

Thanks to the previous theorem we get that $\mathcal{D}$ (the subcategory of $\mathcal{D}$-reflexive modules) is a thick subcategory of of the module category, and hence $\mathcal{D}^b_{\mathcal{D}}(S\text{-Mod})$ and $\mathcal{D}^b_{\mathcal{D}}(\text{Mod-}R)$ exist and we get

2.5 Corollary. *The adjunction* $(\mathbf{R}F, \mathbf{R}G)$ *induced a duality*

$$\mathbf{R}F : \mathcal{D}^b_{\mathcal{D}}(S\text{-Mod}) \rightleftarrows \mathcal{D}^b_{\mathcal{D}}(\text{Mod-}R) : \mathbf{R}G.$$

A very important example of an adjoint pair satisfying the assumptions of the previous discussion is given by the Hom-functors associated to a cotilting bimodule. Let R, S be two arbitrary rings and let ${}_SU_R$ be a cotilting bimodule. Let $\Delta_R = \text{Hom}_R(-, U)$ and $\Delta_S = \text{Hom}_S(-, U)$, or even Δ in case the ring over which U is regarded is clear. Notice that, since the injective dimension of U is at most one, then the cohomogical dimension of both the Δ is at most one. Moreover, since $\text{Ext}^i(U^\alpha, U) = 0$, for any $i > 0$, then projective modules are mapped into acyclic modules (see []). Hence for the right adjoint pair $\mathbf{R}\Delta_S : \mathcal{D}^b(S\text{-Mod}) \rightleftarrows \mathcal{D}^b(\text{Mod-}R) : \mathbf{R}\Delta_R$ the results previously stated hold. In particular, denoted by $\hat{\omega}\colon \text{id}_{\mathcal{D}(R)} \longrightarrow \mathbf{R}\Delta^2$ both the units of this adjunction, a module M is $\mathcal{D}$-reflexive if $\hat{\omega}(M)$ is a quasi-isomorphism.

Eventually, useful information about the interaction between the adjunctions (Δ_S, Δ_R) and $(\mathbf{R}\Delta_S, \mathbf{R}\Delta_R)$, together with the derived functors $\Gamma_R = \text{Ext}^1_R(-, U)$ and $\Gamma_S = \text{Ext}^1_S(-, U)$, are provided by the following (see [**?**, Theorem 1.2] for notation, in particular)

2.6. Proposition [**?**, Proposition 1.7]**.** *Let ${}_SU_R$ be a cotilting bimodule. For any $M \in \text{Mod-}R$ there is a diagram with exact row*

$$\begin{array}{ccccccccc} & & & & M & & & & \\ & & & & \Big\downarrow{\scriptstyle H^0(\hat{\omega}_M)} & & & & \\ 0 & \longrightarrow & \Gamma^2(M) & \xrightarrow{\alpha_M} & R^0\Delta^2(M) & \xrightarrow{\beta_M} & \Delta^2(M) & \longrightarrow & 0 \end{array}$$

in which $\beta_M \circ H^0(\hat{\omega}_M) = \omega_M$*. Moreover,*

$$R^i \Delta^2(M) = \begin{cases} 0 & \text{if } i \neq -1, 0, \\ \Delta\Gamma(M) & \text{if } i = -1. \end{cases}$$

Indeed, this result tells us that a right R-module M is $\mathcal{D}$-reflexive if and only if $\Delta\Gamma(M) = 0$ (whence $\mathbf{R}\Delta^2(M)$ is a stalk complex) and $H^0(\hat{\omega}_M)$ is R-linear isomorphism. If the sequel we will refer to the $\mathcal{D}$-reflexive modules as modules involved in a *cotilting duality*, in the sense of previous proposition.

3. When R is right artinian

By Corollary 2.5 we get that over a right artinian ring R, if ${}_SU_R$ is a cotilting bimodule, then U_R is of finite length. In addition we show that U_R is also product complete. First we need the following lemma, which is a specialisation of the Reiten–Ringel Condition [**?**, p. 522].

3.1 Lemma. *Let* ${}_SU$ *be a* Σ*-pure-injective cotilting module. If* $0 \to Y_0 \to Y \to X \to 0$ *is a short exact sequence with* $Y_0, Y \in {}^{\perp_1}U$*,* Y_0 *finitely generated, and* $X \in {}^{\perp_0}U$*, then* Y *is a finitely presented* S*-module.*

Proof. First let us prove that ${}_SX$ is finitely generated. Since ${}_SU$ is Σ-pure-injective, then by [**?**, Corollary 6.14] and [**?**, Corollary 4.4] the subcategory ${}^{\perp_1}U$ coincides with the class of direct summands of $\mathcal{F}_1$-filtered modules. Thus, there exists a $\mathcal{F}_1$-filtered module N such that $N = Y \oplus Y'$; by Hill Lemma 2.1 (???) there exists a finitely presented module X' such that $Y_0 \leq X' \leq N$ with N/X' being again $\mathcal{F}_1$-filtered, hence in particular in ${}^{\perp_1}U$. From the short exact sequence $0 \to X'/Y_0' \to N/Y_0 \overset{\varphi}{\to} N/X \to 0$ since $X \cong Y/Y_0 \leq N/Y_0$ is in ${}^{\perp_0}U$ by assumption, we obtain $X \leq \operatorname{Ker}\varphi$, hence in particular $X \leq_\oplus X'/Y_0$ and X is finitely generated as claimed. By extension-closure, Y is finitely generated, and in particular it is a finitely generated submodule of the $\mathcal{F}_1$-filtered module N. By Lemma 2.1 there exists a finitely presented module X'' such that $Y \leq X'' \leq N$ and since by [**?**, Corollary 5.4] the ring S is left coherent, we get that Y is finitely presented. □

3.2. Proposition. *Let* R *be a right Artinian ring and* ${}_SU_R$ *be a cotilting bimodule. Then* U_R *is finitely generated and product complete.*

Proof. By Remark 2.7 [], in order to conclude that U_R is product complete we have to show that S is left coherent and right perfect, and that ${}_SU$ is finitely presented. Since U_R is of finite length, then S is semiprimary and so left and right perfect [**?**, Corollary 28.8]. Moreover ${}_SU$ is Σ-pure-injective and hence S is left coherent by [**?**, Corollary 5.4].

