

## ***Sets completely separated by functions in Bishop Set Theory***

Iosif Petrakis

**Abstract** Within Bishop Set Theory, a reconstruction of Bishop’s theory of sets, we study the so-called completely separated sets, that is sets equipped with a positive notion of an inequality, induced by a given set of real-valued functions. We introduce the notion of a global family of completely separated sets over an index-completely separated set, and we describe its Sigma- and Pi-set. The free completely separated set on a given set is also presented. Purely set-theoretic versions of the classical Stone-Čech theorem and the Tychonoff embedding theorem for completely regular spaces are given, replacing topological spaces with function spaces and completely regular spaces with completely separated sets.

### **1 Introduction**

Constructive mathematics (CM), although hard to define, is not only mathematics within intuitionistic logic. It also encompasses certain “attitudes” towards the definition of mathematical concepts, in order to reveal the computational content of mathematical proofs of theorems concerning these concepts. Some of these attitudes, which are not uniformly and consistently followed by the practitioners of CM though, are the following:

- *Predicativity*, that is the avoidance of circular, or impredicative definitions, which often amounts to the avoidance of quantification over proper classes in the definition of a mathematical concept.
- The addition of *witnessing information* so that even weak choice principles, such as countable choice<sup>1</sup>, are avoided in proofs. Bishop’s moduli of convergence, of continuity, of differentiability etc., in [3] are examples of added witnesses to convergence, continuity, differentiability etc.
- Preference for *positive definitions* of concepts over negative ones, that is definitions using negation. The most characteristic example of the positive approach to mathematical concepts within CM, which goes back to Brouwer, is the notion of inequality, or apartness relation on a set.
- Preference for *function-based concepts* over set-based ones, as functions suit better to CM than sets. Characteristic examples of this attitude are the definition of an integrable complemented set through the integrability of its partial characteristic function in constructive measure theory (see [4, 5]) and the reduction of the topology of open sets to the topology of functions in constructive topology of Bishop spaces (see [7, 15], [28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39] and [45]).

In this paper we highlight the combination of the last two attitudes in the study of sets with an inequality, determined in a positive way from a given set of functions. As many results shown here reveal an

2010 Mathematics Subject Classification: Primary 03F65

Keywords: constructive mathematics, Bishop set theory, apartness relation

analogy between these sets with the completely regular topological spaces, we call them *completely separated sets*. Although the notion of an inequality induced by real-valued functions is implicitly used by Bishop [3], p. 66, in his definition of a complemented subset, an elaborate study of sets equipped with such an inequality is missing. Ruitenburg’s work [50], in this journal, on inequalities in CM is set-based.

The various distinctions revealed within CM are also reflected in the various categories of sets that can be defined within CM. Besides the standard category **Set** of sets  $(X, =_X)$  and functions, we have the category **SetIneq** of sets with an inequality (see Definition 2.1) and strongly extensional functions as arrows (see Definition 3.1). This category is the “universe” of Bishop-Cheng measure theory (BCMT), introduced in [4] and extended significantly in [5]. In [41] we also introduce the category of strong sets **SetStrong**, a subcategory of **SetIneq**, where the inequalities considered are equivalent to the *strong negation* of the corresponding equalities, a positive and strong counterpart to the standard *weak negation* (see also [16]). In the category **SetComplSep** of sets completely separated by real-valued functions on them (see Definition 4.1), a subcategory of **SetIneq**, the given inequalities and equalities are equivalent to ones induced by a given set of real-valued functions. The importance of these inequalities lies in the complete avoidance of negation in their definition and in the proof of their basic properties. The category **SetComplSep** is the “universe” of Bishop measure theory<sup>2</sup>(BMT), developed in [3].

