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Natural deduction calculi for classical and intuitionistic S5

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ABSTRACT

We propose an indexed natural deduction system for the modal logic **S5**, ideally following Wansing's previous work in the context of tableaux sequents. The system, given both in the classical and intuitionistic versions (called $\mathcal{N}_5^{\mathbf{c}}$ and $\mathcal{N}_5^{\mathbf{i}}$ respectively), is designed to match as much as possible the structure and properties of the standard system of natural deduction for first-order logic, exploiting the formal analogy between modalities and quantifiers. We study a (syntactical) normalization theorem for both $\mathcal{N}_5^{\mathbf{c}}$ and $\mathcal{N}_5^{\mathbf{i}}$ and its main consequences, the sub-formula principle and the consistency theorem. In particular, we propose an intuitionistic encoding of classical **S5** (via a suitable extension of the Gödel translation for first-order classical logic). Moreover, via the **BHK** interpretation of intuitionistic proofs, we propose a suitable Curry–Howard isomorphism for $\mathcal{N}_5^{\mathbf{i}}$. By translation into the natural deduction system given by Galmiche and Salhi in [(2010b). Label-free proof systems for intuitionistic modal logic is5. In E. M. Clarke & A. Voronkov (Eds.), *Logic for programming, artificial intelligence, and reasoning* (pp. 255–271). Springer Berlin Heidelberg.], we prove the equivalence of $\mathcal{N}_5^{\mathbf{i}}$ w.r.t. an Hilbert-style axiomatization of **IS5**. However, when considering the sheer provability of labelled formulas, our system is comparable to the one presented by Simpson in [(1993). The proof theory and semantics of intuitionistic modal logic [PhD thesis], University of Edinburgh, UK.]. Nevertheless, it remains uncertain whether it is feasible to establish a translation between the corresponding derivations.

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1. Introduction

It is a fact that developing a good proof theory for modal logics is a difficult task. The problem is not in having deductive systems. In fact, all the main modal logics enjoy an axiomatic presentation or in some case a presentation in terms of the so-called semantic tableaux (D'Agostino et al., 1999). The challenging point is in having a concrete structural proof theory, in which the objects of study are (not only) modal formulas but also modal proofs.

For example, a well-defined proof theory cannot ignore the syntactical study of cut elimination/normalization theorem and its consequences, such as following Gentzen

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and Prawitz, the sub-formula property and the consistency theorem (see Troelstra & Schwichtenberg, 2000).

This was clear to Prawitz when in his famous book 'Natural deduction: a proof theoretical study' (Prawitz, 1965) he tried to develop the proof theory for one of the most important modal system, the logic **S4**.

Unfortunately, the ideas developed by Prawitz are rarely useful for modal systems other than **S4**. For example for the logic **S5**, one of the first normal modal systems, whose Kripke models are characterised by the fact that the accessibility relation is an equivalence relation. Despite of its (apparently) simplicity, **S5** lacks of a fully satisfactory natural deduction. For instance, in Martins and Martins (2008), the authors arrived to formulate a natural deduction for **S5** following the classical canons of Gentzen and Prawitz's natural deduction only at the cost of a rather unnatural system, with an objectively complex normalization proof.

In this paper, we address again the problem and propose an *indexed natural deduction system* for **S5**.

Before describing our proposal, we contextualise it with respect to the (pertinent) state of the art of modal proof theory. It is since the early 90s that we have witnessed a real (re)birth of interest in *structural modal proof theory*. In particular, two new and different approaches brought the attention of the logicians' community.

The first approach could be defined as a *geometric* one: the leading idea is to equip formulas with a notion of index or position that provide a sort of 'spatial coordinate'. Examples are 2-sequents (Baratella & Masini, 2003, 2019; Guerrini et al., 1998; Martini & Masini, 1996; Martini et al., 2021; Masini, 1992, 1993), (linear) nested sequents (Brünnler, 2009; Lellmann, 2015; Lellmann & Pimentel, 2019; Poggiolesi, 2009), indexed natural deduction systems, hyper-sequents and their numerous variations (Avron, 1996; Ciabattoni et al., 2014; Wansing, 1999).

The second approach, the so-called Labelled Deductive Systems, is based on the first-order translation of modal logics: that is, the rules that model the accessibility relationship are explicitly imported into the syntactical deductive instruments. Thanks to the direct handling of the accessibility relation in the syntax, Labelled Deductive Systems are very useful in domain-specific applications of logical system, from temporal logic (Masini et al., 2010; Masini, Viganò, & Volpe, 2011) to logic for quantum computation (Masini et al., 2008; Masini, Viganò, & Zorzi, 2011; Viganò et al., 2017) to expert systems (Cristani et al., 2019). Clearly, this versatility has a price: labelled systems require a huge amount of rules and their proof theory often becomes quite exhausting.

The 'geometric' approach bases on very different premises and, in general, geometric systems are sensibly more tractable. Despite this, various logical systems remained in a limbo for a long time. One of these logics is **S5**, usually treated by labelled deductive systems at the cost of a heavy overweight given by the rules formalizing the properties of the accessibility relation.

Yet, a clear solution was provided in the far 70s and was given in the field of semantic tableaux by M. Fitting in Fitting (1977). The idea is simple: each formula is equipped with a label (e.g. a natural number) and each modal rule rewrites the label of the formula. No need of explicit treatment of the accessibility relation since the accessibility relation between labels is universal and hence neglected. Unfortunately, Fitting's proposal is not oriented to proof theory, in particular, no syntactical proof of cut elimination was proposed. We have to wait until the middle of the 90s to get a proof-theoretical status for Fitting's system, when Wansing proposed in Wansing (1995) a labelled tableaux calculus á la Fitting for which he was able to prove a strong cut-elimination theorem.[1]

Our proposal can be seen as an ideal continuation of Wansing's one, but it is also related to other interesting proof-theoretic investigations mainly focused on sequent-based approaches. Among the others, we recall here the following ones. For the classical logic **S5**, it is mandatory to cite the Avron's hypersequent calculus (Avron, 1996), the tree/hypersequent calculi by Poggiolesi defined in Poggiolesi (2008) and Poggiolesi (2009), the hypersequent calculus with restricted contexts by Lellmann (2016), the deep sequent calculus introduced by Brünnler in Brünnler (2009), and finally Renstall's proof nets for **S5** studied in Restall (2007). For the intuitionistic logic **S5**, we cite Simpson (1993), the nested sequent calculus proposed in Strassburger (2010) by Strassburger, and the Galmiche and Sahli's calculi defined in Galmiche and Salhi (2010a) and in Galmiche and Salhi (2010b).

For a detailed comparison with these systems, we address the reader to Sections 6 and 7. In particular, in Sections 6.1 and 6.2, we show the correspondence between the proofs of our natural deduction system and the one given in Galmiche and Salhi (2010b). While in Section 7, we prove that our intuitionistic system is equivalent to Simpson's one (Simpson, 1993), and we analyse Wansing (1995) and other related works.

1.1. Our proposal

We define and study a natural deduction system for **S5** inspired by the indexed formulas used by Wansing (1995) (for a light introduction to Wansing's proposal, see Section 7).

We define both a classical and an intuitionistic version of **S5**, named $\mathcal{N}_5^{\mathbf{c}}$ and $\mathcal{N}_5^{\mathbf{i}}$, respectively. Although our intuitionistic calculus shares some similarities with the one proposed by Simpson (1993), as we will argument in detail in Section 7, our proposal is quite different from Simpson's one, both in terms of derivability and from a proof theoretic point of view. Indeed, the design of Simpson's system is semantically driven and, because of this, it contains relational rules encoding the expected accessibility relation on the Kripke model of the conclusion that one wants to prove. Differently from Simpson, we do not have an explicit syntactical representation of the accessibility relation and a set of rules governing it. In our system, each rule is either an introduction or an elimination of connectives and modal quantifiers. So, our intuitionistic calculus is motivated by means of a **BHK** informal semantics, which is successively formalised into a suitable term calculus, obtaining in this way a Curry–Howard isomorphism between formulas/proofs/reduction and types/lambda-terms/computational steps.

In our opinion, this is the right argument which allows us to say that $\mathcal{N}_5^{\mathbf{i}}$ is truly intuitionistic, regardless of its Kripke-style semantics, which is instead the main argument used by Simpson to assert that his framework is intuitionistic. For the sake of completeness, we have to say that Simpson's purpose was completely different from our one and that he fully achieved it by giving a framework in which one can obtain different modal logics by modulating the relational rules.

Like in previous contributions on this subject, our approach was instead to keep the proof systems as simple as possible and to maintain the structure of the basic formats à la Prawitz, to preserve the general good proof-theoretic properties of Prawitz's natural deduction. The key idea of our system is the decoration of formulas with indexes that play the role of a sort of (first-order) variables. Modalities act then on this indexing by (possibly) changing the index of their formula into another one (with some restrictions), mirroring at the syntactic level what happens at the semantic level in **S5** universal models. We shall show that our classical system $\mathcal{N}_5^c$ is (weakly) complete w.r.t. the standard Hilbert-style axiomatization of **S5**. For both $\mathcal{N}_5^c$ and $\mathcal{N}_5^i$, we shall prove a (syntactical) normalization theorem, as well as its main consequences: the sub-formula principle and the consistency theorem.

1.2. Synopsis

In Section 2, we recall the modal logic **S5**, its Hilbert style axiomatization and we introduce indexed modal formulas; in Section 3, we present the classical natural deduction system $\mathcal{N}_5^c$ and in Section 3.1 we prove it can derive all the formulas of the Hilbert axiomatization of **S5**. The semantics and the soundness theorem for $\mathcal{N}_5^c$ are in Section 4. In Section 5, we prove a weak normalization theorem for $\mathcal{N}_5^c$ and, in Section 5.3, we study its consequences. The intuitionistic natural deduction calculus $\mathcal{N}_5^i$ and its properties are treated in Section 6, where it is also proposed a term calculus, with which we establish a Curry–Howard isomorphism. Finally, in Section 7, we propose a comparison with some related approaches.

2. Preliminary notions

In this section, we recall the modal logic **S5** and its Hilbert-style axiomatization. We also introduce the notion of *indexed formula*.

The language $\mathcal{L}$ of **S5** contains the following symbols:

- a countably infinite set of *propositional symbols*, $AT = \{p_0, p_1, \ldots\}$;
- the *propositional connectives* $\vee, \wedge, \rightarrow, \bot$;
- the *modal operators* $\Box, \Diamond$;
- the *auxiliary symbols* (and).

As usual, $\neg A$ is a shorthand for $A \rightarrow \bot$.

Definition 2.1: The set of *modal formulas* is the least set that contains the propositional symbols and is closed under application of the propositional connectives and the modal operators. A modal formula is atomic if it is either a propositional symbol or the connective $\bot$.

We recall now the Hilbert style axiomatization of **S5**.

Definition 2.2: The logic **S5** is the smallest set X of modal formulas that contains all the instances of the following schemata:

P1. $A \to (B \to A)$
P2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
P3. $((\neg B \to \neg A) \to ((\neg B \to A) \to B))$
K. $\Box(A \to B) \to (\Box A \to \Box B)$
T. $\Box A \to A$
4. $\Box A \to \Box\Box A$
B. $A \to \Box\Diamond A$

and the following closures:

MP if $A, A \to B \in X$ then $B \in X$;
NEC if $A \in X$ then $\Box A \in X$.

When $A \in$ **S5**, we shall write $\vdash_H A$.

2.1. Indexes and indexed formulas

Let $\mathcal{T}$ be a denumerable set whose elements we call *tokens*, ranged by meta-variables x, y, z, possibly indexed.

Definition 2.3: An *indexed-formula* (briefly *i-formula*) is an expression of the form A^x, where A is a modal formula and $x \in \mathcal{T}$.

Given a set/sequence Γ of i-formulas, and a token x, with a little abuse of language we write $x \in \Gamma$ ($x \notin \Gamma$) to denote that Γ contains (does not contain) an *i*-formula A^x.
The notion of sub-formula is quite standard:

Definition 2.4 (Sub-formulas): Given an *i-formula* A^x, we recursively define the set $\mathrm{Sf}(A^x)$ of its sub-formulas as follows:

- $\mathrm{Sf}(A^x) = \{A^x\}$ if A is atomic;
- $\mathrm{Sf}(A \vee B^x) = \mathrm{Sf}(A^x) \cup \mathrm{Sf}(B^x) \cup \{A \vee B^x\}$;
- $\mathrm{Sf}(A \wedge B^x) = \mathrm{Sf}(A^x) \cup \mathrm{Sf}(B^x) \cup \{A \wedge B^x\}$;
- $\mathrm{Sf}(A \to B^x) = \mathrm{Sf}(A^x) \cup \mathrm{Sf}(B^x) \cup \{A \to B^x\}$;
- $\mathrm{Sf}(\Box A^x) = \mathrm{Sf}(A^y) \cup \{\Box A^x\}$ for any $y \in \mathcal{T}$;
- $\mathrm{Sf}(\Diamond A^x) = \mathrm{Sf}(A^y) \cup \{\Diamond A^x\}$ for any $y \in \mathcal{T}$.

The notion of subformula is then extended to set of formula Γ:

$$\mathrm{Sf}(\Gamma) = \bigcup_{A^x \in \Gamma} \mathrm{Sf}(A^x)$$

3. Classical natural deduction for S5

We study now a natural deduction system for **S5**. In this section, we start by giving the rules of the classical system $\mathcal{N}_5^{\mathbf{c}}$, whose intuitionistic version $\mathcal{N}_5^{\mathbf{i}}$ will be studied in Section 6. In Section 4, we shall then analyse the semantics of **S5**, and in Section 5 its normalization.