Since R_R is artinian and hence finitely cogenerated, from the inclusion $R_R \hookrightarrow U^\alpha$ for some cardinal α we obtain a short exact sequence $0 \to R_R \to U^n \to M \to 0$ for a suitable finitely generated right R-module M and $n \in \mathbb{N}$. By applying Δ_R, the exact sequence $0 \to \Delta(M) \to S^n \to U \to \Gamma(M) \to 0$ in S-Mod yields a short exact sequence $0 \to K \to U \to \Gamma(M) \to 0$ with K a finitely generated left S-module. Since M_R is $\mathcal{D}$-reflexive by Corollary 2.6 **??**, then $\Delta\Gamma(M) = 0$ i.e. $\Gamma(M)$ is in ${}^{\perp_0}U$. Now, by Lemma 3.1, we get that ${}_SU$ is finitely presented. □

Proposition 3.2 shows that if R is left artinian, the existence of a cotilting bimodule ${}_SU_R$ forces U_R to be finitely generated and product complete. In the sequel we will show that the converse also holds true.

3.3. Proposition. *Let R be a right artinian ring and U_R a finitely generated cotilting module with $S = \mathrm{End}_R(U)$. If U_R is product complete, then ${}_SU_R$ is a cotilting bimodule. Moreover, S is left coherent and semiprimary, and ${}_SU$ is finitely presented and product complete.*

Therefore, by Proposition **??** and Theorem 3.9 we obtain the announced generalisation of Azumaya's duality. The proof will be postponed...

3.4. Theorem. *A right artinian ring R has a cotiling duality if and only if it admits a finitely generated product complete cotilting module.*

As we recalled before, the main difference between the Morita and the cotilting settings is that, in the latter, over a right artinian ring R the endomorphism ring of a finitely generated product complete cotilting module U_R needs not to be left artinian.

3.5. Proposition. *Let R be a right artinian ring with a cotilting duality induced by U_R, and let $S = \mathrm{End}_R(U)$. The following are equivalent:*

(a) *S is left artinian;*
(b) *S is left noetherian;*
(c) *$\Delta\Gamma(N) = 0$ for every finitely generated left S-module N.*

Proof. (a) ⇔ (b) The ring S is semiprimary as the endomorphism ring of the finite length module U_R. Then by Hopkins–Levitzki Theorem S is left artinian if and only if it is left noetherian.
(b) ⇒ (c) Since any finitely generated module ${}_SN$ is finitely presented, then it is $\mathcal{D}$-reflexive by [**?**, Lemma 3.1(1, c)], hence in particular $\Delta\Gamma(N) = 0$ (see Proposition 2.5 2.6).
(c) ⇒ (b) First, let us prove that the finitely generated left S-modules are precisely the $\mathcal{D}$-reflexives. Let N be a finitely generated S-module and let $0 \to K \to S^n \to N \to 0$ be an acyclic (with respect to the functor Δ_S) presentation of N . If N is torsionfree, then it is $\mathcal{D}$-reflexive by [**?**, Lemma 3.1(1, a)]. If N is torsion, by hypothesis also $\Gamma(N)$ is torsion, and we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & S^n & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \omega_K} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle H^0(\hat{\omega}_N)} & & \\ 0 & \longrightarrow & \Delta^2(K) & \longrightarrow & \Delta^2(S^n) & \longrightarrow & \Gamma^2(N) & \longrightarrow & 0 \end{array}$$

where $H^0(\hat{\omega}_N)$ is epic and therefore also $\Gamma^2(N)$ is torsion and finitely generated. By hypothesis we have $\Delta\Gamma(\Gamma^2(N)) = 0$ i.e. that $\Gamma^3(N)$ is torsion in turn.

Moreover, from the commutative diagram with exact rows where the first is a $\varDelta_R$-acyclic presentation of $\varGamma(N)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_R^n & \longrightarrow & \varDelta(K) & \longrightarrow & \varGamma(N) & \longrightarrow & 0 \\ & & \cong\downarrow & & \downarrow\omega_{\varDelta(K)} & & \downarrow H^0(\hat{\omega}_{\varGamma(N)}) & & \\ 0 & \longrightarrow & \varDelta^2(U^n) & \longrightarrow & \varDelta^3(K) & \longrightarrow & \varGamma^3(N) & \longrightarrow & 0 \end{array}$$

we obtain that $H^0(\hat{\omega}_{\varGamma(N)})$ is monic and $\operatorname{Coker}\omega_{\varDelta(K)} \cong \operatorname{Coker} H^0(\hat{\omega}_{\varGamma(N)})$. Notice that $\omega_{\varDelta(K)}$ is a splitting monomorphism, so the first term in the latter isomorphism is torsionfree while the second is torsion. Hence we get that they are both zero and in particular $\omega_{\varDelta(K)}$ is an isomorphism. By adjunction it follows that also ω_K is an isomorphism, hence K is $\mathcal{D}$-reflexive . By thickness of the subcategory of $\mathcal{D}$-reflexive modules, N is $\mathcal{D}$-reflexive. Finally, let N be an arbitrary finitely generated left S-module. Then $\operatorname{Im}\omega_N \cong N/\operatorname{Ker}\omega_N$ is finitely generated torsionfree, hence $\mathcal{D}$-reflexive by the previous argument. Notice that, thanks to Theorem 3.4, we are in the setting of [**?**], so that the functors Δ's map finitely generated in finitely presented modules. Hence $\operatorname{Im}\omega_N$ is a finitely generated submodule of the finitely presented module $\Delta^2(N)$, and hence it is also finitely presented. Hence we get that $\operatorname{Ker}\omega_N$ is finitely generated and torsion, so that $\mathcal{D}$-reflexive again. Thus we conclude that N is $\mathcal{D}$-reflexive.

Conversely, assume that N is a $\mathcal{D}$-reflexive module. Recall that we already know that S is left coherent by Theorem 3.4. The complex $\mathbf{R}\varDelta_S(N)$ in $\mathcal{D}(R)$ is $\mathcal{D}$-reflexive and hence it has $\mathcal{D}$-reflexive cohomologies. Since R is right artinian, its cohomologies are finitely generated [**?**, Proposition 3.6] and it is quasi-isomorphic to a complex formed by finitely generated projective terms. Then $\mathbf{R}\varDelta^2(N)$ is a complex of finitely presented modules, hence $N \cong R^0\varDelta^2(N)$ is finitely generated.

We are now ready to conclude that S is left noetherian. Indeed, given any short exact sequence $0 \to I \to {}_SS \to N \to 0$ in S-Mod, N is finitely generated and hence $\mathcal{D}$-reflexive. By thickness, I is is $\mathcal{D}$-reflexive and hence finitely generated. □

3.1. **Proof of Theorem 3.4.** . In order to prove the theorem, let us fix these assumptions: R is a right artinian ring, U_R a finitely generated product complete cotilting module with $S = \operatorname{End}_R(U)$. Then ${}_SU$ is finitely presented and S is left coherent. By [**?**, Propositions 1.2, 2.2], the functor $\varDelta_R$ carries finitely generated R-modules to finitely presented left S-modules. In order to prove that ${}_SU_R$ is a cotilting bimodule, we will show first that it is a faithful balanced bimodule.