We work within *Bishop Set Theory* (BST), an informal, constructive theory of totalities and assignment routines, elaborated in [33, 37, 38, 40], that serves as a “completion” of Bishop’s original theory of sets in [3, 5]. BST highlights fundamental notions that were suppressed by Bishop in his account of the set theory underlying Bishop-style constructive mathematics (BISH), and serves as an intermediate step between Bishop’s informal theory of sets and an *adequate* and *faithful* formalisation of BISH, in Feferman’s sense [11]. To assure faithfulness, we use concepts or principles that appear, explicitly or implicitly, in BISH. The features of BST in [37] that somehow “complete” Bishop’s original theory of sets are: the explicit use of an open-ended universe  $\mathbb{V}_0$  of predicative sets, a clear distinction between sets and proper classes, such as  $\mathbb{V}_0$ , the explicit use of dependent operations, and the elaboration of the theory of set-indexed families of sets. Similarly to Martin-Löf Type Theory (MLTT) (see for example [17, 18]), BST behaves like a high-level programming language<sup>3</sup>.

We structure this paper as follows:

- In section 2 we introduce the category **SetIneq** of sets with an inequality and strongly extensional functions, and we define the Sigma-and the Pi-set of a family of sets with an inequality over an index-set with an inequality.
- In section 3 we define the canonical equality  $=_{(X,F)}$  and inequality  $\neq_{(X,F)}$  induced on a set  $(X, =_X)$  by an extensional subset  $F$  of the real-valued functions  $\mathbb{F}(X)$  on  $X$ . The extensionality and the tightness of the inequality  $\neq_{(X,F)}$  avoid completely the use of negation. Basic properties of these inequalities are shown.
- In section 4 we introduce the category **SetComplSep** of completely separated sets, a full subcategory of **SetIneq**, and the category **SetAffine** of affine sets, a subcategory of **SetComplSep** with affine arrows only. We define the notion of a family of completely separated sets over an index-completely separated set<sup>4</sup>, we describe its corresponding Pi-set, and we provide a sufficient condition, in order to get a restricted form of a Sigma-set for it (Proposition 4.6). By introducing the notion of a global family of completely separated sets over an index-completely separated set, we manage to describe its Sigma-set as a completely separated set, and to generalise the notion of a strongly extensional function to dependent functions. The second projection of the Sigma-set of such a global family is shown to be a strongly extensional dependent function (Corollary 4.10(ii)).
- In section 5 we define the free completely separated set  $\varepsilon\mathbf{X}$  on a given set  $(X, =_X)$ , we prove its universal property, and we show that the functor  $\text{Free}: \mathbf{Set} \rightarrow \mathbf{SetComplSep}$  is left adjoint to the corresponding forgetful functor  $\text{Frg}: \mathbf{SetComplSep} \rightarrow \mathbf{Set}$ . Theorem 5.2 corresponds to the type-theoretic fact that the setoid  $(X, =_X)$ , where  $=_X$  is the equality type family on the type  $X$  in intensional MLTT, is the free setoid on the type  $X$ .
- In section 6 we prove a purely set-theoretic version of the Stone-Čech theorem in classical topology, according to which, to any topological space corresponds a completely regular one such that the two spaces have isomorphic rings of continuous, real-valued functions, and the corresponding functor is

a reflector. Replacing topological spaces with function spaces, that is triplets  $(X, =_X; F)$  as above, and completely regular spaces with completely separated sets, we correspond to any function space a completely separated set with the same carrier set and (separating) set of functions (Theorem 6.2). The corresponding functor  $\rho: \mathbf{FunSpace} \rightarrow \mathbf{SetAffine}$  is shown to be a reflector, hence left adjoint to the corresponding embedding functor  $\text{Emb}: \mathbf{SetAffine} \rightarrow \mathbf{FunSpace}$ . Moreover,  $(\rho, \text{Emb})$  form an equivalence of categories (Proposition 6.3).