Logical rules

$$\dfrac{\begin{matrix}\vdots & \vdots\\ A^x & B^x\end{matrix}}{A\wedge B^x}\,(\wedge I)\qquad \dfrac{\begin{matrix}\vdots\\ A\wedge B^x\end{matrix}}{A^x}\,(\wedge_1 E)\qquad \dfrac{\begin{matrix}\vdots\\ A\wedge B^x\end{matrix}}{B^x}\,(\wedge_2 E)$$

$$\dfrac{\begin{matrix}\vdots\\ A^x\end{matrix}}{A\vee B^x}\,(\vee_1 I)\qquad \dfrac{\begin{matrix}\vdots\\ B^x\end{matrix}}{A\vee B^x}\,(\vee_2 I)\qquad \dfrac{\begin{matrix} & [A^x] & [B^x]\\ \vdots & \vdots & \vdots\\ A\vee B^x & C^y & C^y\end{matrix}}{C^y}\,(\vee E)$$

$$\dfrac{\begin{matrix}[A^x]\\ \vdots\\ B^x\end{matrix}}{A\to B^x}\,(\to I)\qquad \dfrac{\begin{matrix}\vdots & \vdots\\ A\to B^x & A^x\end{matrix}}{B^x}\,(\to E)$$

$$\dfrac{\begin{matrix}[\neg A^x]\\ \vdots\\ \perp^y\end{matrix}}{A^x}\,(\perp_c)\qquad \dfrac{\begin{matrix}\vdots\\ \perp^y\end{matrix}}{A^x}\,(\perp_i)$$

In the $\perp_i$ and $\perp_c$ rules, whenever A^x is $\perp^x$, we require $x \neq y$.

Note that the rules for logical connectives $\wedge$ and $\to$ and the $\vee I$ rule can be only applied between formulas with the same index. The rule $\vee E$ mimics instead the original rule by Prawitz and has no restriction on the index of the conclusion C^y.

The role of the indexes is clear in the rules for the modalities $\Box$ and $\Diamond$:

$$\dfrac{\begin{matrix}\vdots\\ A^x\end{matrix}}{\Box A^y}\,(\Box I)\qquad \dfrac{\begin{matrix}\vdots\\ \Box A^x\end{matrix}}{A^y}\,(\Box E)$$

In the rule $\Box I$, one has $x \neq y$ and $x \notin \Gamma$, where Γ is the set of (open) assumptions on which A^x depends. The token x of $\Box I$ is called *proper token* of the rule.

$$\dfrac{\begin{matrix}\vdots\\ A^x\end{matrix}}{\Diamond A^y}\,(\Diamond I)\qquad \dfrac{\begin{matrix} & [A^x]\\ \vdots & \vdots\\ \Diamond A^z & C^y\end{matrix}}{C^y}\,(\Diamond E)$$

In the rule $\Diamond E$, one requires $x \neq z$, $x \neq y$ and $x \notin \Gamma$, where Γ is the set of (open) assumptions on which C^y depends, with the exception of the discharged assumptions A^x.

The token x of $\Diamond E$ is called *proper token* of the rule.

Given an instance of a rule R we define the concept of *principal premise(s)* of R (that is a trivial extension of the classical notion in order to take into account also the case of modal rules).

- If R is an introduction or a $\perp_c$ or a $\perp_i$ rule, then the premise(s) of R is(are) the principal premise(s) of R;
- if R is an elimination rule, then the principal premise is the formula that has the main connective object of the elimination. More precisely, w.r.t. the previous stated rules:
  - the principal premise of $\wedge E$ is $A \wedge B^x$;
  - the principal premise of $\rightarrow E$ is $A \rightarrow B^x$;
  - the principal premise of $\vee E$ is $A \vee B^x$;
  - the principal premise of $\Box E$ is $\Box A^x$;
  - the principal premise of $\Diamond E$ is $\Diamond A^z$.

A premise is called *minor* if it is not principal.

As usual with $\Gamma \vdash A^x$ we denote that there is a derivation with conclusion A^x whose open assumptions belong to Γ.

By token renaming, the following proposition is easily provable (see Troelstra & van Dalen, 1988b, Vol. 2, pag. 529 for the analogous statement for *proper parameters* in the case of first-order logic).

Proposition 3.1: *Let $\Gamma \vdash A^x$. There exists a deduction of A^x from Γ in the system $\mathcal{N}$ such that*

(1) *each proper token is the proper token of exactly one instance of a $\Box I$ or of a $\Diamond E$ rule;*
(2) *the proper token of any instance of a $\Box I$ rule occurs only in the sub-derivation above that instance of the rule;*
(3) *the proper token of any instance of a $\Diamond E$ rule occurs only in the sub-derivation above the minor premise of that instance of the rule.*

Definition 3.2 (Token condition): A deduction satisfying the conditions of Proposition 3.1.(1)–(3) is said to satisfy the *proper token condition*.

Remark 3.1: By Proposition 3.1, we can always assume that, by a suitable renaming of proper tokens, all deductions that we take into account in the rest of the paper satisfy the token condition.

If Π is a deduction, let $\Pi[x/y]$ denote the tree obtained by replacing each token x in Π with the token y.

Remark 3.2: Under reasonable assumptions, token substitution preserves deduction correctness and token condition. Indeed,

(1) if Π is a deduction satisfying the token condition and
(2) x, y are not proper tokens of Π,

then $\Pi[x/y]$ is a deduction satisfying the token condition.

Remark 3.3: Note that if the last rule of Π is $\bot_i$, and the last formula is $\bot^x$ for some x, it might be the case that, after the token substitution, the side condition of this application of $\bot_i$ is no longer satisfied (that is, its premise and conclusion are both $\bot^z$, for the same z). In such a case by $\Pi[x/y]$, we mean the deduction obtained by deleting the last – incorrect – application of $\bot_i$.

Remark 3.4: Finally, we want to make sense of the operation $\Pi[x/y]$ even when the conditions of Remark 3.2 are not satisfied. Note that if Π is a deduction satisfying the token condition, we can replace any proper token in Π by a fresh token, to obtain a deduction Π' of the same formula from the same assumptions, and such that x and y satisfy all the conditions of Remark 3.2. Hence, we define $\Pi[x/y]$ as this $\Pi'[x/y]$. In the following, we will implicitly assume that by $\Pi[x/y]$ we actually mean $\Pi'[x/y]$, for some Π' as above.

3.1. From the Hilbert-style calculus to $\mathcal{N}_5^c$

We prove now a theorem which ensures that all **S5** theorems are derivable in $\mathcal{N}_5^c$.

Proposition 3.3: *For each token y:*

(a) (1) $\vdash A \to (B \to A)^y$;
(2) $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))^y$;
(3) $\vdash ((\neg B \to \neg A) \to ((\neg B \to A) \to B))^y$;
(4) $\vdash \Diamond A \leftrightarrow \neg\Box\neg A^y$;
(5) $\vdash \Box(A \to B) \to (\Box A \to \Box B)^y$;
(6) $\vdash \Box A \to A^y$;
(7) $\vdash \Box A \to \Diamond A^y$;
(8) $\vdash \Box A \to \Box\Box A^y$;
(9) $\vdash A \to \Box\Diamond A^y$.

(b) *if* $\vdash A^y$, *then* $\vdash \Box A^y$; *if* $\vdash A \to B^y$ *and* $\vdash A^y$ *then* $\vdash B^y$.

Proof: (a) The proofs of (1)–(3) are exactly as for standard natural deduction for propositional logic, simply label all the formulas in the classical standard deductions with the same token y;

(4)

$$\dfrac{[\Diamond A^y] \qquad \dfrac{\dfrac{\dfrac{[A^x] \qquad \dfrac{[\Box\neg A^y]}{\neg A^x}\,\Box E}{\bot^x}\,\to E}{\bot^y}\,\bot_i}{\neg\Box\neg A^y}\,\to I}{\dfrac{\neg\Box\neg A^y}{\Diamond A \to \neg\Box\neg A^y}\,\to I}\,\Diamond E
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\Diamond A^u] \qquad \dfrac{[A^x]}{\Diamond A^u}\,\Diamond I}{\bot^u}\,\to E}{\bot^x}\,\bot_i}{\neg A^x}\,\to I}{\Box\neg A^u}\,\Box I \qquad [\neg\Box\neg A^u]}{\bot^u}\,\to E}{\dfrac{\Diamond A^u}{\neg\Box\neg A \to \Diamond A^u}\,\to I}\,\bot_c$$

(5)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Box A^y]}{A^x}\,\Box E \qquad \dfrac{[\Box(A \to B)^y]}{A \to B^x}\,\Box E}{B^x}\,\to E}{\Box B^y}\,\Box I}{\Box A \to \Box B^y}\,\to I}{\Box(A \to B) \to (\Box A \to \Box B)^y}\,\to I$$

(6)

$$\dfrac{\dfrac{[\Box A^y]}{A^y}\,\Box E}{\Box A \to A^y}\,\to I$$

(7)

$$\dfrac{\dfrac{\dfrac{[\Box A^y]}{A^x}\,\Box E}{\Diamond A^y}\,\Diamond I}{\Box A \to \Diamond A^y}\,\to I$$

(8)

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Box A^y]}{A^z}\,\Box E}{\Box A^u}\,\Box I}{\Box\Box A^y}\,\Box I}{\Box A \to \Box\Box A^y}\,\to I$$

(9)

$$\dfrac{\dfrac{\dfrac{[A^y]}{\Diamond A^x}\,\Diamond I}{\Box\Diamond A^y}\,\Box I}{A \to \Box\Diamond A^y}\,\to I$$

(b) Let us suppose that $\vdash A^y$. Then there is a proof Π with conclusion A^y. Now, pick a fresh variable z, and therefore we have a proof $\Pi[y/z]$ of A^z. Then apply $\Box I$ and obtain a proof of $\vdash \Box A^y$. Finally, closure under **MP** is trivially ensured by rule $(\to E)$.

■

From Proposition 3.3, we easily the theorem:

Theorem 3.4: *For each token y, if* $\vdash_H A$*, then* $\vdash A^y$.

4. Semantics of S5

4.1. The standard universal semantics of S5

We know that **S5** is sound and complete with respect to all the Kripke models whose accessibility relation is an equivalence relation, but also with respect to all the Kripke models whose accessibility relation is *universal*, namely where all the possible worlds are mutually accessible. Consequently the accessibility relation may be dropped in the definition of models (see the book Chellas, 1980, p. 178).

Definition 4.1 (Universal Model): A *Universal Model* is a pair $\mathcal{W} = < W, V >$ s.t.

(1) W is a non empty set of possible words;
(2) $V : W \to 2^{AT}$.

In the following, variables w, v (possibly indexed) range over the set W of a given universal model $\mathcal{W} = < W, V >$.

We recall the notion of *satisfiability* of modal formulas w.r.t. universal models.

Definition 4.2 (Satisfiability):

- $\mathcal{W}, w \not\models \bot$
- $\mathcal{W}, w \models p \Leftrightarrow p \in V(w)$ for $p \in AT$
- $\mathcal{W}, w \models A \to B \Leftrightarrow \mathcal{W}, w \models A \Rightarrow \mathcal{W}, w \models B$
- $\mathcal{W}, w \models A \vee B \Leftrightarrow \mathcal{W}, w \models A$ or $\mathcal{W}, w \models B$
- $\mathcal{W}, w \models A \wedge B \Leftrightarrow \mathcal{W}, w \models A$ and $\mathcal{W}, w \models B$
- $\mathcal{W}, w \models \Box A \Leftrightarrow \forall v \in W, \mathcal{W}, v \models A$
- $\mathcal{W}, w \models \Diamond A \Leftrightarrow \exists v \in W, \mathcal{W}, v \models A$

We write $\mathcal{W} \models A$ to denote that, for each $w \in W$, we have $\mathcal{W}, w \models A$, and $\models A$ to denote that $\mathcal{W} \models A$ holds for each universal model $\mathcal{W}$.

4.2. Semantics and soundness of $\mathcal{N}_5^c$

The semantics for indexed formulas requires the definition of an evaluation function that interprets tokens into possible worlds. Note that we treat modal operators as much as possible like quantifiers. Keeping in mind first-order systems, the following definitions may be considered quite standard.

Definition 4.3 (Interpretation, satisfiability and logical consequence): (1) An *interpretation* is a pair $< \mathcal{W}, \rho_{\mathcal{W}} >$ where $\mathcal{W} = < W, V >$ is a universal model and $\rho_{\mathcal{W}} : \mathcal{T} \to W$ is a map from token to worlds, that we call *evaluation function*.

(2) The satisfiability relation for indexed formulas is defined as

$$\rho_{\mathcal{W}} \Vdash A^x \Leftrightarrow \mathcal{W}, \quad \rho_{\mathcal{W}}(x) \models A.$$

(3) Finally the notion of *logical consequence* is defined as

$$\Gamma \Vdash A^x \Leftrightarrow \forall \mathcal{W}, \quad \forall \rho_{\mathcal{W}}.(\rho_{\mathcal{W}} \Vdash \Gamma \Rightarrow \rho_{\mathcal{W}} \Vdash A^x).$$

Proposition 4.4: *Let $\mathcal{W}$ be an universal model and let $\circ \in \{\Box, \Diamond\}$,*

$$\rho_{\mathcal{W}} \Vdash \circ A^x \Leftrightarrow \forall \rho'_{\mathcal{W}}, \quad \forall z \in \mathcal{T}, \rho'_{\mathcal{W}} \Vdash \circ A^z.$$

As an easy consequence of this proposition, we have

Proposition 4.5: (1) *if $\Gamma \Vdash A^x$ and $x \notin \Gamma$ then for each $z \in \mathcal{T}$, $\Gamma \Vdash \Box A^z$;*
(2) *if $\Gamma \Vdash \Box A^x$ then for each $z \in \mathcal{T}$, $\Gamma \Vdash A^z$;*
(3) *if $\Gamma \Vdash A^x$ then for each $z \in \mathcal{T}$, $\Gamma \Vdash \Diamond A^z$;*
(4) *if $\Gamma' \Vdash \Diamond A^y$, $\Gamma'', A^x \Vdash B^z$, $z \neq x \neq y$ and $x \notin \Gamma''$ then $\Gamma', \Gamma'' \Vdash B^z$.*

Thanks to Proposition 4.5, by an easy induction on proofs which strictly mimics the standard proof of soundness for first-order natural deduction we can state the following theorem.