Step one: ${}_SU_R$ is faithful balanced. Since U_R is product complete, any right R-module M admits a minimal left $\operatorname{Add}U$-approximations, i.e an exact sequence $0 \to M \to U^{(\alpha)} \to K \to 0$, where $K \in {}^{\perp_1}U$. In particular, since R_R is finitely cogenerated, it admits a coresolution

$$0 \longrightarrow R_R \xrightarrow{f_0} U_0 \xrightarrow{f_1} U_1 \xrightarrow{f_2} \cdots$$

where $U_i \in \operatorname{add} U_R$ and $\operatorname{Coker} f_i \in {}^{\perp_1}U$. Thus, by [**?**, Proposition 1] we conclude that U is a faithfully balanced S-R-bimodule and $\operatorname{Ext}^n_S(U,U) = 0$ for all $n > 0$.

Step two : $\operatorname{Cogen}{}_SU \subseteq \operatorname{Ker}\Gamma_S$ *and* $\text{i.d.}_S U \leq 1$. Assume M be a finitely generated module (finitely) cogenerated by U_R. Since $\operatorname{Add}U$ is contravariantly finite, then there is an exact sequence $0 \to M \to U^n \to L \to 0$, with $L \in {}^{\perp_1}U$. In particular, by [**?**, Lemma 1.1], any torsionless module finitely presented by U_R is reflexive and so, by [**?**, Theorem 1.4] we get that

$$\operatorname{Cogen}{}_SU \cap \operatorname{gen}{}_SS \subseteq \operatorname{Ker}\Gamma_S\ .$$

Now, since U_R is of finite length, ${}_SU$ is pure-injective and therefore, once an arbitrary $N \in \operatorname{Cogen}{}_SU$ is expressed as the direct limit of its finitely generated submodules, by the above display N is the direct limit of modules in $\operatorname{Ker}\Gamma_S$, hence

$$\operatorname{Cogen}{}_SU \subseteq \operatorname{Ker}\Gamma_S$$

by pure-injectivity. In order to conclude that ${}_SU$ is a partial cotilting module, it remains to show that $\text{i.d.}_S U \leq 1$. Let $0 \to K \to S^{(\alpha)} \to N \to 0$ be a short exact sequence in S-Mod. Since ${}_SS$ is cogenerated by U, then also K is cogenerated by ${}_SU$, so that $\operatorname{Ext}^2_S(N,U) \cong \operatorname{Ext}^1_S(K,U) = 0$.

3.6 Lemma. *For any finitely generated module* $N \in S\text{-Mod}$ *such that* $\operatorname{Hom}_S(N,U) = 0 = \operatorname{Ext}^1_S(N,U)$, *it is* $N = 0$.

Proof. Suppose $N \neq 0$ and $\operatorname{Hom}_S(N,U) = 0 = \operatorname{Ext}^1_S(N,U)$ and let us conclude with a contradiction. Let $0 \to K \to S^n \to N \to 0$ be a short exact sequence in S-Mod, $n \in \mathbb{N}$. The module K is the direct union of its finitely generated submodules K_i's, for some directed set I. Then the S^n/K_i's form a direct system of modules whose direct limit is N [**?**, Lemma 8.7]. Notice that any K_i is reflexive w.r.t. ${}_SU$, being finitely generated and cogenerated by ${}_SU$ [] (QUA COSA CITIAMO? FORSE CF?). For any $i \leq j$ in I, we have the following commutative diagrams with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \longrightarrow & K & \longrightarrow & C_i & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & K_j & \longrightarrow & K & \longrightarrow & C_j & \longrightarrow & 0 \end{array} \qquad \text{and} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & C_i & \longrightarrow & S^n/K_i & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & C_j & \longrightarrow & S^n/K_j & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

where each C_i is the cokernel of the inclusion $K_i \hookrightarrow K$. Now, the $\Delta(C_i)$'s form an inverse system of submodules of the right R-module of finite length $\Delta(K) \cong U^n_R$, and, moreover, $\varprojlim \Delta(C_i) \cong \Delta(\varinjlim C_i) = 0$. Since R is right artinian, we conclude that $\Delta(C_i) = 0$ eventually for $i \geq \bar{\imath}$. Therefore, we can assume the C_i's hence the S^n/K_i's belong to $\operatorname{Ker}\Delta_S$ (for $i \geq \bar{\imath}$). Notice that the $\Gamma_S(S^n/K_i) \neq 0$, otherwise from the short exact sequence $0 \to K_i \to S^n \to S^n/K_i \to 0$, we will get $\Delta_S(K_i) = \Delta_S(S^n)$, from which $K_i \cong \Delta^2(K_i) = \Delta^2(S^n) \cong S^n$. But this would imply $S^n/K_i = 0$, contradiction to $N = \varinjlim S^n/K_i$.

Now, for each $i \leq j \leq k$ in I, set $K_{ji} = K_j/K_i$ and consider the commutative diagram with exact row

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \longrightarrow & K_j & \longrightarrow & K_{ji} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_i & \longrightarrow & K_k & \longrightarrow & K_{ki} & \longrightarrow & 0 \end{array}$$

from which, by applying Δ_S, we obtain in Mod-R

Passing to the inverse limits, we get the exact sequence

$$0 \longrightarrow \varprojlim_{j\in I} X_{ij} \longrightarrow \Delta(K_i) \longrightarrow \overbrace{\varprojlim_{j\in I} \Gamma(K_{ji})}^{\cong \Gamma(\varinjlim K_{ji}) = \Gamma(C_i)} \longrightarrow 0\ .$$

Since U_R is pure-injective [**?**, Theorem 2.8] so that $\operatorname{Cogen} U$ is closed under direct limits, we obtain the short exact sequence Since $\Delta(K_i)$ is of finite length, then $\varprojlim_{j\in I} X_{ij} = \bigcap_{j\in I} X_{ij} = X_{i\bar{\jmath}}$ for a suitable index $\bar{\jmath} \in I$, whence $\Gamma(C_i) = \Gamma(K_{\bar{\jmath}i})$. Yet, since both C_i and $C_{\bar{\jmath}}$ are in $\operatorname{Ker} \Delta_S$, from $0 \to K_{\bar{\jmath}i} \to C_i \to C_{\bar{\jmath}} \to 0$ we get the exact row

$$0 \longrightarrow \Delta(K_{\bar{\jmath}i}) \longrightarrow \Gamma(C_{\bar{\jmath}}) \longrightarrow \Gamma(C_i) \longrightarrow \Gamma(K_{\bar{\jmath}i}) \longrightarrow 0\ .$$