- In section 7 we prove a purely set-theoretic version of the Tychonoff embedding theorem in classical topology, according to which, a  $T_1$  topological space is completely regular if and only if it is topologically embedded into a product of  $[0, 1]$ . According to Theorem 7.5, if  $(X, =_X; F)$  is a function space, and if the induced inequality  $\neq_{(X,F)}$  on  $X$  is tight, then there is an affine embedding of the completely separated set  $(X, =_X, \neq_{(X,F)}; F)$  into the completely separated set  $\mathbf{R}^F$ . Conversely, if  $e: (X, =_X; F) \rightarrow (\mathbb{R}^F, =_{\mathbb{R}^F}; \otimes_{f \in F} \{\text{id}_{\mathbb{R}}\})$  is an affine embedding in  $\mathbf{FunSpace}$ , the induced inequality  $\neq_{(X,F)}$  on  $X$  is tight. The latter result provides a criterion for the generation of a completely separated set from a given function space.

For all notions and results from BST that are used here without explanation or proof, we refer to [37]. For all categorical notions and facts that are used here without explanation or proof, we refer to [2]. A formulation of category theory within BST is presented in [44].

## 2 Sets with an inequality

In this section we introduce the category  $\mathbf{SetIneq}$  of sets with an inequality. We use a terminology on inequalities that is closer to [19], rather than to [5]. In [5], p. 72, an inequality is what we call here an apartness relation. In [51], p. 31, an inequality is considered to be always symmetric. As the canonical inequality defined though strong negation in [41] is not in general symmetric, an abstract inequality here is a relation on a set that contradicts its given equality. In BST  $\text{falsum}$  is defined as the formula  $\perp := 0 =_{\mathbb{N}} 1$ .

**Definition 2.1** If  $(X, =_X)$  is a set, let the following formulas with respect to a relation  $x \neq_X y$ :

- (Ineq<sub>1</sub>)  $\forall x, y \in X (x =_X y \ \& \ x \neq_X y \Rightarrow \perp)$ ,
- (Ineq<sub>2</sub>)  $\forall x, x', y, y' \in X (x =_X x' \ \& \ y =_X y' \ \& \ x \neq_X y \Rightarrow x' \neq_X y')$ ,
- (Ineq<sub>3</sub>)  $\forall x, y \in X (\neg(x \neq_X y) \Rightarrow x =_X y)$ ,
- (Ineq<sub>4</sub>)  $\forall x, y \in X (x \neq_X y \Rightarrow y \neq_X x)$ ,
- (Ineq<sub>5</sub>)  $\forall x, y \in X (x \neq_X y \Rightarrow \forall z \in X (z \neq_X x \vee z \neq_X y))$ ,
- (Ineq<sub>6</sub>)  $\forall x, y \in X (x =_X y \vee x \neq_X y)$ .

If Ineq<sub>1</sub> is satisfied, we call  $\neq_X$  an inequality on  $X$ , and the structure  $\mathbf{X} := (X, =_X, \neq_X)$  a set with an inequality. We also write  $|\mathbf{X}| := X$ . If (Ineq<sub>6</sub>) is satisfied, then  $\mathbf{X}$  is called *discrete*. An inequality is called extensional, if (Ineq<sub>2</sub>) holds, and it is called tight, if (Ineq<sub>3</sub>) is satisfied. An inequality satisfying (Ineq<sub>4</sub>) and (Ineq<sub>5</sub>) is called an apartness relation on  $X$ . If  $\mathbf{Y} := (Y, =_Y, \neq_Y)$  is a set with an inequality, a function<sup>5</sup>  $f: X \rightarrow Y$  is strongly extensional, if

$$\forall x, y \in X (f(x) \neq_Y f(y) \Rightarrow x \neq_X y).$$

Let  $\mathbb{F}^{\neq}(\mathbf{X}, \mathbf{Y})$  be the set<sup>6</sup> of strongly extensional functions from  $\mathbf{X}$  to  $\mathbf{Y}$ , equipped with the pointwise equality. Furthermore, let  $\mathbb{V}_0^{\neq}$  be the proper class of sets with an inequality, equipped with the equality

$$\mathbf{X} =_{\mathbb{V}_0^{\neq}} \mathbf{Y} :\Leftrightarrow \exists f \in \mathbb{F}^{\neq}(\mathbf{X}, \mathbf{Y}) \exists g \in \mathbb{F}^{\neq}(\mathbf{Y}, \mathbf{X}) ((f, g): X =_{\mathbb{V}_0} Y).$$

Let  $\mathbf{SetIneq}$  be the category of sets with an inequality and strongly extensional functions.