Theorem 4.6 (Soundness 1): *If $\Gamma \vdash A^x$, then $\Gamma \Vdash A^x$.*

Corollary 4.7: *If $\vdash A^x$, then $\vdash_H A$.*

5. Weak normalization and its consequences

In this section, we prove that each derivation Π in $\mathcal{N}_5^c$ can be reduced to another derivation Π' in normal form (see later for the definition of normal form).

We write

$$\begin{array}{c} B^z \\ \Pi \\ A^x \end{array}$$

to say that Π is a deduction of A^x having some (possibly zero) occurrences of formula B^z among its assumptions, and we write

$$\begin{array}{c} \Pi \\ A^x \end{array} R$$

to say that Π is a deduction of formula A^x whose last rule is R.

Following Prawitz, from now on we restrict the study of the normalization to the complete basis $\rightarrow, \Box$ and $\bot$.

Moreover, we can assume that the conclusion of the $\bot_c$ rule is atomic. To show this fact let us apply repeatedly the following transformations (e.g. see Prawitz, 1965, p.41):

$$\dfrac{\begin{array}{c}[\neg(A \rightarrow B)^x] \\ \Pi \\ \bot^y\end{array}}{A \rightarrow B^x} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[A^x]^3 [A \rightarrow B^x]^1}{B^x} \quad [\neg B^x]^2}{\bot^y}}{\neg(A \rightarrow B)^x}\,(1) \\ \Pi \\ \bot^y\end{array}}{B^x}\,(2)}{A \rightarrow B^x}\,(3)$$

$$\dfrac{\begin{array}{c}[\neg(\Box A)^x]\\ \Pi\\ \bot^y\end{array}}{\Box A^x} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[\Box A^x]^1}{A^y} \quad [\neg A^y]^2}{\bot^y}}{\neg(\Box A)^x}\,(1)\\ \Pi\\ \bot^z\end{array}}{A^y}\,(2)}{\Box A^x}\,(3)$$

To define the *normal form* for a deduction, we need first to introduce the notions of *contractions*, *reduction steps*, and *reduction sequence* (see, e.g. Girard, 1987.)

5.1. Proper contractions

The relation $\rhd$ of *proper contractibility* between deductions is defined as follows:

$$\dfrac{\dfrac{\begin{array}{c}[A^x]\\ \Pi_1\\ B^x\end{array}}{A \to B^x} \quad \begin{array}{c}\Pi_2\\ A^x\end{array}}{B^x} \quad \rhd \quad \begin{array}{c}\Pi_2\\ A^x\\ \Pi_1\\ B^x\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\Pi\\ A^x\end{array}}{\Box A^y}}{A^z} \quad \rhd \quad \begin{array}{c}\Pi[x/z]\\ A^z\end{array}$$

Remark 5.1: It is easy to verify that contractions transform deductions into deductions.

Definition 5.1 (Reducibility between Deductions): (1) The relation $\succ$ of *immediate reducibility* between deductions is the 'context closure' of $\rhd$, defined as follows: $\Pi_1 \succ \Pi_2$ if and only if there exist deductions Π_3 and Π_4 such that $\Pi_3 \rhd \Pi_4$ and Π_2 is obtained by replacing Π_3 with Π_4 in Π_1.

(2) The relation $\overset{*}{\succ}$ of *reducibility* is the transitive and reflexive closure of $\succ$.

Remark 5.2: It is easy to verify that reductions transform deductions into deductions. Moreover, applying what we have written in remark 3.1, we assume that if $\Pi \succ \Pi'$ then Π' verify the token condition, if this is not the case simply rename the proper tokens of Π' to obtain a derivation that verifies the token condition. This approach is standard in the first-order natural deduction, where instead of proper tokens there are the so-called proper parameters (see, e.g. Troelstra & van Dalen, 1988b pp. 529–530).

5.2. Normalization

Definition 5.2 (Normal forms and normalizable deductions): A deduction Π is

(1) in *normal form* if there is no deduction Π' such that $\Pi \succ \Pi'$;

(2) *normalizable* if there is a deduction Π' s.t. $\Pi \overset{*}{\succ} \Pi'$ and Π' is in normal form.

Definition 5.3 (Degree of a formula): (1) The *degree* $\deg(A)$ of a modal formula A is recursively defined as

(a) (a)$\deg(A) = 0$ if A is an atomic formula;
(b) (b)$\deg(\neg A) = \deg(\Box A) = \deg(\Diamond A) = \deg(A) + 1$;
(c) (c)$\deg(A \wedge B) = \deg(A \vee B) = \deg(A \rightarrow B) = \max\{\deg(A), \deg(B)\} + 1$.

(2) The *degree* $\deg(A^x)$ of formula A^x is just $\deg(A)$.

Definition 5.4 (cut): (1) A *cut* in a derivation Π is a formula A^x which is conclusion of an introduction rule $I*$ of a connective $*$, and principal premise of an elimination rule $E*$ of the same connective.
(2) A cut A^x in Π is *maximal* if $\deg(A^x) = \max\{\deg(A'^x) : A'^x$ is a cut in $\Pi\}$.

Let $C[\Pi]$ be the set of cuts of Π. For the normalization theorem we will use the lexicographic ordering between pairs of natural numbers.[2]

Theorem 5.5 (Weak normalization): *For each derivation Π there exists a derivation Π' s.t. $\Pi \overset{*}{\succ} \Pi'$ and Π' is in normal form.*

Proof: The proof is on well ordering induction on pairs (d, n) of natural numbers. We associate to each derivation Π a pair (called *rank*) $\#[\Pi] = (d, n)$ s.t.

- $d = \max\{\deg(A^x) : A^x \in C[\Pi]\}$;
- $n = \sum_{A^x \in C[\Pi], \deg(A^x) = d} d$.

We then prove the following **claim**:

$$\#[\Pi] > (0, 0) \Rightarrow \exists \Pi'(\Pi \overset{*}{\succ} \Pi' \,\&\, \#[\Pi'] < \#[\Pi]).$$

(1) Let us suppose that $\#[\Pi] > (0, 0)$;
(2) pick a maximal cut A^x in Π s.t. the sub-derivation $\Pi*$ ending with A^x *does not contain any other maximal cut*;
(4) perform the relevant contraction.

The resulting derivation Π' has a smaller rank w.r.t Π i.e. $\#[\Pi'] < \#[\Pi]$ for the lexicographic order. In fact, if there exists a unique maximal cut of degree d, then after the contraction lowers the first number of the pair $\#[\Pi]$ (such a contraction cannot create new maximal cuts); if the maximal cut is not unique, then after the reduction lowers the second number of the pair $\#[\Pi]$ (once again such a contraction cannot create new maximal cuts). So the result is established by a double induction.

Using the **claim**, since the lexicographic order is well founded, for each derivation Π there exists a derivation Π' s.t. $\Pi \overset{*}{\succ} \Pi'$ and $\#[\Pi'] = (0, 0)$, i.e. the thesis. ∎

5.3. Consequences of the normalization

In this section, we prove a Consistency Theorem, as a main consequence of normalization. In the following, we strictly follow Prawitz (1965).

Definition 5.6 (Thread): A finite sequence $(A_i{}^{x_i})_{i \leq m}$ of formulas in a deduction is a *thread* if:

(1) for all $i < m$, $A_i{}^{x_i}$ is immediately above $A_{i+1}{}^{x_{i+1}}$;
(2) $A_m{}^{x_m}$ is the end-formula of the deduction;
(3) $A_0{}^{x_0}$ is an assumption (either discharged or undischarged).

A *branch* is an initial segment of a thread satisfying some properties:

Definition 5.7 (Branch): By a branch in a deduction we understand an initial segment $A_1{}^{x_1} \ldots A_n{}^{x_n}$ of a thread $A_1{}^{x_1} \ldots A_n{}^{x_n} A_{n+1}{}^{x_{n+1}} \ldots A_m{}^{x_m}$ in a deduction such that $A_n{}^{x_n}$ is either

(1) the first formula occurrence in the thread that is the minor premise of an application of $\to E$ rule;
(2) the last formula occurrence in the thread (the end-formula of the deduction) if there is no such minor premise in the thread.

By a *main branch,* we understand a branch that is also a thread and contains no minor premise of $\to E$ rule.

The following proposition tells that a normal deduction can be split into two subsequences such that the formula occurrences that are major premises of some elimination rule precede all the formula occurrences that are premises of an introduction rule of a $\perp_c$ rule:

Proposition 5.8: *Let Π be a normal deduction and let $b = A_1{}^{x_1} \ldots A_i{}^{x_n}$ a branch in Π. Then there exists a modal formula occurrence $A_i{}^{x_i}$, called the minimum formula in b, that splits b in two (possibly empty) parts with the following properties:*

(1) *each $A_j{}^{x_j}$, $j < i$, is a major premise of an $\to E$ rule or the premise of a $\Box E$ rule and contains $A_{j+1}{}^{x_{j+1}}$ as a sub-formula; we call the sequence $(A_j{}^{x_j})_{j \leq i}$ an elimination sequence;*
(2) *each $A_j{}^{x_j}$, $i < j$ and $j \neq n$ is a premise of an $\to I$ rule or of a $\Box I$ rule and is a sub-formula of $A_{j+1}{}^{x_{j+1}}$; we call the sequence $(A_j{}^{x_j})_{i \leq j}$ an introduction sequence;*
(3) *the last formula $A_n{}^{x_n}$, provided $n \neq i$, is premise of an $\to I$ rule or premise of an $\Box I$ rule or the $\perp_c$ rule.*

Proof: The formula occurrences in b that are major premises of an $\to E$ rule precede all the formula occurrences that are premises of an $\to I$ rule or a $\perp_c$ rule. Otherwise, there is a first formula occurrence in b, call it $A_k{}^{x_k}$ that is a major premise of an $\to E$ rule but succeeds a premise of an $\to I$ rule or a $\perp_c$ rule: $A_k{}^{x_k}$ would be a maximum formula, and this contradicts the assumption that Π is normal. The formula occurrences in b that are major premises of an $\Box E$ rule precede all the formula occurrences that are premises of an $\Box I$ rule or a $\perp_C$ rule. Otherwise we can reason as in the previous case.

Let $A_l{}^{x_l}$ be the first formula occurrence in b that is premise of an $\rightarrow I$ rule or premise of the $\Box I$ rule or the $\perp_C$ rule, or let $A_l{}^{x_l} = A_n{}^{x_n}$: by definition, $A_l{}^{x_l}$ is a minimum formula and satisfies clauses 1) and 3). Moreover, every $A_j{}^{x_j}$ such that $j < i < n$ is a premise of an $\rightarrow I$ rule or of an $\Box I$ rule or of the $\perp_C$ rule. We can exclude the latter possibility, since a premise of the $\perp_C$ rule is an occurrence of the formula $\perp$ and can be a consequence of an $\rightarrow E$ rule only. Hence $A_l{}^{x_l}$ satisfies also clause 2). ■

In the proof of the following result, we use the notion of *order between branches*:

Definition 5.9 (Order of a branch): Let Π be a derivation in normal form, we associate to each branch $b \in \Pi$ a number $\mathbf{o}(b)$, called **order**, as follows.

(1) if b is a main branch, then $\mathbf{o}(b) = 0$
(2) if b ends with the minor premise of an application of a $\rightarrow E$ rule and the major premise of the rule belongs to a branch of order n, then $\mathbf{o}(b) = n + 1$.

Corollary 5.10 (Sub-formula Property): *Given a normal deduction Π of a formula A^x from a set Γ of assumptions, we call Δ the set of all the formula occurrences in Π except for the assumptions discharged by a $\perp_C$ rule and for occurrences of $\perp$ that stand below such assumptions. Then each modal formula occurrence $A_i{}^{x_i}$ in Δ is either a sub-formula of A^x or a formula in Γ.*

Proof: It is enough to prove that the sub-formula property holds for all formula occurrences $A_i{}^{x_i}$ in a branch of order k, on the assumption that the property holds for all $A_i{}^{x_i}$ in branch of order $h < k$. Consider a branch b $= A_1{}^{x_1} \ldots A_i{}^{x_n}$ and its minimum formula $A_m{}^{x_m}$. The property clearly holds for $A_n{}^{x_n}$: in fact, $A_n{}^{x_n}$ is either the conclusion A or the major premise of an $\rightarrow E$ rule (then it is of the shape $A_l \rightarrow B_l{}^{x_l}$) and belongs to a branch of order $h = k - 1 < k$. Consider now a $A_i{}^{x_i}$ in the introduction sequence of b. By Proposition 5.8, we know $A_i{}^{x_i}$ is a premise of an $\rightarrow I$ rule and is a sub-formula of $A_{i+1}{}^{x_{i+1}}$ and we can conclude that the property holds. If $A_i{}^{x_i}$ does not follows in the previous cases, then either it is an assumption Δ or it has been discharged by an application of $\rightarrow I$. In the latter case, the consequence of the rule has the shape $A_l \rightarrow B_l{}^{x_l}$ and either belongs to the introduction sequence of a branch or belongs to some branch of order $h < k$. In this case, we can conclude by Proposition 5.8.

Finally, if $A_i{}^{x_i}$ has been discharged by an application of the $\perp_C$ rule, either $A_i{}^{x_i}$ is a premise of an $\rightarrow I$ rule or is a major premise of an $\rightarrow E$ rule. In the former case, one has $A_i{}^{x_i} = A_n{}^{x_n}$, whereas in the latter case either $A_i{}^{x_i}$ is of the shape $\perp^{x_i}$ or $A_i{}^{x_i}$ is the minor premise of a $\rightarrow E$ rule and so $A_i{}^{x_i} = A_n{}^{x_n}$. ■

As an immediate consequence, we have the following *Consistency Theorem*:

Theorem 5.11 (Consistency): *For each token x, $\nvdash \perp^x$.*

6. Intuitionistic system and normalization

In this section, we extend our proposal to the intuitionistic version of **S5**, obtaining the system $\mathcal{N}_5^{\mathbf{i}}$.

From a semantic perspective, Intuitionistic Modal logic is a very delicate topic. In particular, there is no standard extension of intuitionistic Kripke-style semantics for modal logics. A very good explanation of the problems in having a Kripke semantics for modal intuitionistic logics can be found in Simpson's PhD Thesis (Simpson, 1993).