The modules $\Gamma(C_i)$ and $\Gamma(K_{\bar{\jmath}i})$ have the same finite length, for they are isomorphic factors of the module of finite length $\Delta(K_i)$, so in turn $\Delta(K_{\bar{\jmath}i}) \cong \Gamma(C_{\bar{\jmath}})$ by exactness. Finally, since $\Gamma(C_{\bar{\jmath}}) \cong \Gamma(S^n/K_{\bar{\jmath}})$ is torsion as well, then $\Gamma(S^n/K_{\bar{\jmath}}) = 0$ i.e. $S^n/K_{\bar{\jmath}} = 0$ by the previous argument, contradiction. □

3.7 Lemma. *Let $(Y_i)_{i\in I}$ be a monomorphic direct system of left S-modules in $\mathcal{F}_1$ such that $Y_j/Y_i \in \mathcal{T}_1$, for any $i \leq j$. Then $\varinjlim Y_i \in \operatorname{Cogen}{}_S U$.*

Proof. By hypothesis on the Y_j/Y_i's, applying Δ_S we obtain an monomorphic inverse system $(\Delta(Y_i))_{i\in I}$ in Mod-R, whose inverse limit is $\bigcap_{i\in I} \Delta_S(Y_i)$. Since R is right artinian and the $\Delta_S(Y_i)$'s are submodules of the module of finite length U_R^n, there exists $\bar{\jmath} \in I$ such that $\Delta(Y_i) = \Delta(Y_{\bar{\jmath}})$ for all $i \geq \bar{\jmath}$. Since each Y_i is $\mathcal{D}$-reflexive for $Y_i \cong R^0\Delta^2(Y_i)$ (see Proposition 2.6), then $\varinjlim Y_i = Y_{\bar{\jmath}} \in \operatorname{Cogen}{}_S U$. □

3.8 Lemma. *Given a short exact sequence $0 \to Y_0 \to Y \to X \to 0$ in S-Mod with $Y_0 \in \mathcal{F}_1$, $Y \in \varinjlim \mathcal{F}_1$ and $X \in \varinjlim \mathcal{T}_1$, then $Y \in \operatorname{Cogen}{}_S U$.*

Proof. Write $X = \varinjlim X_i$ for the monomorphic direct system of its finitely generated submodules. Since $(\mathcal{T}_1, \mathcal{F}_1)$ is a torsion pair in S-Mod and $\operatorname{Hom}_S(X, \varinjlim \mathcal{F}_1) = 0$, with a usual argument we may assume that actually $X_i \in \mathcal{T}_1$. Now, each X_i is of the form Y_i/Y_0 for some $Y_i \in \mathcal{F}_1$ containing Y_0, hence we obtain the monomorphic direct system of short exact sequence $0 \to Y_0 \to Y_i \to X_i \to 0$, in which $Y_j/Y_i \cong X_j/X_i \in \mathcal{T}_1$ for all $i \leq j$. By the previous Lemma, $Y = \varinjlim Y_i \in \operatorname{Cogen}{}_S U$. □

3.9. Theorem. *Let R be a right artinian ring admitting a finitely generated cotilting module U_R with $S = \operatorname{End}_R(U)$. If U_R is product complete, then ${}_S U_R$ is a cotilting bimodule.*

Proof. First notice that, by [**?**, Proposition 3.9], since U_R is finitely generated and product complete, then ${}_SU$ is finitely presented and S is left coherent. We will often use the fact [**?**, Propositions 1.2, 2.2] that under our assumptions the functor Δ_R carries finitely generated R-modules to finitely presented left S-modules. In order to prove that ${}_SU_R$ is a cotilting bimodule, we will show first that ${}_SU_R$ is a partial cotilting bimodule, and then that $\operatorname{Ker}\Gamma_S \subseteq \operatorname{Cogen}{}_SU$, whence U is cotilting over S.

Step one. It is well known that a module cogenerated by a cotilting module is also copresented by this latter [**?**, Proposition 1.8]. In particular, since U_R is finitely generated and product complete by hypothesis, R_R admits a coresolution

$$0 \longrightarrow R_R \xrightarrow{f_0} U_0 \xrightarrow{f_1} U_1 \xrightarrow{f_2} \cdots$$

where $U_i \in \operatorname{add} U_R$ and $\operatorname{Coker} f_i \in \Gamma_R$. Thus, by [**?**, Proposition 1] we conclude that U is a faithfully balanced S-R-bimodule and $\operatorname{Ext}^n_S(U,U) = 0$ for all $n > 0$. It follows that, for every finitely generated projective module P_R, $\Delta_R(P)$ is Δ_S-acyclic (in the sense of [**?**, Definition 1.4]); that is, $R^n\Delta_S(\Delta_R(P)) = 0$ for all $n \neq 0$, hence any finitely generated right R-module is $\mathcal{D}$-reflexive. In particular, any torsionfree R-module in $\operatorname{gen} U_R$ is reflexive. By the proof of [**?**, Theorem 1.4] we conclude that

$$\operatorname{Cogen}{}_SU \cap \operatorname{gen}{}_SS \subseteq \operatorname{Ker}\Gamma_S \ .$$

Now, since U_R is of finite length, then in particular ${}_SU$ is pure-injective [**?**, Theorem 4.1]; therefore, once an arbitrary $N \in \operatorname{Cogen}{}_SU$ is expressed as the direct limit of its finitely generated submodules, by the above display N is the direct limit of modules in $\operatorname{Ker}\Gamma_S$, hence

$$\operatorname{Cogen}{}_SU \subseteq \operatorname{Ker}\Gamma_S$$

by pure-injectivity. In order to conclude that ${}_SU$ is a partial cotilting module, it remains to show that $\text{i.d.}_S U \leq 1$. Let $0 \to K \to S^{(\alpha)} \to N \to 0$ be a short exact sequence in S-Mod. Since ${}_SS$ is (finitely) cogenerated by U, then also K is cogenerated by ${}_SU$, so that $\operatorname{Ext}^2_S(N,U) \cong \operatorname{Ext}^1_S(K,U) = 0$.