An equality is always extensional on  $X \times X$ . There are non-extensional properties on a set, that is formulas  $P(x)$ , where  $x$  is a variable of the set  $X$ , such that  $P(x)$  and  $x =_X y$  do not imply  $P(y)$ . For example, let  $n \in \mathbb{N}$ ,  $q \in \mathbb{Q}$  and  $P_q(x)$ , where  $x$  is a variable of set  $\mathbb{R}$ , defined by  $P_q(x) := x_n =_{\mathbb{Q}} q$ . If  $y \in \mathbb{R}$  such that  $y =_{\mathbb{R}} x$ , then it is not necessary that  $y_n =_{\mathbb{Q}} q$ , if  $x_n =_{\mathbb{Q}} q$ . It is not easy to give examples of non strongly extensional functions, although we cannot accept in BISH that all functions are strongly extensional. For example, the strong extensionality of all functions from a metric space to itself is equivalent to Markov's

principle (see [10], p. 40). Even to show that a constant function between sets with an inequality is strongly extensional, one needs intuitionistic, and not minimal, logic.

**Remark 2.2** An apartness relation on a set  $(X, =_X)$  is an extensional inequality.

**Proof** Let  $x, y \in X$  with  $x \neq_X y$ , and  $x', y' \in X$  with  $x' =_X x$  and  $y' =_X y$ . By (Ineq<sub>5</sub>) we get  $x' \neq_X x$ , which is excluded from (Ineq<sub>1</sub>), or  $x' \neq_X y$ , which has to be the case. Hence,  $y' \neq_X x'$ , or  $y' \neq_X y$ . Since the latter is excluded similarly, we get  $y' \neq_X x'$ , hence by (Ineq<sub>4</sub>) we get  $x' \neq_X y'$ .  $\square$

For the canonical equalities of all sets mentioned here we refer to [37], Chapter 2, and to [5].

**Definition 2.3** Let  $\mathbf{X} := (X, =_X, \neq_X)$  and  $\mathbf{Y} := (Y, =_Y, \neq_Y)$  be in **SetIneq**, and let  $(A, i_A) \subseteq X$ . The canonical inequalities on the product  $X \times Y$ , the function space  $\mathbb{F}(X, Y)$  and  $A$  are given, respectively, by

$$\begin{aligned} (x, y) \neq_{X \times Y} (x', y') &: \Leftrightarrow x \neq_X x' \vee y \neq_Y y', \\ f \neq_{\mathbb{F}(X, Y)} g &: \Leftrightarrow \exists x \in X [f(x) \neq_Y g(x)], \\ a \neq_A a' &: \Leftrightarrow i_A(a) \neq_X i_A(a'). \end{aligned}$$

The canonical inequality on the set of reals  $\mathbb{R}$  is given by  $a \neq_{\mathbb{R}} b : \Leftrightarrow |a - b| > 0 \Leftrightarrow a > b \vee a < b$ , which is a special case of the canonical inequality on a metric space  $(Z, d)$ , given by  $z \neq_{(Z, d)} z' : \Leftrightarrow d(z, z') > 0$ . Let  $\mathbf{R} := (\mathbb{R}, =_{\mathbb{R}}, \neq_{\mathbb{R}})$ ,  $\mathbf{Z} := (Z, =_{(Z, d)}, \neq_{(Z, d)})$ , and  $\mathbf{N} := (\mathbb{N}, =_{\mathbb{N}}, \neq_{\mathbb{N}})$ . If  $\mathcal{2} := \{x \in \mathbb{N} \mid x =_{\mathbb{N}} 0 \vee x =_{\mathbb{N}} 1\}$ , let  $\mathbf{2} := (\mathbf{2}, =_{\mathbf{2}}, \neq_{\mathbf{2}})$ , where the (in)equality on  $\mathcal{2}$  is induced by the (in)equality on  $\mathbb{N}$ .