It is our firm opinion that an intuitionistic logic should not be characterised by a semantics, which in some way could be arbitrary. What deeply identify a logic is rather the set of the properties of its proofs.

In other words, an intuitionistic logic should be directly characterised by its **BHK** 'semantics', also called *functional interpretation* of proofs.

Moreover, as technically discussed in Section 6.10, the **BHK** semantics of proofs induces a purely functional calculus, opening in our case the perspective of a theory of intuitionistic **S5** types for computational systems.

Following these motivations, we begin this section by stating that, in analogy with first-order intuitionistic logic, the calculus obeys to the following **BHK** interpretation:

Definition 6.1 (BHK interpretation of proofs): (1) A proof $A \wedge B^x$ is a pair (a, b) where a is a proof of A^x and b is a proof of B^x;

(2) a proof of $A \vee B^x$ is a pair (i, c) where $i \in \{0, 1\}$ and if $i = 0$ then c is a proof of A^x otherwise c is a proof of B^x;
(3) a proof of $A \to B^x$ is a construction f that transform each proof a of A^x in a proof $f(x)$ of B;
(4) there is no proof of $\perp^x$;
(5) a proof of $\Box A^x$ is a construction that for each y gives a proof $f(y)$ of A^y;
(6) a proof of $\Diamond A^x$ is a pair (y, a) such that a is a proof of A^y.

To be consistent with the **BHK** interpretation of proofs, exactly as for first-order calculus, the deductive system for intuitionistic **S5** is obtained from its classical version by dropping the *reductio ab absurdum* rule $\perp_c$, *but not* the ex falso quod libet rule $\perp_i$.

Definition 6.2: The intuitionistic natural deduction system $\mathcal{N}_5^{\mathbf{i}}$ for intuitionistic **S5** is obtained by dropping the $\perp_c$ rule in $\mathcal{N}_5^{\mathbf{c}}$. The resulting system is called $\mathcal{N}_5^{\mathbf{i}}$.

In Section 6.3, we will prove a weak normalization theorem and its main consequences in the intuitionistic setting: the *disjunction property* and the *witness property*.

Before this, in the next two subsections we shall prove the equivalence of $\mathcal{N}_5^{\mathbf{i}}$ w.r.t. an Hilbert-style axiomatization of **IS5**, by translation into the natural deduction system given by Galmiche and Salhi (2010b).

6.1. IS5: the intuitionistic Hilbert calculus of S5

In this section, we recall the intuitionistic modal logic **IS5** via an Hilbert-style axiomatization and prove the equivalence of $\mathcal{N}_5^{\mathbf{i}}$ w.r.t **IS5**.

The language of **IS5** is the same of classical **S5**. Let IPL be a sound and complete axiomatization of intuitionistic propositional logic with modus ponens as unique inference rule (see, e.g. Troelstra & van Dalen, 1988a).

Definition 6.3: The logic **IS5** is the smallest set X of modal formulas that contain all the instances of the following schemata (see, e.g. Galmiche & Salhi, 2010b; Simpson, 1993):

(0) all the axioms of IPL;
(1) $\Box(A \to B) \to (\Box A \to \Box B)$;
(2) $\Box(A \to B) \to (\Diamond A \to \Diamond B)$;
(3) $\Diamond\bot \to \bot$;
(4) $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$;
(5) $(\Diamond A \to \Box B) \to \Box(A \to B)$;
(6) $\Box A \to A$;
(7) $A \to \Diamond A$;
(8) $\Diamond A \to \Box\Diamond A$;
(9) $\Diamond\Box A \to \Box A$.

and the following closures:

MP if $A, A \to B \in X$ then $B \in X$;
NEC if $A \in X$ then $\Box A \in X$.

When $A \in$ **IS5**, we shall write $\vdash_{\mathbf{IS5}} A$.

To avoid misunderstandings, in this section we will denote with $\vdash_{\mathcal{N}_5^{\mathbf{i}}}$ the derivability relation of $\mathcal{N}_5^{\mathbf{i}}$.

Proposition 6.4: *For each instance A of an axiom of **IS5** and for each token y,* $\vdash_{\mathcal{N}_5^{\mathbf{i}}} A^y$.

Proof: (0) for each propositional axiom of IPL and for each token y, $\vdash_{\mathcal{N}_5^{\mathbf{i}}} A^y$. Simply take a proof Π in standard intuitionistic natural deduction and decorate each occurrence of a formula B in Π with B^y;

(1) $\vdash_{\mathcal{N}_5^{\mathbf{i}}} \Box(A \to B) \to (\Box A \to \Box B)^y$;

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Box A^y]}{A^x}\,\Box E \qquad \dfrac{[\Box(A \to B)^y]}{A \to B^x}\,\Box E}{B^x}\,\to E}{\Box B^y}\,\Box I}{\Box A \to \Box B^y}\,\to I}{\Box(A \to B) \to (\Box A \to \Box B)^y}\,\to I$$

(2) $\vdash_{\mathcal{N}_5^{\mathbf{i}}} \Box(A \to B) \to (\Diamond A \to \Diamond B)^y$;

$$
\begin{prooftree}
\AxiomC{$[\Diamond A^y]$}
\AxiomC{$[\Box(A \to B)^y]$}
\RightLabel{$\Box E$}
\UnaryInfC{$A \to B^x$}
\AxiomC{$[A^x]$}
\RightLabel{$\Diamond I$}
\BinaryInfC{B^x}
\RightLabel{$\Diamond E$}
\UnaryInfC{$\Diamond B^y$}
\RightLabel{$\Diamond E$}
\BinaryInfC{$\Diamond B^y$}
\RightLabel{$\to I$}
\UnaryInfC{$\Diamond A \to \Diamond B^y$}
\RightLabel{$\to I$}
\UnaryInfC{$\Box(A \to B) \to (\Diamond A \to \Diamond B)^y$}
\end{prooftree}
$$

(3) $\vdash_{\mathcal{N}_5^{\mathbf{i}}} \Diamond\bot \to \bot^y$;

$$
\begin{prooftree}
\AxiomC{$[\Diamond\bot^y]$}
\AxiomC{$[\bot^z]$}
\RightLabel{$\bot_i$}
\UnaryInfC{$\bot^y$}
\RightLabel{$\Diamond E$}
\BinaryInfC{$\bot^y$}
\RightLabel{$\to I$}
\UnaryInfC{$\Diamond\bot \to \bot^y$}
\end{prooftree}
$$

(4) $\vdash_{\mathcal{N}_5^{\mathbf{i}}} \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)^y$;

$$
\begin{prooftree}
\AxiomC{$[\Diamond(A \vee B)^y]$}
\AxiomC{$[A \vee B^z]$}
\AxiomC{$[A^z]$}
\RightLabel{$\Diamond I$}
\UnaryInfC{$\Diamond A^y$}
\RightLabel{$\vee_1$}
\UnaryInfC{$\Diamond A \vee \Diamond B^y$}
\AxiomC{$[B^z]$}
\RightLabel{$\Diamond I$}
\UnaryInfC{$\Diamond B^y$}
\RightLabel{$\vee_2$}
\UnaryInfC{$\Diamond A \vee \Diamond B^y$}
\RightLabel{$\vee E$}
\TrinaryInfC{$\Diamond A \vee \Diamond B^y$}
\RightLabel{$\Diamond E$}
\BinaryInfC{$\Diamond A \vee \Diamond B^y$}
\RightLabel{$\to I$}
\UnaryInfC{$\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)^y$}
\end{prooftree}
$$

(5) $\vdash_{\mathcal{N}_5^{\mathbf{i}}} (\Diamond A \to \Box B) \to \Box(A \to B)^y$;

$$
\begin{prooftree}
\AxiomC{$[A^x]$}
\RightLabel{$\Diamond I$}
\UnaryInfC{$\Diamond A^y$}
\AxiomC{$[\Diamond A \to \Box B^y]$}
\RightLabel{$\Box E$}
\BinaryInfC{$\Box B^y$}
\RightLabel{$\to I$}
\UnaryInfC{B^x}
\RightLabel{$\to I$}
\UnaryInfC{$A \to B^x$}
\RightLabel{$\Box I$}
\UnaryInfC{$\Box(A \to B)^y$}
\RightLabel{$\to I$}
\UnaryInfC{$(\Diamond A \to \Box B) \to \Box(A \to B)^y$}
\end{prooftree}
$$

(6) $\vdash_{\mathcal{N}_5^{\mathbf{i}}} \Box A \to A^y$;

$$
\begin{prooftree}
\AxiomC{$[\Box A^y]$}
\RightLabel{$\Box E$}
\UnaryInfC{A^y}
\RightLabel{$\to I$}
\UnaryInfC{$\Box A \to A^y$}
\end{prooftree}
$$

(7) $\vdash_{\mathcal{N}_5^{\mathbf{i}}} A \to \Diamond A^y$;

$$\dfrac{\dfrac{[A^y]}{\Diamond A^y}\,\Diamond I}{A \to \Diamond A^y}\,\to I$$

(8) $\vdash_{\mathcal{N}_5^{\mathbf{i}}} \Diamond A \to \Box\Diamond A^y$;

$$\dfrac{\dfrac{[\Diamond A^y] \qquad \dfrac{\dfrac{[A^x]}{\Diamond A^x}\,\Diamond I}{\Box\Diamond A^y}\,\Box I}{\Box\Diamond A^y}\,\Diamond E}{\Diamond A \to \Box\Diamond A^y}\,\to I$$

(9) $\vdash_{\mathcal{N}_5^{\mathbf{i}}} \Diamond\Box A \to \Box A^y$;

$$\dfrac{\dfrac{[\Diamond\Box A^y] \qquad \dfrac{\dfrac{\Box A^x}{A^z}\,\Box E}{\Box A^y}\,\Box I}{\Box A^y}\,\Diamond E}{\Diamond\Box A \to \Box A^y}\,\to I$$

■

Proposition 6.5:
If $\vdash_{\mathcal{N}_5^{\mathbf{i}}} A^y$, *then* $\vdash_{\mathcal{N}_5^{\mathbf{i}}} \Box A^y$;
if $\vdash_{\mathcal{N}_5^{\mathbf{i}}} A \to B^y$ *and* $\vdash_{\mathcal{N}_5^{\mathbf{i}}} A^y$ *then* $\vdash_{\mathcal{N}_5^{\mathbf{i}}} B^y$.

Proof: Let us suppose that $\vdash_{\mathcal{N}_5^{\mathbf{i}}} A^y$. Then there is a proof Π with conclusion A^y. Now, pick a fresh variable z, and therefore we have a proof $\Pi[y/z]$ of A^z. Then apply $\Box I$ and obtain a proof of $\vdash_{\mathcal{N}_5^{\mathbf{i}}} \Box A^y$. Finally, closure under **MP** is trivially ensured by rule $(\to E)$. ■

As an immediate consequence, we have the following theorem:

Theorem 6.6: *For each token y, if* $\vdash_{\mathbf{IS5}} A$, *then* $\vdash_{\mathcal{N}_5^{\mathbf{i}}} A^y$.

We also have the reverse direction, by translation into the $\mathrm{ND}_{\mathrm{IS5}}$ system proposed by Galmiche and Salhi (2010b). In the next section, we shall see that every proof of $\mathcal{N}_5^{\mathbf{i}}$ can be translated into a proof of $\mathrm{ND}_{\mathrm{IS5}}$, and vice versa. So, if we call $\vdash_{\mathrm{ND}_{\mathrm{IS5}}}$ the derivability relation of the approach of $\mathrm{ND_{IS5}}$, we will show that (see Lemma 6.8)

$$\vdash_{\mathcal{N}_5^{\mathbf{i}}} A^y \Rightarrow \vdash_{\mathrm{ND_{IS5}}} A$$

for each token y. Since Galmiche and Salhi prove the equivalence of their system w.r.t. **IS5**

$$\vdash_{\mathbf{IS5}} A \Leftrightarrow \vdash_{\mathrm{ND_{IS5}}} A$$

then we can state the following:

Theorem 6.7: *For each token y, if $\vdash_{\mathcal{N}_5^{\mathbf{i}}} A^y$ then $\vdash_{\mathbf{IS5}} A$.*

6.2. MC-sequents

Galmiche and Salhi (2010b) gave a label-free natural deduction system $\mathrm{ND_{IS5}}$ proving that it is sound and complete w.r.t. **IS5**. The authors did not prove normalization for $\mathrm{ND_{IS5}}$, focusing on the cut elimination of a sequent-style counterpart, which is proved equivalent to the natural deduction system.

The rules of the systems are given in terms of MC-sequents

$$\Gamma_1; \ldots ; \Gamma_n \vdash \Gamma_0 \vdash A$$

where $\Gamma_0, \ldots , \Gamma_n$ are multisets of formulas, and $\Gamma_1; \ldots ; \Gamma_n$ is at its turn a multiset (or equivalently, it is an unordered sequence). When $\Gamma_0 = \varnothing$ it can be ignored and the MC-sequent can be written as $\Gamma_1; \ldots ; \Gamma_n \vdash \ \vdash A$.

We shall now prove that every proof of $\mathcal{N}_5^{\mathbf{i}}$ can be translated into a proof of $\mathrm{ND_{IS5}}$, and vice versa.

From indexed sequents to MC-sequents

Given a judgement $\Gamma \vdash A^x$, since the formulae in a context Γ are not ordered, we can rearrange it in sub-contexts Γ^x in which we collect all the formulas with the token x, and take $\Gamma = {\Gamma_1}^{x_1}, {\Gamma_2}^{x_2}, \ldots , {\Gamma_n}^{x_n}$, where $x_i = x_j$ only if $i = j$.

According to the latter convention, every judgment of $\mathcal{N}_5^{\mathbf{i}}$ can be written as

$${\Gamma_1}^{x_1}, \ldots {\Gamma_i}^{x_i}, \ldots , {\Gamma_n}^{x_n} \vdash A^{x_i}$$

where $x_1, \ldots , x_n$ are distinct tokens and ${\Gamma_j}^{x_j}$ is a (possibly empty) multiset of formulas indexed by the token x_j, for $1 \leq j \leq n$. We remark the particular case in which $\Gamma_i = \varnothing$, corresponding to a proof in which the tokens of all the open assumptions differ from the token x_i of the conclusion.