Step two. Let us reverse the inclusion obtained in the previous step by proving formerly that $\operatorname{Ker}\Gamma_S \cap \operatorname{gen}{}_SS \subseteq \operatorname{Cogen}{}_SU$. Let $0 \to K \to S^n \to N \to 0$ be a short exact sequence with $\Gamma_S(N) = 0$. Then $0 \to \Delta(N) \to U_R^n \to \Delta(K) \to 0$ is a short exact sequence of finitely generated torsionfree right R-modules, hence (by [**?**]) $\Delta^2(N)$ and $\Delta^2(K)$ are finitely presented left S-modules. Therefore, in the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & S^n & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \omega_K} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \omega_N} & & \\ 0 & \longrightarrow & \Delta^2(K) & \longrightarrow & \Delta^2(S^n) & \longrightarrow & \Delta^2(N) & \longrightarrow & 0 \end{array}$$

we have that ω_N is epic, and $\operatorname{Ker}\omega_N \cong \operatorname{Coker}\omega_K$ is finitely generated. Moreover, since $\Delta^2(N) \in \operatorname{Ker}\Gamma_S$, we have $\operatorname{Ker}\omega_N \in \operatorname{Ker}\Delta_S \cap \operatorname{Ker}\Gamma_S$ hence $\operatorname{Ker}\omega_N = 0$ by Lemma 3.6; that is, N is reflexive and finitely presented. Observe that in particular we proved a stronger position than the claimed:

$$\operatorname{Ker} \Gamma_S \cap \operatorname{gen}{}_S S \subseteq \operatorname{Cogen}{}_S U \cap S\text{-mod} .$$

Now, since ${}_S U$ is a partial cotilting module, then $(\operatorname{Ker} \Delta_S, \operatorname{Cogen}{}_S U)$ and $(\mathcal{T}_1, \mathcal{F}_1)$ are torsion pairs in S-mod. Moreover, since S is left coherent, by [?, (4.4) p. 1666] also $(\varinjlim \mathcal{T}_1, \varinjlim \mathcal{F}_1)$ is a torsion pair in S-Mod. In particular, we have $\varinjlim \mathcal{T}_1 \subseteq \operatorname{Ker} \Delta_S$, $\operatorname{Cogen}{}_S U \subseteq \operatorname{Ker} \Gamma_S$ and $\varinjlim \mathcal{F}_1 = \operatorname{Ker} \Gamma_S$ (the third position follows since ${}_S U$ is pure-injective —thus $\operatorname{Ker} \Gamma_S$ is closed under submodules— together with the last display). In order to conclude that ${}_S U$ is cotilting, let us show that $\varinjlim \mathcal{F}_1 \subseteq \operatorname{Cogen}{}_S U$. First, let $N \in \varinjlim \mathcal{F}_1 \cap \operatorname{Ker} \Delta_S$ and fix $x \in N \setminus \{0\}$. Then we have the pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Sx & \longrightarrow & Q & \longrightarrow & r(X) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Sx & \longrightarrow & N & \longrightarrow & X & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y & = & Y & &
\end{array}$$

where r is the torsion radical associated with the torsion class $\varinjlim \mathcal{T}_1$, so that $Y \in \varinjlim \mathcal{F}_1$. Since $Sx \in \mathcal{F}_1$, $Q \in \varinjlim \mathcal{F}_1$ and $r(X) \in \varinjlim \mathcal{T}_1$, Lemma 3.8 applies on the first row so $Q \in \operatorname{Cogen}{}_S U$. In particular, there exists an S-linear map $f \colon Q \to U$ such that $f(x) \neq 0$, and since $\Gamma_S(Y) = 0$, then f can be extended to a nonzero map $\bar{f} \colon N \to U$ with $\bar{f}(x) \neq 0$, whence $N \in \operatorname{Cogen}{}_S U$ and therefore $N = 0$. Finally, let $N \in \varinjlim \mathcal{F}_1$, and consider the short exact sequence $0 \to t(N) \to N \to Y \to 0$, where t is the torsion radical associated with the torsion class $\operatorname{Ker} \Delta_S$. Then $t(N) \in \varinjlim \mathcal{F}_1 \cap \operatorname{Ker} \Delta_S$ and $Y \in \operatorname{Cogen}{}_S U$, hence by the previous argument we get $t(N) = 0$ and thus $N \in \operatorname{Cogen}{}_S U$. □

4. $\mathcal{D}$-Reflexive Modules and Artinian Rings

Cotilting *bimodules* generalise Morita bimodule: [?, Example 1.3] exhibits an artinian ring which does not admit a Morita bimodule but admits a faithfully balanced cotilting bimodule. In [?, Proposition 3.6] authors provided a first instance of analogy between Azumaya bimodules and cotilting bimodules. Namely, they proved that given a right artinian ring R, the $\mathcal{D}$-reflexive right R-modules w.r.t. a cotilting bimodule ${}_S U_R$ are the finitely generated ones, and U_R is finitely generated, hence of finite length. However, a crucial lack of symmetry occurs at once, for in the cotilting setting the ring S needs not to be left artinian:

4.1. Example. Consider the matrix ring $R = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{smallmatrix}\right)$. Then R is right artinian and not left artinian. Since R is hereditary, then by [?, Proposition 1.2] the regular bimodule ${}_R R_R$ is a cotilting bimodule.

This asymmetry affects also $\mathcal{D}$-reflexive left S- and right R-modules.

4.2. Proposition. *Let ${}_S U_R$ be a cotilting bimodule.*

(i) *If the class of $\mathcal{D}$-reflexive right R-modules coincides with the subcategory of finitely generated right R-modules, then R is right noetherian.*
(ii) *If both the classes of $\mathcal{D}$-reflexive modules coincide with the corresponding classes of finitely generated modules, then both U_R and ${}_SU$ are Σ-pure-injective.*

Proof. (i) Let $0 \to K \to R_R \to M \to 0$ be a short exact sequence in Mod-R. Since R_R and M are both $\mathcal{D}$-reflexive, then also K is, hence it is finitely generated by hypothesis.
(ii) Since by part (i) the ring R is right noetherian, by [**?**, Theorem 6.3] in order to prove that U_R is Σ-pure-injective it suffices to show that [**?**, Condition 5.1] holds true for the torsionfree class ${}^{\perp_1}U_R$. Hence, let $0 \to Y_0 \to Y \to X \to 0$ be a short exact sequence in Mod-R, where $Y_0, Y \in {}^{\perp_1}U$ and Y_0 is finitely generated, and $X \in {}^{\perp_0}U$. By hypothesis and by [**?**, Lemma 3.1(1, a)], $\Delta(Y_0)$ is finitely generated hence a $\mathcal{D}$-reflexive left S-module. Thus, $\Delta\Gamma(X) = 0$ and $H^0(\hat{\omega}_X)\colon X \to \Gamma^2(X)$ is a monomorphism, for the following commutative diagram with exact rows,

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & Y_0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0 \\ & & & & \downarrow\cong & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Delta\Gamma(X) & \longrightarrow & \Delta^2(Y_0) & \longrightarrow & \Delta^2(Y) & \longrightarrow & \Gamma^2(X) & \longrightarrow & 0 \end{array}$$