Let us now give a map $[\![\mathcal{J}]\!]$ translating any judgment $\mathcal{J}$ of $\mathcal{N}_5^{\mathbf{i}}$ into an MC-sequent, by defining:

$$[\![{\Gamma_1}^{x_1}, \ldots , {\Gamma_n}^{x_n} \vdash A^{x_i}]\!] = \Gamma_1; \ldots ; \Gamma_{i-1}; \Gamma_{i+1}; \ldots , \Gamma_n \vdash \Gamma_i \vdash A$$

The rules of the $\mathrm{ND_{IS5}}$ natural deduction system of Galmiche and Salhi (2010b) translate almost directly into the rules of $\mathcal{N}_5^{\mathbf{i}}$, by replacing the judgements corresponding to the premises and the conclusion of each rule with their translations. Let us see in detail the translation of the $\Box I$ and $\Box E$ rules, the other ones are similar.

- We start with the $\Box I$ rule

$$\frac{A^x}{\Box A^y}\,\Box I$$

In which we recall that $y \neq x$ and $x \notin \{x_1, \ldots, x_n\}$, where $\{x_1, \ldots, x_n\}$ is the set of the tokens of the open assumptions in the context. In accord with these constraints, let us assume w.l.o.g. that $y = x_n$ and take $x = x_{n+1}$. The translation of the rule is a $\Box I$ rule of $\mathrm{ND}_{\mathrm{IS5}}$

$$\frac{[\![{\Gamma_1}^{x_1}, \ldots, {\Gamma_n}^{x_n} \vdash A^{x_{n+1}}]\!]}{[\![{\Gamma_1}^{x_1}, \ldots, {\Gamma_n}^{x_n} \vdash \Box A^{x_n}]\!]}\,\Box_I$$

whose premise and conclusion are just the translation of the corresponding judgements of the $\mathcal{N}_5^{\mathbf{i}}$ proof. Since $\Gamma_{n+1} = \varnothing$, we obtain

$$\frac{\Gamma_1; \ldots; \Gamma_n \vdash\ \vdash A}{\Gamma_1; \ldots; \Gamma_{n-1} \vdash \Gamma_n \vdash \Box A}\,\Box_I$$

- For the translation of the $\Box E$ rule

$$\frac{A^x}{\Box A^y}\,\Box I$$

let us assume w.l.o.g. that $x = x_n$ and $y = x^j$, with $1 \leq j \leq n$. The resulting rule is obtained as above by

$$\frac{[\![{\Gamma_1}^{x_1}, \ldots, {\Gamma_n}^{x_n} \vdash A^{x_n}]\!]}{[\![{\Gamma_1}^{x_1}, \ldots, {\Gamma_n}^{x_n} \vdash \Box A^{x^j}]\!]}\,\Box_E^k$$

but we must distinguish two cases, corresponding to the two $\Box_E$ rules of $\mathrm{ND}_{\mathrm{IS5}}$ (for $k = 1$ or $k = 2$, respectively):

(1) when $j = n$, we have $k = 1$ and the resulting $\mathrm{ND}_{\mathrm{IS5}}$ rule is

$$\frac{\Gamma_1; \ldots; \Gamma_{n-1} \vdash \Gamma_n \vdash A}{\Gamma_1; \ldots; \Gamma_{n-1} \vdash \Gamma_n \vdash \Box A}\,\Box_E^1$$

(2) when $j \neq n$, we have $k = 2$ and, by assuming w.l.o.g. that $j = n-1$, the resulting $\mathrm{ND}_{\mathrm{IS5}}$ rule is

$$\frac{\Gamma_1; \ldots; \Gamma_{n-2}; \Gamma_{n-1} \vdash \Gamma_n \vdash A}{\Gamma_1; \ldots; \Gamma_{n-2}; \Gamma_n \vdash \Gamma_{n-1} \vdash \Box A}\,\Box_E^2$$

By applying the above translation to all the rules, we can then conclude that every formula provable in $\mathcal{N}_5^{\mathbf{i}}$ can be proved in $\mathrm{ND}_{\mathrm{IS5}}$.

Lemma 6.8: *Let us write* $\vdash_{\mathrm{ND}_{\mathrm{IS5}}} A$ *to denote that* $\vdash\ \vdash A$ *is derivable in* $\mathrm{ND}_{\mathrm{IS5}}$.

$$\vdash_{\mathcal{N}_5^{\mathbf{i}}} A^y \Rightarrow \vdash_{\mathrm{ND}_{\mathrm{IS5}}} A$$

As already remarked (see of Theorem 6.7), since $\vdash_{\mathrm{ND}_{\mathrm{IS5}}}$ is sound w.r.t. **IS5**, the previous lemma proves that $\mathcal{N}_5^{\mathbf{i}}$ too is sound.

From MC-sequents to indexed sequents

Analogously, we have an inverse translation translating every MC-sequents into an indexed judgement of $\mathcal{N}_5^{\mathbf{i}}$. Namely:

$$\rho[\Gamma_1;\ldots;\Gamma_n \vdash \Gamma_0 \vdash A] \quad = \quad {\Gamma_1}^{x_1};\ldots;{\Gamma_n}^{x_n} \vdash {\Gamma_0}^{x_0} \vdash A^{x_0}$$

where x_0, x_1, x_n are distinct tokens. By replacing each MC-sequent $\mathcal{S}$ in an $\mathrm{ND}_{\mathrm{IS5}}$ proof with the conclusion of the corresponding $\rho[\mathcal{S}]$ judgement of $\mathcal{N}_5^{\mathbf{i}}$, we can see by induction of the length of the proof that the translation of every rule of the $\mathrm{ND}_{\mathrm{IS5}}$ deduction becomes a valid $\mathcal{N}_5^{\mathbf{i}}$ rule whose conclusion is the conclusion of an $\mathcal{N}_5^{\mathbf{i}}$ subproof proving the corresponding judgement.

Lemma 6.9: *For each token y*

$$\vdash_{\mathrm{ND}_{\mathrm{IS5}}} A \quad \Rightarrow \quad \vdash_{\mathcal{N}_5^{\mathbf{i}}} A^y$$

Remark 6.1: We remark that, thanks to the latter Lemma 6.9, we have that

$$\vdash_{\mathbf{IS5}} A \Rightarrow \vdash_{\mathrm{ND}_{\mathrm{IS5}}} A \Rightarrow \vdash_{\mathcal{N}_5^{\mathbf{i}}} A^y$$

for each token y. However, to show our calculus at work, we thought it has been interesting to give a direct proof of the axioms of **IS5** (see Proposition 6.4).

6.3. Weak normalization

In view of the proof of normalization, it is useful to restrict the $\bot_i$ to atomic conclusions.

We call the resulting system $\mathcal{N}_5^{\mathbf{ia}}$.

To show that the constraint for the $\bot_i$ rule preserves the logical power of the system, it suffices to apply the following transformation rules:

$$\dfrac{\begin{array}{c}\Pi\\ \bot^y\end{array}}{A \to B^x} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\Pi\\ \bot^y\end{array}}{B^x}}{A \to B^x} \qquad \dfrac{\begin{array}{c}\Pi\\ \bot^y\end{array}}{A \vee B^x} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\Pi\\ \bot^y\end{array}}{B^x}}{A \vee B^x}$$

$$\dfrac{\begin{array}{c}\Pi\\ \bot^y\end{array}}{A \wedge B^x} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\Pi\\ \bot^y\end{array}}{A^x} \qquad \dfrac{\begin{array}{c}\Pi\\ \bot^y\end{array}}{B^x}}{A \wedge B^x}$$

$$\begin{array}{c}\Pi\\ \bot^y\\ \hline \Box A^x\end{array} \rightsquigarrow \begin{array}{c}\Pi\\ \bot^y\\ \hline A^z\\ \hline \Box A^x\end{array}\ (z\text{ fresh}) \qquad \begin{array}{c}\Pi\\ \bot^y\\ \hline \Diamond A^x\end{array} \rightsquigarrow \begin{array}{c}\Pi\\ \bot^y\\ \hline A^x\\ \hline \Diamond A^x\end{array}$$

As for the classical case, we write

$$\begin{array}{c}B^z\\ \Pi\\ A^y\end{array}$$

to say that Π is a deduction of A^y having some (possibly zero) occurrences of formula B^z among its assumptions, and we write

$$\begin{array}{c}\Pi\\ A^y\end{array} R$$

to say that Π is a deduction of formula A^y whose last rule is R.

6.4. Proper contractions

The relation $\triangleright$ of *proper contractibility* between deductions is defined as follows :[3]

$$\begin{array}{c}\begin{array}{cc}\Pi_1 & \Pi_2\\ A^y & B^y\end{array}\\ \hline A \wedge B^y\\ \hline A^y\end{array} \triangleright \begin{array}{c}\Pi_1\\ A^y\end{array} \qquad \begin{array}{c}\begin{array}{cc}\Pi_1 & \Pi_2\\ A^y & B^y\end{array}\\ \hline A \wedge B^y\\ \hline B^y\end{array} \triangleright \begin{array}{c}\Pi_2\\ B^y\end{array}$$

$$\begin{array}{c}\begin{array}{ccc}\begin{array}{c}\Pi_1\\ A^y\\ \hline A \vee B^y\end{array} & \begin{array}{c}[A^y]\\ \Pi_2\\ C^z\end{array} & \begin{array}{c}[B^y]\\ \Pi_3\\ C^z\end{array}\end{array}\\ \hline C^z\end{array} \triangleright \begin{array}{c}\Pi_1\\ A^y\\ \Pi_2\\ C^z\end{array} \qquad \begin{array}{c}\begin{array}{ccc}\begin{array}{c}\Pi_1\\ B^y\\ \hline A \vee B^y\end{array} & \begin{array}{c}[A^y]\\ \Pi_2\\ C^z\end{array} & \begin{array}{c}[B^y]\\ \Pi_3\\ C^z\end{array}\end{array}\\ \hline C^z\end{array} \triangleright \begin{array}{c}\Pi_1\\ B^y\\ \Pi_3\\ C^z\end{array}$$

$$\begin{array}{c}\begin{array}{cc}\begin{array}{c}[A^y]\\ \Pi_1\\ B^y\\ \hline A \rightarrow B^y\end{array} & \begin{array}{c}\Pi_2\\ A^y\end{array}\end{array}\\ \hline B^y\end{array} \triangleright \begin{array}{c}\Pi_2\\ A^y\\ \Pi_1\\ B^y\end{array}$$

$$\begin{array}{c}\Pi\\ A^x\\ \hline \Box A^y\\ \hline A^z\end{array} \triangleright \begin{array}{c}\Pi[x/z]\\ A^z\end{array} \qquad \begin{array}{c}\begin{array}{cc}\begin{array}{c}\Pi_1\\ A^z\\ \hline \Diamond A^y\end{array} & \begin{array}{c}[A^x]\\ \Pi_2\\ C^v\end{array}\end{array}\\ \hline C^v\end{array} \triangleright \begin{array}{c}\Pi_1\\ A^z\\ \Pi_2[x/z]\\ C^v\end{array}$$

6.5. Commutative contractions

In this section, we denote by

$$\dfrac{\begin{matrix}\Pi_1\\ C^z\end{matrix} \quad \Pi_2}{D^\gamma}\,R$$

a deduction ending with an elimination rule R whose major premise is a formula C^z. We further extend the relation $\rhd$ by adding the following *commutative contractions*:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\vee B^y\end{matrix} \quad \begin{matrix}[A^y]\\ \Pi_2\\ C^z\end{matrix} \quad \begin{matrix}[B^y]\\ \Pi_3\\ C^z\end{matrix}}{C^z} \qquad \Pi_4}{D^v}\,R \quad \rhd \quad \dfrac{\begin{matrix}\Pi_1\\ A\vee B^y\end{matrix} \quad \dfrac{\begin{matrix}[A^y]\\ \Pi_2\\ C^z\end{matrix} \quad \Pi_4}{D^v}\,R \quad \dfrac{\begin{matrix}[B^y]\\ \Pi_3\\ C^z\end{matrix} \quad \Pi_4}{D^v}\,R}{D^v}$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Diamond A^y\end{matrix} \quad \begin{matrix}[A^x]\\ \Pi_2\\ C^z\end{matrix}}{C^z} \qquad \Pi_3}{D^v}\,R \quad \rhd \quad \dfrac{\begin{matrix}\Pi_1\\ \Diamond A^y\end{matrix} \quad \dfrac{\begin{matrix}[A^x]\\ \Pi_2\\ C^z\end{matrix} \quad \Pi_3}{D^v}\,R}{D^v}$$

6.6. Simplification contraction

$$\dfrac{\begin{matrix}\Pi\\ A\vee B^x\end{matrix} \quad \begin{matrix}\Pi_1\\ C^z\end{matrix} \quad \begin{matrix}\Pi_2\\ C^z\end{matrix}}{C^z}\,\vee E \quad \rhd \quad \begin{matrix}\Pi_i\\ C^z\end{matrix}$$

if the rule does not discharge assumptions in Π_1, Π_2;

$$\dfrac{\begin{matrix}\Pi\\ \Diamond A^x\end{matrix} \quad \begin{matrix}\Pi_1\\ C^z\end{matrix}}{C^z}\,\Diamond E \quad \rhd \quad \begin{matrix}\Pi_1\\ C^z\end{matrix}$$

if the rule does not discharge assumptions in Π_1.

Remark 6.2: Following the approach highlighted in Remark 5.2 for classical logic, we define reducibility and normal forms for $\mathcal{N}_5^{\mathbf{ia}}$ as in Definitions 5.1 and 5.2.

6.7. Normalization

Definition 6.10 (Segments and Endsegments): Let A^y be an indexed formula.

(1) A finite sequence $({A^y}_i)_{i\leq m}$ of occurrences of A^y in a deduction Π is a *segment* (of length $m+1$) if:

(a) A^y_0 is not a conclusion of $\vee E$ or $\Diamond E$;
(b) A^y_m is not a minor premise of $\vee E$ or $\Diamond E$;
(c) for all $i < m$, A^y_i is a minor premise of $\vee E$ or $\Diamond E$ with conclusion A^y_{i+1}.

(2) A segment in a deduction is an *endsegment* if its last formula is the last formula of the deduction.

We will denote segments by σ, possibly indexed. When we want to highlight that a segment is made of occurrences of a formula A^y we will write $\sigma[A^y]$. With $|\sigma|$ we denote the length of the segment σ.

Given a deduction $\begin{array}{c}\Pi\\ A^y\end{array}$ R, with little abuse of language we will say that a deduction Π' is a *(main) premise of rule R* to mean that Π' is a sub-deduction of Π whose end-formula is a (main) premise of the displayed application of R.

Definition 6.11 (Major/Minor Premises and Conclusions): Let $\sigma[A^y] = A^y{}_0 \dots A^y_m$ and let R be a segment and an instance of a deduction rule in Π, respectively. We say that:

- σ is the *(major/minor) premise* of R, if $A^y{}_m$ is the (major/minor) premise of R;
- σ is *conclusion* of R, if $A^y{}_0$ is the conclusion of R.

The *degree* of a segment $\sigma[A^y]$ is defined by $\delta(\sigma[A^y]) = \deg(A)$, where $\deg(A)$ is the degree of the formula A (see Definition 5.3).

Definition 6.12 (cut): (1) A *cut* in a derivation Π is a segment σ which is conclusion of an introduction rule $I*$ of a connective $*$, and principal premise of an elimination rule $E*$ of the same connective.
(2) A cut σ in Π is *maximal* if $\delta(\sigma) = \max\{\delta(\sigma') : \sigma' \text{ is a cut in } \Pi\}$.
(3) A *(maximal) cut formula* is a (maximal) cut segment of length 1.

Let $C[\Pi]$ be the set of cuts of Π. For the normalization theorem, we will use the lexicographic ordering between pairs of natural numbers.[4]

Theorem 6.13 (normalization): *For each derivation Π, there exists a derivation Π' s.t. $\Pi \overset{*}{\succ} \Pi'$ and Π' is in normal form.*

Proof: **Phase 1** : In this phase, we do not consider simplification contractions.

The proof is on well ordering induction on pairs (d, n) of natural numbers. We associate to each derivation Π a pair (called *rank*) $\#[\Pi] = (d, n)$ s.t.

- $d = \max\{\delta(\sigma) : \sigma \in C[\Pi]\}$;
- $n = \sum_{\sigma \in C[\Pi], \delta(\sigma)=d} d$.

We then prove the following **claim**:

$$\#[\Pi] > (0,0) \Rightarrow \exists \Pi' (\Pi \overset{*}{\succ} \Pi' \ \& \ \#[\Pi'] < \#[\Pi]).$$

(1) Let us suppose that $\#[\Pi] > (0,0)$;
(2) pick a maximal cut σ in Π s.t. the sub-derivation $\Pi*$ ending with σ (i.e. ending with the last occurrence of σ) does not contain any other maximal cut segment;

(3) perform all possible commutative contractions with respect to the segment under consideration;
(4) perform the relevant contraction.

The resulting derivation Π' has a smaller rank w.r.t Π, i.e. $\#[\Pi'] < \#[\Pi]$ for the lexicographic order. In fact, if there exists a unique maximal cut of degree d, then after the contraction lowers the first number of the pair $\#[\Pi]$ (such a contraction cannot create new maximal cuts); if the maximal cut is not unique, then after the reduction lowers the second number of the pair $\#[\Pi]$ (once again such a contraction cannot create new maximal cuts). So the result is established by a double induction.

Using the **claim**, since the lexicographic order is well founded, for each derivation Π there exists a derivation Π' s.t. $\Pi \overset{*}{\succ} \Pi'$ and $\#[\Pi'] = (0, 0)$, i.e. the thesis.

Phase 2: Simplifications.

We proceed by applying inductively the simplification contractions, until no simplification may be performed. Please note that such a simplifications do not produce new redexes. ■

6.8. Consequences of the normalization

In this section, we study the structure of intuitionistic normal derivations. We essentially retrace what we have done in Section 5.3. The proof is a plain variation of the classical case.

Recall the definition of *segment* and *endsegment* from Definition 6.10. A *maximum segment* σ is a segment that begins with a consequence of an application of an introduction rule or the $\bot_i$ rule and ends with a major premise of an elimination rule. Moreover, we say a segment s is a *top segment* (resp. *end segment*) or a *consequence* (resp. *major* or *minor premise*) of an application r of a rule when the first (resp. last) formula is σ is a top formula (resp. the end formula) or a consequence (resp major or minor premise) of r.

We can adapt what we proved for the classical case by taking the segments instead of the formula occurrences and replacing the branches with sequence of formula occurrences. We formalise this by the notion of *path*.

Definition 6.14 (Path): A sequence of labelled formulas $\pi = {A_1}^{y_1} \dots {A_n}^{y_n}$ is a *path* in a deduction Π such that:

(1) ${A_1}^{y_1}$ is an hypothesis that is not discharged by an application of $\vee E$ or $\Diamond E$;
(2) for each $i < n$, ${A_i}^{y_i}$ is not the minor premise of an application of $\rightarrow E$ and either (i) ${A_i}^{y_i}$ is not the major premise of an $\vee E$ or $\Diamond E$ and ${A_{i+1}}^{y_{i+1}}$ is the formula occurrence immediately below ${A_i}^{y_i}$ or (ii) ${A_i}^{y_i}$ is the major premise of an application r of $\vee E$ or $\Diamond E$ and ${A_{i+1}}^{y_{i+1}}$ is an assumption of Π discharged by r;
(3) ${A_n}^{y_n}$ is either the minor premise of $\rightarrow E$, or the end formula of Π, or a major premise of an application r of $\vee E$ or $\Diamond E$ such that r does not discharges assumptions.

Note that in a normal deduction Π the last formula in a path is always a minor premise of $\rightarrow E$ or the end formula in Π.

Any path π can be (uniquely) divided into consecutive segments $\sigma_1 \dots \sigma_k$. We provide some useful definitions.

Proposition 6.15: *Given a normal form derivation, for each path we have that no introduction rule can precede an elimination one.*

Proof: Let $\pi = A_1{}^{x_1} \dots A_m{}^{x_m}$ be a path and let $\sigma_1 \dots \sigma_n$ the sequence of segments in π . We reason by contradiction. Suppose the statements are not true. In this case, there would be an σ_k conclusion of the last introduction rule, call it R_I, that precedes in the path the first elimination rule R_E. Note that σ_k cannot be the main premise of an elimination rule (otherwise the derivation would not be in normal form). This implies that, between R_I and R_E there must be another rule which can only be $\bot_i$. But this is impossible, since this case $\bot_k$ would consist only of a sequence of $\bot$ which cannot be the result of an introduction rule. ■

Moreover, analogously to the classical case, the following proposition tells that a path in a normal deduction can be split into an 'elimination sequence ' and an 'introduction sequence'.

Proposition 6.16: *Let Π be a normal deduction and let $\pi = A_1{}^{x_1} \dots A_i{}^{x_n}$ a path in Π and let $\sigma_1 \dots \sigma_n$ the sequence of segments in p. Then there exists a segment σ_i, called the minimum segment in p, that splits p in two (possibly empty) parts with the following properties:*

(1) *each σ_j, $j < i$, is a major premise of an elimination rule and the formula occurring in s_j is sub-formula of the one occurring in s_{j+1}; we call the sequence $\{\sigma_j\}_{j\leq i}$ the elimination part of the path;*
(2) *σ_i, provided $i \neq n$, is a premise of an introduction rule or of the $\bot_i$ rule;*
(3) *each σ_j, $i < j$ and $j \neq n$ is a premise of an introduction rule and the formula occurring in σ_j is a sub-formula of the one occurring in σ_{j+1}; we call the sequence $\{\sigma_j\}_{i\leq j}$ the introduction part of the path;*

Proof: As a consequence of Proposition 6.15, we have that in every path all the elimination rules must precede those of introduction.

Let R_E be the last elimination rule in the path. Its conclusion is σ_n, or it is the premise of an introduction or it is $\bot$ the premise of $\bot_i$. ■

As for the branches in the classical case, we can assign an order to paths and prove a Sub-formula property.

Since in the intuitionistic system we consider all the connectives and rules, we split the proof of the subformula property.

We adapt to paths the notion of order between branches:

Definition 6.17 (Order of a path): Let Π be a derivation in normal form, we associate to each path $\pi \in \Pi$ a number $\mathbf{o}(\pi)$, called **order**, as follows.

(1) if π ends with the conclusion of Π, then $\mathbf{o}(\pi) = 0$ and a path of order 0 is said a *main path*;
(2) if a path π ends in the minor premise of an instance of $\to E$ whose major premise belongs to a path π' with $\mathbf{o}(\pi') = n$ then $\mathbf{o}(\pi) = n + 1$.

Lemma 6.18: *Let Π be a derivation in normal form. Then each (occurrence of a) formula A^x in Π belongs to some path.*

Proof: The proof proceeds by induction on the height of Π and by cases with respect to the last rule. We show the case where the last rule is $\Diamond E$.

$$\dfrac{\begin{array}{c}\Pi_1\\ \Diamond A^x\end{array} \qquad \begin{array}{c}[A^y]\\ \Pi_2\\ C^z\end{array}}{C^z}\ \Diamond E$$

Let's examine the shape of the paths in Π.

Since Π is in normal form (even with respect to simplification contractions) the rule $\Diamond E$ discharges the hypothesis A^y and therefore no track can stop at the major premise of the rule.

π is a path of Π if:

- π is a path in Π_1 with $\mathbf{o}(\pi) > 0$ or a path in Π_2 with $\mathbf{o}(\pi) > 0$;
- $\pi = \pi_1, \pi_2$ with π_1 a path of Π_1 with $\mathbf{o}(\pi_1) = 0$ and π_2 a path of Π_2 starting with an occurrence of the discharged hypothesis A^y and with $\mathbf{o}(\pi_1) > 0$
- $\pi = \pi_1, \pi_2, C^z$ with π_1 a path of Π_1 with $\mathbf{o}(\pi_1) = 0$ and π_2 a path of Π_2 starting with an occurrence of the discharged hypothesis A^y and with $\mathbf{o}(\pi_2) = 0$
- $\pi = \pi_2, \gamma$ with π_2 a path of Π_2 that doesn't start with an occurrence of the downloaded assumption A^y and with $\mathbf{o}(\pi_2) = 0$.

By applying the inductive hypotheses to the sub-derivations Π_1 we have the thesis.

Cases related to other rules are treated in the same way (i.e. examining all possible path cases). ■

Let us call HP(Π) the set of undischarged hypotheses of Π.

Lemma 6.19: *Let* $\begin{array}{c}\Pi\\ A^x\end{array}$ *be a derivation in normal form: for any n and for any C^z, if π is a path with $\mathbf{o}(\pi) = n$ and $C^z \in \pi$ then $C^z \in \mathrm{SF}(\mathrm{HP}(\Pi) \cup \{A^x\})$.*

Proof: The proof proceeds by induction on the order of the paths. Let $\pi = A_0^{x_0}, \ldots, A_m^{x_m}$.

base: if $\mathbf{o}(\pi) = 0$ the theorem trivially holds for $A_m^{x_m}$ and therefore for all the formulas belonging to the I–part. Now consider $A_0^{x_0}$. If $A_0^{x_0}$ was not discharged, then all formulas in the E–part and the minimum segment verify the assertion to be proved. If instead $A_0^{x_0}$ was discharged (by $\rightarrow I$), then there is a formula $A_k^{x_k} = A_0 \rightarrow B^{x_k}$ in the I -part. Since all the formulas of the I-part and of the minimum segment are subformulas of $A_0^{x_0}$, then they are also subformulas of $A_k^{x_k}$ which is in its turn a subformula of $A_m^{x_m}$.

inductive step: The reasoning to be done is very similar to that of the base case with the addition of the application of the inductive hypothesis.

Let $\mathbf{o}(\pi) = n + 1$.

If $A^z = A_m^{x_m}$ then A^z is a minor premise of a rule $\rightarrow E$, and therefore is a subformula of a formula in a path π' such that $\mathbf{o}(\pi') = n$. Then we apply the inductive hypothesis and conclude.

If A^z belongs to the I-part, then it is subformula of $A_m^{x_m}$ and concludes.

Now consider $A_0^{x_0}$. If $A_0^{x_0}$ was not discharged, then all formulas in E-part and minimum segment verify the assertion to be proved.

If instead $A_0^{x_0}$ has been discharged (by $\rightarrow I$), then either in the *I*-part of π there is a formula $A_k^{x_k} = A_0 \rightarrow B^{x_k}$ and we conclude as for the base case, or $A_0 \rightarrow B^{x_k}$ belongs to a path π' such that $\mathbf{o}(\pi') = n$.

We therefore apply the inductive hypothesis to π' and since all the formulas of the *I*-part and of the minimum segment are subformulas of $A_0^{x_0}$, we conclude. ■

Corollary 6.20 (Sub-formula Property): *Every formula $A_i{}^{x_i}$ occurring in a normal deduction Π of A^x from a set Γ of assumptions, is a sub-formula of A^x or of some formula in Γ.*

Proof: We simply observe that for Lemma 6.18 for each formula there is a path that contains it, therefore we apply the previous Lemma 6.19 and conclude. ■

As an immediate consequence, we have the following Consistency Theorem:

Theorem 6.21 (Consistency): *For each token y, $\nvdash_{\mathcal{N}_5^1} \bot^y$.*

Proof: By the subformula property, every formula in a derivation should be a subformula of $\bot$, so no rule could be used and therefore no derivation of $\bot$ exists with all hypotheses discharged. ■

Proposition 6.22: *Let $\begin{array}{c}\Pi\\ A^x\end{array}$ a normal form derivation. If A^x is not an atomic formula and the set of non-discharged hypotheses of Π is empty, then the last rule of Π is an introduction one.*

Proof: Obviously the last rule cannot be $\bot_i$. Let us suppose that the last rule of Π is an elimination rule. Now let us consider a path of order 0. Therefore all the formulas in such a path belong to the elimination part, and therefore there must be at least one undischarged hypothesis, absurd. ■

As a consequence, we have the following theorem.