Moreover, since $\Delta(Y_0) \to \Gamma(X)$ is an epimorphism, also $\Gamma(X)$ is finitely generated hence $\mathcal{D}$-reflexive, and torsion. Since $\mathbf{R}\Delta(\hat{\omega}_{\Gamma(X)}) \circ \hat{\omega}_{\mathbf{R}\Delta(\Gamma(X))} = \mathrm{id}_{\mathbf{R}\Delta(\Gamma(X))}$, then also $\mathbf{R}\Delta(\Gamma(X)) = \Gamma^2(X)[-1]$ is $\mathcal{D}$-reflexive. Therefore, since $X \hookrightarrow \Gamma^2(X)$ and R is right noetherian, X is finitely generated i.e. $\mathcal{D}$-reflexive, so by extension-closure also Y is finitely generated. Same argument for ${}_SU$. □

However, we shall see that in case a cotilting duality entails the finitely generated modules, then the rings involved are necessarily artinian, hence a fundamental feature of Azumaya duality would be preserved.

4.3. Theorem. *Let ${}_SU_R$ be a cotilting bimodule. The following are equivalent:*
(a) *R is right artinian and S is left artinian;*
(b) *the $\mathcal{D}$-reflexive modules either over R and S are the finitely generated ones.*

If one of these conditions holds true, then U_R and ${}_SU$ are product complete.

Proof. "(a) ⇒ (b)" It follows by [**?**, Proposition 3.6].
"(b) ⇒ (a)" By the previous Proposition, both the rings R and S are noetherian and U_R and ${}_SU$ are Σ-pure-injective. Moreover, since U_R and ${}_SU$ are finitely generated being $\mathcal{D}$-reflexive, they are noetherian modules. By [**?**, Lemma 4.3], since U_R and ${}_SU$ σ-pure injective and noetherian over their endomorphism ring, they are endofinite. In particular, ${}_SU$ and U_R have finite length. Now, applying Δ_R on the epimorphism $R^n \to U_R \to 0$ ($n \in \mathbb{N}$) we obtain the monomorphism ${}_SS \hookrightarrow {}_SU^n$, hence S is left artinian being finitely U-cogenerated. Analogously one proves that R is right artinian.

Finally, since U_R and ${}_SU$ are of finite length and Σ-pure injective, again by [**?**, Lemma 4.3] they are endofinite hence product complete. □

4.4 Corollary. *If ${}_SUR$ is a cotilting bimodule and S and R are left artinian and right artinian, respectively, then*

Let us pass to discuss the representability of a derived duality by a cotilting bimodule, namely the remaining feature (besides Theorem 4.3) to generalise Azumaya's Theorem 1.1.

Assume that $F : S\text{-Mod} \rightleftarrows \text{Mod-}R : G$ is a right adjoint pair of contravariant functors of cohomological dimension 1 such that F and G map projectives in acyclic, so that we are in the assumption of Section[] and hence

$$\mathbf{R}F : \mathcal{D}^b(S\text{-Mod}) \rightleftarrows \mathcal{D}^b(\text{Mod-}R) : \mathbf{R}G$$

is an adjunction.

In view of Azumaya's characterization of an adjoint pair of functors giving a duality between the subcategories of finitely generated modules, it is natural to ask when the adjoint pair $(\mathbf{R}F, \mathbf{R}G)$ induces a duality between the bounded derived subcategories of complexes with finitely generated cohomologies. In other words, we are interested in characterizing $\mathbf{R}F$ and $\mathbf{R}G$ such that the image of a bounded complex with finitely generated cohomologies is again a bounded complex with finitely generated cohomologies, and the associated unit is a quasi-isomorphism.

4.5. Theorem. *Let $F : S\text{-Mod} \rightleftarrows \text{Mod-}R : G$ be two additive contravariant functors such that*

$$\mathbf{R}F : \mathcal{D}^b_{\text{fg}}(S\text{-Mod}) \rightleftarrows \mathcal{D}^b_{\text{fg}}(\text{Mod-}R) : \mathbf{R}G$$

is a duality. Assume that the cohomological dimension of F and G is at most 1 *and that F and G map projectives in acyclics. Then*

(i) *The duality is representable by a faithfully balanced bimodule ${}_SU_R$ which is finitely generated on both sides;*
(ii) *${}_SU_R$ is a cotilting bimodule;*
(iii) *R is right artinian and S is left artinian.*
(iv) *The $\mathcal{D}$-reflexive coincide with the finitely generated*

LO ENUNCEREI COSI' E POI METTEREI UN COMMENTO/CONFRONTO CON AZUMAYA (dopo la dimostrazione)

Proof. First notice that R turns out to be right noetherian and S left noetherian. Indeed, by assumption, any finitely generated right R-module is $\mathcal{D}$-reflexive. So let $I \leq R$ a right ideal od R. Then in the exact triangle $I \to R_R \to R/I \stackrel{+}{\to}$ of $\mathcal{D}^b(R)$ both R_R and R/I are finitely generated, hence $\mathcal{D}$-reflexive, so that I is $\mathcal{D}$-reflexive as well. In turn, we obtain that $\mathbf{R}G(I)$ belongs to $\mathcal{D}^b_{\text{fg}}(S\text{-Mod})$, and therefore $\mathbf{R}FG(I) \cong I$ belongs to $\mathcal{D}^b_{\text{fg}}(\text{Mod-}R)$; that is, $I = R^0F(G(I))$ is finitely generated. Similar argument to show that S is left noetherian. Thus, the categories $\mathcal{D}^b_{\text{fg}}(\text{Mod-}R)$ and $\mathcal{D}^b_{\text{fg}}(S\text{-Mod})$ are triangulated and equivalent respectively to $\mathcal{D}^b(\text{mod-}R)$ and $\mathcal{D}^b(S\text{-mod})$ [**?**, Proposition 4.8]. Therefore, from now on we will deal with these latter derived categories.

(i) Since R_R and ${}_SS$ are projective over themselves, their stalks are acyclic with respect to any additive functor, hence $R^nG(R_R) = 0 = R^nF({}_SS)$ for all $n \neq 0$; that is, $\mathbf{R}G(R_R)$ and $\mathbf{R}F({}_SS)$ are stalk complexes concentrated in degree zero. Set ${}_SV = \mathbf{R}G(R_R)$ and $U_R = \mathbf{R}F({}_SS)$; notice that by hypothesis U_R and ${}_SV$ are finitely generated. As usual, U_R admits a natural structure of left S-module compatible with the R-module structure (at level of module categories it is the one given by $sx = F(\dot{s})(x)$); analogously, V is an S-R-bimodule. Let us show that the bimodules ${}_SU_R$ and ${}_SV_R$ are isomorphic. By the (right) adjunction of $\mathbf{R}F$ and $\mathbf{R}G$, for every $N^\bullet \in \mathcal{D}^b(S\text{-mod})$ and every $M^\bullet \in \mathcal{D}^b(\text{mod-}R)$ we have

$$\mathrm{Hom}_{\mathcal{D}^b(\text{mod-}R)}(M^\bullet, \mathbf{R}F(N^\bullet)) \cong \mathrm{Hom}_{\mathcal{D}^b(S\text{-mod})}(N^\bullet, \mathbf{R}G(M^\bullet)),$$

so the claim readily follows by choosing $M^\bullet = R_R$ and $N^\bullet = {}_SS$ [**?**]. Moreover, the bimodule ${}_SU_R$ is faithfully balanced since

$$\begin{aligned} R \cong \mathrm{Hom}_R(R,R) &= \mathrm{Hom}_{\mathcal{D}^b(\text{mod-}R)}(R,R) \\ &\cong \mathrm{Hom}_{\mathcal{D}^b(\text{mod-}R)}(R, \mathbf{R}F \circ \mathbf{R}G(R_R)) \\ &\cong \mathrm{Hom}_{\mathcal{D}^b(S\text{-mod})}({}_SU, \mathbf{R}G(R)) = \mathrm{End}_{\mathcal{D}^b(S\text{-mod})}(U) = \mathrm{End}_S(U), \end{aligned}$$

and analogously for the other ring isomorphism. Now, let us prove that the duality is representable by ${}_SU_R$; that is, there are natural isomorphisms $\mathbf{R}F \cong \mathbf{R}\mathrm{Hom}_S(-,U)$ and $\mathbf{R}G \cong \mathbf{R}\mathrm{Hom}_R(-,U)$. For instance, for any $M^\bullet \in \mathcal{D}^b(\text{mod-}R)$, the complex $\mathbf{R}G(M^\bullet)$ is computed as the complex $0 \to G(P_0) \to G(P_1) \to G(P_2) \to \cdots$ for a homotopically projective resolution $P_\bullet \to M^\bullet$ formed by finitely generated projective terms, and for every $n \geq 0$ we have

$$\begin{aligned} G(P_n) &\cong \mathrm{Hom}_S(S, G(P_n)) \cong \mathrm{Hom}_{\mathcal{D}^b(S\text{-mod})}(S, \mathbf{R}G(P_n)) \\ &\cong \mathrm{Hom}_{\mathcal{D}^b(\text{mod-}R)}(\mathbf{R}F \circ \mathbf{R}G(P_n), \mathbf{R}F({}_SS)) \\ &\cong \mathrm{Hom}_{\mathcal{D}^b(\text{mod-}R)}(P_n, U_R) = \mathrm{Hom}_R(P_n, U_R) \end{aligned}$$

naturally; that is, the complex $\mathbf{R}G(M^\bullet)$ is naturally isomorphic to $\mathrm{Hom}_R(P_\bullet, U)$, i.e. to $\mathbf{R}\mathrm{Hom}_R(M^\bullet, U)$, as desired. Again, a similar argument shows that $\mathbf{R}F \cong \mathbf{R}\mathrm{Hom}_S(-,U)$ naturally.

(ii) Notice that the hypotheses on the functors $\Delta_R = \mathrm{Hom}_R(-,U)$ and $\Delta_S = \mathrm{Hom}_S(-,U)$ imply that the injective dimensions of ${}_SU_R$ are at most 1, and also $\mathrm{Ext}^1_R(U,U) = 0 = \mathrm{Ext}^1_S(U,U)$; for short, $\Gamma(U^\alpha) = 0$. For instance:

$$\begin{aligned} \mathrm{Ext}^1_R(U,U) &= \mathrm{Hom}_{\mathcal{D}^b(\text{mod-}R)}(U, U[1]) \\ &\cong \mathrm{Hom}_{\mathcal{D}^b(S\text{-mod})}(\mathbf{R}G(U), \mathbf{R}G(U)[1]) = \mathrm{Ext}^1_S(S,S)\ . \end{aligned}$$

The conclusion of the proof of that ${}_SU_R$ is a cotilting bimodule is the outcome of the following three steps.

Step one: Let us prove that

$$\mathrm{Ker}\,\Gamma_R \cap \text{mod-}R = \mathrm{Cogen}\,U_R \cap \text{mod-}R = \mathrm{cogen}\,U_R \cap \text{mod-}R,$$

and analogously over S. Using another notation, let us prove that the first two members classes, say $\bar{\mathcal{F}}_1$ and $\mathcal{F}_1$ respectively, both coincide with the class of finitely presented finitely U-cogenerated modules. Notice that $\mathcal{T}_1 \cap \bar{\mathcal{F}}_1 = 0$ since for all $M \in \text{mod-}R$ by part (i) we have $\mathbf{R}G(M) = \mathbf{R}\mathrm{Hom}_R(M,U) \neq 0$. If $M \in \bar{\mathcal{F}}_1$, then by Proposition 2.6 we have $M \cong R^0\Delta^2(M) = \Delta^2(M)$. From the exact sequence $0 \to K \to S^n \to \Delta_R(M) \to 0$ we then obtain the inclusion $\Delta^2(M) \hookrightarrow U_R^n$, so that M_R is a finitely presented finitely U-cogenerated module. Conversely, Γ_R vanishes on the finitely presented modules which embed in some U_R^n ($n \in \mathbb{N}$) because its kernel is closed under subobjects.

On the other hand, it remains to prove that $\mathcal{F}_1 \subseteq \operatorname{cogen} U_R \cap \text{mod-}R$. Every $M \in \mathcal{F}_1$ is reflexive since $M \cong R^0\Delta^2(M)$ and ω_M is a monomorphism. Now, we have $\Delta^2(M) = \Delta_S(\Delta_R(M)) \hookrightarrow U_R^n$ since $\Delta_R(M) \in S\text{-mod}$.

Step two: Let us prove that both the $(\mathcal{T}_1, \mathcal{F}_1)$'s are torsion pairs in mod-R and S-mod. For instance, let $M \in \text{mod-}R$, then $\operatorname{Im}\omega_M$ is a finitely presented finitely U_R-cogenerated module, and since $\Gamma_R(\operatorname{Im}\omega_M) = 0$, then $\Delta_R(\operatorname{Im}\omega_M) = 0$; that is, $\operatorname{Ker}\omega_M \in \mathcal{T}_1$. In particular, the torsion pairs are faithful i.e. the regular modules are torsionfree.