Theorem 6.23 (Disjunction and Witness Properties): (1) *If $\vdash A \vee B^x$ then either $\vdash_{\mathcal{N}_5^1} A^x$ or $\vdash_{\mathcal{N}_5^1} B^x$.*

(2) *If $\vdash_{\mathcal{N}_5^1} \Diamond A^x$ then there exist y s.t. $\vdash_{\mathcal{N}_5^1} A^y$.*

6.9. A translation of the classical calculus into the intuitionistic one

We end the section by showing that intuitionistic $\mathcal{N}_5^{\mathbf{i}}$ is expressive enough to 'encode' classical **S5**. As a by-product, we obtain a syntactical proof of consistency for the classical systems.

In this section, to gain in readability, we refer by $\vdash_{\mathcal{N}_5^{\mathbf{c}}}$ and $\vdash_{\mathcal{N}_5^{\mathbf{i}}}$ to the derivability relations of $\mathcal{N}_5^{\mathbf{c}}$ and $\mathcal{N}_5^{\mathbf{i}}$ respectively.

We adapt now Gödel's *double negation translation* to our system.

We must note a very recent work of Lin and Ma (2022), where Gödel's translation (as well Kolmogorov and Kuroda's translations) is used in a refined way to obtain to prove proof-theoretical results on $\mathrm{IK}_t \oplus \mathrm{S}$.

Our approach is less pretentious, having as its sole purpose the original one of Gödel's translation, i.e. to show that intuitionistic logic can codify classical reasoning (see, e.g. Troelstra & Schwichtenberg, 2000 for the case of on modal logics).

As usual, $A \leftrightarrow B$ is a shorthand for $(A \to B) \wedge (B \to A)$.

Definition 6.24 (Translation map): We inductively define a map g between modal formulas as follows:

$g(\bot) = \bot$;
$g(A) = \neg\neg A$ for atomic A distinct from $\bot$;
$g(A \vee B) = \neg(\neg g(A) \wedge \neg g(B))$;
$g(A \sharp B) = g(A) \sharp g(B)$ when $\sharp$ is a binary connective distinct from $\vee$;
$g(\Box A) = \Box g(A)$;
$g(\Diamond A) = \neg\Box\neg g(A)$;

Proposition 6.25: *For every modal formula A and every token y,*

$$\vdash_{\mathcal{N}_5^{\mathbf{c}}} (A \leftrightarrow g(A))^y.$$

Proof: By induction on the complexity of A. ■

Definition 6.26 (Negative Formulas): A modal formula is *negative* if it is constructed from $\bot$ or from atomic formulas by means of $\Box, \wedge, \to$.

Lemma 6.27: *Let A be a negative formula constructed from doubly negated atomic formulas or from $\bot$. Then, for all tokens y*

$$\vdash_{\mathcal{N}_5^{\mathbf{i}}} (A \leftrightarrow \neg\neg A)^y.$$

Proof: By induction on the complexity of A.

- For the basis, recall that if A is either $\bot$ or a doubly negated atomic formula then A is provably equivalent to $\neg\neg A$ in an intuitionistic framework.
- Concerning the induction step, we only examine some nontrivial cases.

$\Box A$: Suppose the statement true for A. Then

$$\vdash_{\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}} (\Box A \leftrightarrow \Box\neg\neg A)^y$$

for all positions s. Therefore, to prove the nontrivial implication $\vdash_{\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}} (\neg\neg\Box A \to \Box A)^y$, it suffices to show that $\vdash_{\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}} (\neg\neg\Box A \to \Box\neg\neg A)^y$. The latter holds since

$$\vdash_{\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}} (\Diamond\neg A \to \neg\Box A)^y \quad \text{and} \quad \vdash_{\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}} (\neg\Diamond\neg A \to \Box\neg\neg A)^y$$

are true for all token s, even with no assumption on A.

$A \to B$: The direction $(A \to B)^y \to \neg\neg(A \to B)^y$ is trivial. For the other direction, suppose by i.h. that $\vdash_{\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}} (B \leftrightarrow \neg\neg B)^y$. The thesis $\neg\neg(A \to B)^y \to (A \to B)^y$ follows as shown in the derivation below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \to B^y] \quad [A^y]}{B^y}\,{\to E} \qquad [\neg B^y]}{\bot^y}\,{\to E}}{\neg(A \to B)^y}\,{\to I} \qquad [\neg\neg(A \to B)^y]}{\bot^y}\,{\to E}}{\neg\neg B^y}\,{\to I} \qquad \overset{i.h.}{\neg\neg B \to B^y}}{B^y}\,{\to E}}{A \to B^y}\,{\to I}}{\neg\neg A \to B^y \to A \to B^y}\,{\to I}$$

■

Remark 6.3: For every modal formula A, the formula $g(A)$ satisfies the assumptions of Lemma 6.27.

Remark 6.4: The following holds for any set Γ of formulas and formulas A^y and B^z: if $\Gamma, A^y \vdash_{\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}} B^z$ then $\Gamma, \neg B^z \vdash_{\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}} \neg A^y$.

We can now prove the following:

Proposition 6.28: *For every family $\{B_i^{y_i} : i \in I\}$ of formulas and every formula A^y*

$$\{B_i^{y_i} : i \in I\} \vdash_{\mathcal{N}^{\mathbf{c}}_{\mathbf{5}}} A^y \;\Leftrightarrow\; \{g(B_i)^{y_i} : i \in I\} \vdash_{\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}} g(A)^y.$$

Proof: $(\Leftarrow)$ Straightforward from Proposition 6.25.

$(\Rightarrow)$ By induction on the height of a deduction of A^y in **S5**. We only examine some nontrivial cases of the induction step.

$(\Diamond E)$ Suppose

$$\dfrac{\begin{array}{c}\dots\ B_i^{y_i}\ \dots\\ \vdots\\ \Diamond C^z\end{array}\qquad \begin{array}{c}[C^x]\ \dots B_i^{y_i}\dots\\ \vdots\\ A^y\end{array}}{A^y}$$

in $\mathcal{N}_{\mathbf{5}}^{\mathbf{c}}$. Then (inductively) we get the deductions

$$\begin{array}{c}\dots\ g(B_i)^{y_i}\ \dots\\ \vdots\\ \neg\Box\neg g(C)^z\end{array}\qquad \text{and} \qquad \begin{array}{c}g(C)^x\ \dots g(B_i)^{y_i}\dots\\ \vdots\\ g(A)^y\end{array}$$

in $\mathcal{N}_{\mathbf{5}}^{\mathbf{i}}$. By Remark 6.4, Remark 6.3 and Lemma 6.27 we get the following deduction in $\mathcal{N}_{\mathbf{5}}^{\mathbf{i}}$ (we leave to the reader to check that all side conditions of deduction rules are fulfilled):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[\neg g(A)^y]\ \dots g(B_i)^{y_i}\dots\\ \vdots\\ \neg g(C)^{zx}\end{array}}{\Box\neg g(C)^z}\qquad \begin{array}{c}\dots\ g(B_i)^{y_i}\ \dots\\ \vdots\\ \neg\Box\neg g(C)^z\end{array}}{\bot^z}}{\bot^y}}{\neg\neg g(A)^y}\qquad \begin{array}{c}\vdots\\ \neg\neg g(A)\to g(A)^y\end{array}}{g(A)^y}$$

$(\vee E)$ Suppose

$$\dfrac{\begin{array}{c}\dots B_i^{y_i}\dots\\ \vdots\\ B\vee C^z\end{array}\qquad \begin{array}{c}[B^z]\ \dots B_i^{y_i}\dots\\ \vdots\\ A^y\end{array}\qquad \begin{array}{c}[C^z]\ \dots B_i^{y_i}\dots\\ \vdots\\ A^y\end{array}}{A^y}$$

in **S5**.

By induction hypothesis and by Remark 6.4, we get the following deductions in **IS5**:

$$\begin{array}{c}\dots g(B_i)^{y_i}\dots\\ \vdots\\ \neg(\neg g(B)\wedge\neg g(C))^z\end{array}\qquad \begin{array}{c}\neg g(A)^y\ \dots g(B_i)^{y_i}\dots\\ \vdots\\ \neg g(B)^z\end{array}\qquad \begin{array}{c}\neg g(A)^y\ \dots g(B_i)^{y_i}\dots\\ \vdots\\ \neg g(C)^z\end{array}$$

From these deductions, we can produce the following in **IS5**:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\dots g(B_i)^{y_i}\dots\\ \vdots\\ \neg(\neg g(B)\wedge\neg g(C))^z\end{matrix} \qquad \dfrac{\begin{matrix}[\neg g(A)^y]\ \dots g(B_i)^{y_i}\dots\\ \vdots\\ \neg g(B)^z\end{matrix}\qquad \begin{matrix}[\neg g(A)^y]\ \dots g(B_i)^{y_i}\dots\\ \vdots\\ \neg g(C)^z\end{matrix}}{\neg g(B)\wedge\neg g(C)^z}}{\bot^z}}{\bot^y}}{\neg\neg g(A)^y}$$

We finally get the required deduction in **IS5** from Lemma 6.27.The other cases are easier. ■

Corollary 6.29: *For every formula A^y*

$$\vdash_{\mathcal{N}_5^{\mathbf{c}}} A^y \ \Leftrightarrow\ \vdash_{\mathcal{N}_5^{\mathbf{i}}} g(A)^y.$$

Consistency of **S5** follows immediately from Corollary 6.29.

6.10. Curry–Howard correspondence and Intuitionistic S5

We conclude our treatment of $\mathcal{N}_5^{\mathbf{i}}$ by showing how the **BHK** interpretation, via the natural deduction system, induces a Curry–Howard Isomorphism, in the following (standard sense):

(i) indexed formulas can be interpreted as types;
(ii) derivations can be interpreted as lambda terms;
(iii) reductions can be interpreted as computational steps.

We assume the reader is familiar with the main ideas behind the Curry–Howard isomorphism for standard intuitionistic logic, and typed lambda–calculus.

For the sake of clarity, we start from the negative fragment $\wedge, \rightarrow, \Box$ of $\mathcal{N}_5^{\mathbf{i}}$.

The set of *raw lambda terms* has an alphabet given by

- a denumerable set of variables $a_0, a_1, \dots$ (ranged over by a, b, c);
- a denumerable set of indexes $x_0, x_1, \dots$ (ranged over by x, y, z).

The set T of raw λ-terms is built according to the following abstract syntax:

$$T := x \mid (T_1 T_2) \mid (T_1 x) \mid (\lambda a : A^x.T_1) \mid (\Lambda x.T_1)$$

The reader should note that lambda terms are formed with two different kind of lambda abstractions, λ and Λ

Now we show how to inductively associate lambda terms to the indexed formula in the derivations, i.e. to types.

First, we assign to each occurrence of an hypothesis, a variable a, in such a way that if we have two occurrence $a : B^x$ and $a : C^y$ of undischarged hypotheses, then $B^x = C^y$.

Now, we can inductively decorate the proofs with lambda terms as follows:

$$\dfrac{\begin{array}{c}[a : A^x]\\ \Pi\\ T : B^x\end{array}}{\lambda a : A^x.T : A \to B^x}\ (\to I) \qquad \dfrac{\begin{array}{c}\Pi_1\\ T_1 : A \to B^x\end{array} \quad \begin{array}{c}\Pi_2\\ T_2 : \ A^x\end{array}}{T_1 T_2 : B^x}\ (\to E)$$

$$\dfrac{\begin{array}{c}\Pi\\ T : A^x\end{array}}{\Lambda x.T : \Box A^y}\ (\Box I) \qquad \dfrac{\begin{array}{c}\Pi\\ T : \Box A^x\end{array}}{Ty : A^y}\ (\Box E)$$

The reader is invited to note the similarity of what has just been written with λ–PRED, i.e. the typed lambda calculus for the negative fragment of first-order intuitionistic logic in the so-called Barendregt cube (see, e.g. Barendregt, 1992).

The lambda-abstraction λ is related to the introduction of the arrow type, in an absolutely standard way, while the lambda-abstraction Λ is connected to the introduction of the $\Box$. Obviously we have two different kinds of functional applications, corresponding to the elimination of the $\to$ and of the $\Box$.

Notation 6.1: (1) In the rest of this section, $\Pi, \Pi_1, \ldots$ will denote derivations under the assignment of lambda terms just defined.

(2) As usual, we shall write $\Gamma \vdash T : A^x$ when there exists a derivation $\Pi T : A^x$ s.t. Γ contains the undischarged occurrences of hypotheses in Π. Γ, also called context, is a set of indexed formulas s.t., if $a : B^x, a : C^y \in \Gamma$, then $B^x = C^y$.

(3) We read $\Gamma \vdash T : A^x$ as '*the lambda term T has type A^x, with respect to the context Γ.*' Or, in other words, '*the typed lambda term $T : A^x$ is derivable from the context Γ.*'

Now we take a look to computations, by focusing on the transformations that reductions induce on the lambda terms used to decorate derivations.

By applying the inductive definition of labelling derivations, we have two labelled contractions:

$$\dfrac{\dfrac{\begin{array}{c}[a : A^x]\\ \Pi_1\\ T_1 : B^x\end{array}}{\lambda a : A^x.T_1 : A \to B^x} \quad \begin{array}{c}\Pi_2\\ T_2 : A^x\end{array}}{(\lambda a : A^x.T_1)T_2 : B^x} \quad \triangleright \quad \begin{array}{c}\Pi_2\\ T_2 : A^x\\ \Pi_1[a/T_2]\\ T_1[a/T_2] : B^x\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ T : A^x\end{array}}{\Lambda x.T : \Box A^y}}{(\Lambda x.T)z : A^z} \quad \triangleright \quad \begin{array}{c}\Pi[x/z]\\ T[x/z] : A^z\end{array}$$

The above contractions induce the following β reductions between lambda terms:

$$(\lambda a : A^x.T_1)T \;\rhd\; T_1[a/T]$$

$$(\Lambda x.T_1)y \;\rhd\; T_1[x/y]$$

As usual for reductions between lambda terms, we have the contextual closures $\succ$ of $\rhd$, and the reflexive and transitive closure $\overset{*}{\succ}$ of $\succ$ between terms. The relation $\succ$ represents the computational step between lambda terms.