Now, since both the rings are noetherian, by [**?**, Theorem 1.5] $(\varinjlim \mathcal{T}_1, \varinjlim \mathcal{F}_1)$'s are cotilting torsion pairs in Mod-R and S-Mod; that is, they are cogenerated by some cotilting modules $U' \in \text{Mod-}R$ and $V' \in S\text{-Mod}$.

Step three: Let us prove that the torsion pair $(\varinjlim \mathcal{T}_1, \varinjlim \mathcal{F}_1)$ (e.g. in Mod-R) satisfies the Reiten–Ringel Condition. First, notice that since $\operatorname{Ker}\Delta$ is closed under direct limits and $\mathcal{T}_1 \subseteq \operatorname{Ker}\Delta$ by definition, then $\varinjlim \mathcal{T}_1 = \operatorname{Ker}\Delta$. On the other hand, $\operatorname{Cogen} U \subseteq \varinjlim \mathcal{F}_1$. Since we are over noetherian rings, it suffices to show that if in any short exact sequence $0 \to Y_0 \to Y \to X \to 0$ we have $Y_0 \in \mathcal{F}_1$, $Y \in \varinjlim \mathcal{F}_1$ and $X \in \varinjlim \mathcal{T}_1$, then Y is finitely generated. By Lemma 3.8 the hypotheses **WOULD** imply that $Y \in \operatorname{Cogen} U$. Moreover, $\Delta(Y) \leq \Delta(Y_0)$ so $\Delta(Y)$ is finitely generated hence $\Delta^2(Y)$ is finitely generated in turn. Therefore, since $Y \hookrightarrow \Delta^2(Y)$, we are done.

The claim then implies that the cotilting module U' cogenerating the torsion pair is Σ-pure-injective [**?**, Theorem 6.3], hence any torsionfree module (i.e. in $\varinjlim \mathcal{F}_1$) is $\mathcal{F}_1$-filtered [**?**, Proposition 6.4]. Thus, we have $\varinjlim \mathcal{F}_1 \subseteq {}^{\perp_1}(\mathcal{F}_1^{\perp_1}) \subseteq {}^{\perp_1}U_R$, whence we obtain $\operatorname{Ker}\operatorname{Ext}^1_R(-, U) \cong \operatorname{Ker}^1_R(-, U')$; that is, U_R is equivalent to U' so that the former is a Σ-pure-injective (finitely presented) cotilting bimodule.

(iii) It follows at once by Theorem 4.3.

(iv) It follows from ... since the rings are artinian. □

Let us show how to deduce and lighten the proof of the previous theorem within different hypotheses.

4.6. Theorem. *Let $F : S\text{-Mod} \rightleftarrows \text{Mod-}R : G$ of cohomological dimension* 1. *If R and S are perfect (ma dalla parte giusta, da controllare), then F and G map projectives in acyclics.*
Then statements (i)–(iii) *of Thereom 4.5 hold true.*

Proof.
(1) The hypothesis of that $R^1G(F(Q)) = 0 = R^1F(G(P))$ for all projective modules ${}_SQ$ and P_R ensures that $\mathbf{R}G \circ \mathbf{R}F = \mathbf{R}(GF)$ and $\mathbf{R}F \circ \mathbf{R}G = \mathbf{R}(FG)$ [**?**, Proposition 5.4].

(i) It is identical as in the previous proof.

(ii) Besides the axiom (i) of cotilting module, under the new assumption we also gain axiom (ii); that is U_R and ${}_SU$ are partial cotilting modules. So, let us prove that $\operatorname{Ker}\Delta \cap \operatorname{Ker}\Gamma = 0$. Consider e.g. the case over R, so that let $M \in \operatorname{Ker}\Delta_R \cap \operatorname{Ker}\Gamma_R$ be a nonzero module. Consider a finitely generated submodule $M' \leq M$; from the short exact sequence $0 \to M' \to M \to M'' \to 0$ we get the triangle $\mathbf{R}\Delta(M'') \to \mathbf{R}\Delta(M) \to \mathbf{R}\Delta(M') \stackrel{+}{\to}$, and hence we obtain by hypothesis $\Delta_R(M') = R^0\Delta(M') \cong R^1\Delta(M'') = \Gamma_R(M'')$. Since $\Delta_R(M') \in \operatorname{Ker}\Gamma_S$, we obtain $0 = \operatorname{Ext}^1_S(\Gamma_R(M''), U)^\alpha \cong \operatorname{Ext}^1_S(\Gamma_R(M''), U^\alpha)$ for every cardinal α. This means that from the short exact sequence $0 \to K \stackrel{i}{\to} R^{(\alpha)} \to M'' \to 0$ in Mod-R we obtain the split exact sequence $0 \to U^\alpha \to \Delta_R(K) \to \Gamma_R(M'') \to 0$ in S-Mod, whence $\Delta^2(K) \cong \Delta^2(R^{(\alpha)}) \oplus \Delta\Gamma(M'')$. Therefore, from these facts together with the hypothesis, we have that the morphism $\hat{\omega}_{M''}$ is described by the following cochain map:

$$\begin{array}{ccccccccc}
M'' & & 0 & \longrightarrow & K & \xrightarrow{\;i\;} & R^{(\alpha)} & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle \hat{\omega}_{M''}} & & & & {\scriptstyle \omega_K}\big\downarrow & & \big\downarrow{\scriptstyle \omega_{R^{(\alpha)}}} & & \\
\mathbf{R}\Delta^2(M'') & & 0 & \longrightarrow & \Delta^2(K) & \xrightarrow{\Delta^2(i)} & \Delta^2(R^{(\alpha)}) & \longrightarrow & 0
\end{array}$$

and since $\Delta^2(i)$ is a split epimorphism, $\hat{\omega}_{M''}$ is null homotopic. In particular, $\mathbf{R}\Delta(\hat{\omega}_{M''}) = 0$, contradiction to the adjunction formula $\mathbf{R}\Delta(\hat{\omega}_{M''}) \circ \hat{\omega}_{\mathbf{R}\Delta(M'')} = \mathrm{id}$.
(iii) Since U_R and ${}_SU$ are cotilting modules, by pure-injectivity $\operatorname{Cogen} U$ is closed under direct limits, hence we can retrace the conclusion of part (ii) of the previous Theorem and obtain with the same arguments that the rings R and S are both artinian.
(2) There is nothing to prove. However, notice that within the assumption, by [**?**, Proposition 3.9] U is product complete, hence Σ-pure-injective, so that the third claim in the proof of Theorem 4.5 can be skipped. □

Alla fine concludiamo che se i finitamente generati sono D-riflessivi, con immagine finitamente generata (come Azumaya), allora la dualita' e' rappresentabile da un cotilting, gli anelli sono artiniani e quidni i finitamente generati sono tutti e soli i D-riflessivi