By means of the correctness of the reductions steps between derivations, we get immediately the following theorem.

Theorem 6.30 (Subject reduction): *If $\Gamma \vdash T : A^x$ and $T \succ T_1$ then $\Gamma \vdash T_1 : A^x$.*

Moreover, since derivations normalise (Theorem 6.13), we have also the following normalization theorem for lambda terms:

Theorem 6.31 (Normalization for lambda terms): *If $\Gamma \vdash T : A^x$, then there exists a lambda term T_1 s.t.*

- *T_1 is in normal form (i.e. $T_1 \overset{*}{\succ} T_2 \Rightarrow T_1 = T_2$);*
- *$T \overset{*}{\succ} T_1$.*

6.10.1. Extending the Curry–Howard isomorphism to the $\Diamond$ types.

The extension of the lambda calculus to include $\vee$ and $\bot$ also is similar to the corresponding propositional intuitionistic logic (e.g.see Girard et al., 1989). Then, we show here the new case of $\Diamond$ types only.

Raw lambda terms are extended with two new constructors: Let and pairing $<\_,\_>$. The abstract syntax for raw terms becomes then:

$$T := x \mid (T_1T_2) \mid (T_1x) \mid (\lambda a : A^x.T_1) \mid (\Lambda x.T_1) \mid\; < T_1, T_2 > \;\mid \mathrm{Let}[T_1 \text{ be } T_2] \text{ in } T_3$$

The assignment of terms to derivations is augmented by the following assignments:

$$\frac{\begin{array}{c}\Pi\\ T : A^x\end{array}}{\langle T, x\rangle : \Diamond A^y}\,(\Diamond I) \qquad \frac{\begin{array}{c}\Pi_1\\ T_1 : \Diamond A^z\end{array} \quad \begin{array}{c}[a : A^x]\\ \Pi_2\\ T_2 : C^y\end{array}}{\mathrm{Let}[\langle a, x\rangle \text{ be } T_1] \text{ in } T_2] : C^y}\,(\Diamond E)$$

and consequently we have the reduction:

$$\frac{\dfrac{\begin{array}{c}\Pi_1\\ T_1 : A^z\end{array}}{\langle T_1, z\rangle : \Diamond A^y} \quad \begin{array}{c}[c : A^x]\\ \Pi_2\\ T_2 : C^v\end{array}}{\mathrm{Let}[\langle c : A^x, x\rangle \text{ be } \langle T_1, z\rangle] \text{ in } T_2] : C^v} \quad \rhd \quad \begin{array}{c}\Pi_1\\ A^z\\ \Pi_2[x/z, c/T_1]\\ T_2[x/z, c/T_1] : C^v\end{array}$$

Such a contraction induces the following contraction between lambda terms:

$$\mathrm{Let}[< c : A^x, x > \text{ be } < T_1, z >] \text{ in } T_2] \;\rhd\; T_2[x/z, c/T_1]$$

The notion of $\succ$ and $\overset{*}{\succ}$ are extended accordingly. Moreover, the *subject reduction* and *normalization theorems* still hold true.

7. Other related works

We have examined in a detailed way the connection of our system with those of Galmiche and Salhi (2010b), but other proposals deserve attention.

While it is true that our work is not the first attempt to obtain a natural deduction system for **S5**, we believe that our approach, as mentioned in the introduction, is simpler, more direct, and more faithful to Prawitz's original natural deduction than previous works proposed in the literature. In particular, let us stress that:

(1) our natural deduction does not require the introduction on permutation rules in order to obtain the sub-formula property and a procedure for strong normalization (see the comparison with Simpson Simpson, 1993);
(2) the design of our systems is syntactically driven and the intuitionistic system is obtained by asking that its rules are compatible with some (non artificial) **BHK** interpretation. As for intuitionistic propositional logic, $\mathcal{N}_{\mathbf{5}}^{\mathbf{i}}$ is obtained by keeping the *ex falso quod libet* rule for the elimination of $\perp$ (the $\perp$ rule) and dropping the *reduction ad absurdum* rule (the $\perp_c$ rule) instead, since this is the only rule for which a **BHK** interpretation cannot be found. In other approaches instead, the design of the proof system is semantically driven, and the intuitionistic nature of it is given in terms of the interpretation into some Kripke model (see Simpson, 1993, for instance);
(3) to verify the applicability of the introduction rule for $\Box$ we do not have to verify all the paths in the proofs of the hypothesis, but it suffices to verify (the label of) the hypothesis and (the labels of) the open premises of the proof (see the comparison with Martins & Martins, 2008).

Let us now analyse in more details some other natural deduction approaches and the relevant work of Wansing (1995).

H. Wansing semantic tableau calculus. In Wansing (1995), Wansing develops a tableau calculus for classical **S5** (with $\wedge$ and $\Diamond$ as derived operators). Following a previous work by Fitting (1977), the formula of Wansing's tableaux are labelled by indexes. The derivable assertions are sequents of the form $\mathbf{X} \to \mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ (by using our notation) are *sets of indexed formulas*. The modal rules are:

$$\frac{\mathbf{X} \to \Box A^i, \mathbf{Y}}{\mathbf{X} \to A^k, \mathbf{Y}} \to \Box \qquad\qquad \frac{\mathbf{X}, \Box A^i \to \mathbf{Y}}{\mathbf{X}, A^k \to \mathbf{Y}} \Box \to$$

provided that in $\to \Box$ the index k is new in the branch under extension. Wansing proves then an important cut elimination theorem for his tableaux.

Our work develops and extends Wansing's indexed formula approach to natural deduction. Which is an important step towards an intuitionistic proof system and a corresponding term calculus for which the **BHK** interpretation holds. We remark also that, given a tableau/sequent formulation of a logical system, the existence of a corresponding natural deduction is not all evident, see for instance the case of Linear Logic, which has a very simple formulation in terms of sequents but for which there is no natural deduction and no satisfactory term calculus.

Intuitionism: the proposal of A. Simpson. Simpson (1993) proposes a natural deduction for intuitionistic **S5** whose derivable assertions are labelled formulas $x : A$ very similar to our ones, since the label x corresponds, de facto, to our tokens. The differences with our system become, however, evident if we analyse the rules. Simpson's approach is based on the first-order translation of modal formulas and is semantics driven. In other words, Simpson controls the introduction of modalities by means of additional relational rules where relational formulas xRy are used only as hypotheses and where R is a relation coding the accessibility relation of the Kripke model. The main advantage of Simpson's calculus is that it is modular, by changing the relation R he can obtain a wide spectrum of modal logics. His main disadvantage is that this leads to add a series of relational rules which do not introduce or eliminate any connective. In detail, here it is Simpson's modal and relational rules for intuitionistic **S5**:

$$\dfrac{\begin{matrix}[sRt]\\ \Pi\\ t:A\end{matrix}}{s:\Box A}\,(\Box I)* \qquad \dfrac{\begin{matrix}\Pi\\ s:\Box A\end{matrix}\quad sRt}{t:A}\,(\Box E)$$

$$\dfrac{\begin{matrix}\Pi\\ t:A\end{matrix}\quad sRt}{s:\Diamond A}\,(\Diamond I) \qquad \dfrac{\begin{matrix}\Pi\\ s:\Diamond A\end{matrix}\qquad \begin{matrix}[t:A][sRt]\\ \Pi_1\\ u:B\end{matrix}}{u:B}\,(\Diamond E)^{\circ}$$

$$\dfrac{\begin{matrix}[tRt]\\ \vdots\\ w:A\end{matrix}}{w:A}\,\text{refl} \qquad \dfrac{xRy\ yRz\quad \begin{matrix}[xRz]\\ \vdots\\ w:A\end{matrix}}{w:A}\,\text{trans} \qquad \dfrac{xRy\ xRz\quad \begin{matrix}[yRz]\\ \vdots\\ w:A\end{matrix}}{w:A}\,\text{five}$$

The structure of the proofs is then much more involved and less directly driven by the structure of the formula as in our system. As an example, let us compare the proof (in normal form) of the formula named B, namely $A \to \Box\Diamond A$, in Simpson's framework (right-hand side) and in our system (left-hand side).

$$\dfrac{\dfrac{\dfrac{[A^y]^1}{\Diamond A^x}\,\Diamond I}{\Box\Diamond A^y}\,\Box I}{A \to \Box\Diamond A^y}\,\to I^{(1)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[x:A]^4\quad [xRx]^0}{x:\Diamond A}\,\text{refl}^{(0)} \qquad \dfrac{[xRy]^3\ [xRz]^2 \qquad \dfrac{[z:A]^2\ [yRz]^1}{y:\Diamond A}\,\Diamond I}{y:\Diamond A}\,\text{five}^{(1)}}{y:\Diamond A}\,\Diamond E^{(2)}}{x:\Box\Diamond A}\,\Box I^{(3)}}{x:A \to \Box\Diamond A}\,\to I^{(4)}$$

The main proof theoretic drawback of these relational rules is that a series of corresponding proof permutation rules are needed to recover the sub-formula property (in addition to the standard permutation rules caused by $\vee$ and $\diamond$ elimination rules). As for instance:

$$\dfrac{\dfrac{x_1Ry_1 \quad \cdots \quad x_nRy_n \quad \begin{matrix}[tRu] \\ \vdots \\ w:A\end{matrix}}{w:A}\,r \qquad \Pi}{u:B}\,R \quad \rhd \quad \dfrac{x_1Ry_1 \quad \cdots \quad x_nRy_n \qquad \dfrac{\begin{matrix}[tRu]^n \\ \vdots \\ w:A\end{matrix} \qquad \Pi}{u:B}\,R}{u:B}\,r^{(n)}$$

Proof reduction must take into account these permutations and becomes then more involved, to the point that, instead of giving a direct combinatorial proof, Simpson proves strong normalization indirectly, via a translation into a first-order calculus.

Since Simpson's framework is probably one of the most important and accomplished natural deduction proposal up to know, let us briefly analyse how it relates to our solution in terms of provability. We stress that Simpson's framework is intuitionistic, so this comparison makes sense with our $\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}$ system only.

In Sections 6.1 and 6.2, we have seen that for each token x:

$$\vdash_{\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}} A^x \iff \vdash_{\mathrm{IS5}} A$$

On the other hand, via semantical methods, Simpson proves that for each label x (let us call SiS5 the system of Simpson):

$$\vdash_{\mathrm{SiS5}} x:A \iff \vdash_{\mathrm{IS5}} A$$

and therefore

$$\vdash_{\mathrm{SiS5}} x:A \iff \vdash_{\mathcal{N}^{\mathbf{i}}_{\mathbf{5}}} A^x$$

This means that from the point of view of pure provability of labelled formulas our system is equivalent to Simpson's one.

On the other hand, differently from the comparison we have done with the approach of Galmiche and Salhi (2010b), we do not know if it is possible to have a translation of our derivations in those of Simpson, since the two systems are truly different in the way they handle tokens/labels.

The key point is that Simpson's deductions are constructed by applying appropriate relational rules from time to time, depending on what we intend to prove. In other words, the proof of an assertion codes also its Kripke semantics interpretation. For instance, how could we translate the rules for $\Box$?

$$\dfrac{\begin{matrix}\vdots \\ A^x\end{matrix}}{\Box A^y}\,(\Box I) \qquad \dfrac{\begin{matrix}\vdots \\ \Box A^x\end{matrix}}{A^y}\,(\Box E)$$

The point is that it is not clear at all when and how to introduce the relational rules which allow to verify the side condition of the corresponding Simpson's $\Box$ rules. To mimic Simpson's approach, maybe we should have separate rules for each property of the accessibility relation. Nor, it seems viable to postpone all the relational rules to the end of the translation, since this seems not feasible without leaving undischarged relational hypotheses.

Other proposals. There are also proposals for natural deduction systems for **S5** that do not require indexed formulas or extended judgments. The most important one for intuitionistic **S5** is that of Prawitz (1965), revived in recent times by Martins and Martins (2008) by extending it to full classical **S5**.

Roughly speaking they study weak normalization in a system with an introduction rule for $\Box$ with the shape:

$$\begin{array}{ccc} \Pi_1 & & \Pi_k \\ C_1 & \cdots & C_2 \\ & \Pi & \\ & \dfrac{A}{\Box A}\,\Box I & \end{array}$$

where each C_1 is modally closed (the definition is at p. 77 of Prawitz, 1965).

In other words, to introduce the $\Box$ operator it is necessary to examine all the paths from the premises of the rule to the leaves, namely the assumptions of the proofs, and check that every path contains at least a modally closed formulas.

This formulation is, however, not completely in the spirit of natural deduction, since it is no longer true that the rules depend only on their hypotheses and on their conclusions but also on what is inside the derivations above them. As a consequence, we also have that the proof of the normalization result becomes very complex from a technical point of view.

Finally, we mention the work by Murphy et al. (2004); Murphy VII et al. (2005), in which the authors study modal types for distributed and mobile computing. Even if the authors propose term calculi for **S5**, their interest is not proof-theoretic, but to find a proof system for a modal lambda-calculus suitable for their purposes. Because of this, the authors introduce structural rules which make sense as a type system and have no clear logical meaning instead. At the same time, the authors are less interested in the usual proof theoretic good properties, as normalization and sub-formula.

Notes

1. We remark that Wansing's proof applies not only to tableaux, but to sequent calculus too.
2. $(n, m) < (p, q)$ if either $n < p$ or ($n = p$ and $m < q$)
3. Since the conclusion of $\bot_i$ is always atomic, we do not have contractions associated to such a rule.
4. $(n, m) < (p, q)$ if either $n < p$ or ($n = p$ and $m < q$)

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