The inequalities  $a \neq_{\mathbb{R}} b$ ,  $z \neq_{(Z, d)} z'$ ,  $\neq_{\mathbb{N}}$  and  $\neq_{\mathbf{2}}$  are tight apartness relations. Clearly, the projections  $\text{pr}_X, \text{pr}_Y$  associated to  $X \times Y$ , and also, by definition, the embedding  $i_A^X$  of a subset  $A$  of  $X$ , are strongly extensional functions. The dependent assignment routines described next are primitive objects in BST.

**Definition 2.4** If  $(I, =_I)$  is a set and  $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$  is an assignment routine that corresponds to each element  $i \in I$  a set  $\lambda_0(i)$ , a dependent assignment routine  $\Theta$  assigns to every element  $i \in I$  an element  $\Theta_i$  of  $\lambda_0(i)$ . We denote by  $\mathcal{D}\mathcal{O}(I, \lambda_0)$  the totality of dependent assignment routines over  $I$  and  $\lambda_0$ , which becomes a set with an inequality, if we consider the following pointwise equality and inequality

$$\begin{aligned} \Theta =_{\mathcal{D}\mathcal{O}(I, \lambda_0)} \Theta' &: \Leftrightarrow \forall i \in I (\Theta_i =_{\lambda_0(i)} \Theta'_i), \\ \Theta \neq_{\mathcal{D}\mathcal{O}(I, \lambda_0)} \Theta' &: \Leftrightarrow \exists i \in I (\Theta_i \neq_{\lambda_0(i)} \Theta'_i), \end{aligned}$$

where for the latter we suppose that every  $\lambda_0(i)$  is equipped with an inequality  $\neq_{\lambda_0(i)}$ .

A family of sets indexed by some set  $(I, =_I)$  is an assignment routine  $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$  that behaves like a function, that is if  $i =_I j$ , then  $\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$ . A more explicit definition, which is due to Richman, is included in [5], p. 78 (Problem 2), which is made precise in [37] by highlighting the role of dependent assignment routines in its formulation. In accordance to the second attitude described in the Introduction, this is a proof-relevant definition revealing the witnesses of the equality  $\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$ . We define the notion of a family of sets within **SetIneq** similarly.

**Definition 2.5** If  $\mathbf{I} := (I, =_I, \neq_I)$  is in **SetIneq**, let the *diagonal*  $D(\mathbf{I})$  of  $\mathbf{I}$ , defined by

$$D(\mathbf{I}) := \{(i, j) \in I \times I \mid i =_I j\}.$$

A family of sets with an inequality indexed by  $\mathbf{I}$  is a pair  $\Lambda := (\lambda_0, \lambda_1)$ , where  $\lambda_0 : I \rightsquigarrow \mathbb{V}_0^{\neq}$ ,

$$\bar{\lambda}_0(i) := (\lambda_0(i), =_{\lambda_0(i)}, \neq_{\lambda_0(i)}),$$

for every  $i \in I$ , and  $\lambda_1$ , a *modulus of function-likeness* for  $\lambda_0$ , is a dependent operation

$$\lambda_1 : \bigwedge_{(i, j) \in D(\mathbf{I})} \mathbb{F}^{\neq}(\lambda_0(i), \lambda_0(j)), \quad \lambda_1(i, j) =: \lambda_{ij}, \quad (i, j) \in D(\mathbf{I}),$$

such that the *transport maps*  $\lambda_{ij}$  of  $\Lambda$  satisfy the following conditions:

- (a) For every  $i \in I$ , we have that  $\lambda_{ii} = \text{id}_{\lambda_0(i)}$ .
- (b) If  $i =_I j$  and  $j =_I k$ , the following triangle commutes

$$\begin{array}{ccc}
 \lambda_0(i) & & \\
 \lambda_{ij} \downarrow & \searrow \lambda_{ik} & \\
 \lambda_0(j) & \xrightarrow{\lambda_{jk}} & \lambda_0(k).
 \end{array}$$

If  $\mathbf{X}, \mathbf{Y}$  are in **SetIneq**, the constant  $\mathbf{I}$ -family of sets  $\mathbf{X}$  is the pair  $\mathbf{C}^{\mathbf{X}} := (\lambda_0^{\mathbf{X}}, \lambda_1^{\mathbf{X}})$ , where  $\lambda_0(i) := X$ , for every  $i \in I$ , and  $\lambda_1(i, j) := \text{id}_X$ , for every  $(i, j) \in D(I)$ . The  $\mathbf{2}$ -family  $\Lambda^2(\mathbf{X}, \mathbf{Y})$  of  $\mathbf{X}$  and  $\mathbf{Y}$  is defined by  $\lambda_0(0) := \mathbf{X}$ ,  $\lambda_0(1) := \mathbf{Y}$ ,  $\lambda_{00} := \text{id}_X$  and  $\lambda_{11} := \text{id}_Y$ .

If  $i =_I j$ , then  $(\lambda_{ij}, \lambda_{ji}) : \lambda_0(i) =_{\mathbb{V}_0^{\neq}} \lambda_0(j)$ . Next we describe the Sigma-set (or the exterior union, or the disjoint union) and the Pi-set (or the set of dependent functions) of a given family of sets with an inequality.

**Definition 2.6** Let  $\Lambda := (\lambda_0, \lambda_1)$  be an  $\mathbf{I}$ -family of sets with an inequality. Its Sigma-set

$$\sum_I \Lambda := \left( \sum_{i \in I} \lambda_0(i), =_{\sum_{i \in I} \lambda_0(i)}, \neq_{\sum_{i \in I} \lambda_0(i)} \right) \in \mathbf{SetIneq}$$

is defined by

$$w \in \sum_{i \in I} \lambda_0(i) :\Leftrightarrow \exists i \in I \exists x \in \lambda_0(i) (w := (i, x)),$$

$$(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow i =_I j \ \& \ \lambda_{ij}(x) =_{\lambda_0(j)} y,$$

$$(i, x) \neq_{\sum_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow i \neq_I j \vee (i =_I j \ \& \ \lambda_{ij}(x) \neq_{\lambda_0(j)} y).$$

The Sigma-set of the  $\mathbf{2}$ -family  $\Lambda^2(\mathbf{X}, \mathbf{Y})$  of  $\mathbf{X}$  and  $\mathbf{Y}$  is their *coproduct*. The *first projection* on  $\sum_{i \in I} \lambda_0(i)$  is the (non-dependent) operation<sup>7</sup>

$$\text{pr}_1^\Lambda : \sum_{i \in I} \lambda_0(i) \rightsquigarrow I, \quad \text{pr}_1^\Lambda(i, x) := \mathbf{pr}_1(i, x) := i, \quad (i, x) \in \sum_{i \in I} \lambda_0(i).$$

The *second projection* on  $\sum_{i \in I} \lambda_0(i)$  is the dependent operation

$$\text{pr}_2^\Lambda : \bigwedge_{(i, x) \in \sum_{i \in I} \lambda_0(i)} \lambda_0(i), \quad \text{pr}_2^\Lambda(i, x) := \mathbf{pr}_2(i, x) := x, \quad (i, x) \in \sum_{i \in I} \lambda_0(i).$$

We write  $\text{pr}_1, \text{pr}_2$ , if  $\Lambda$  is clearly understood from the context. The Pi-set of  $\Lambda$

$$\prod_I \Lambda := \left( \prod_{i \in I} \lambda_0(i), =_{\prod_{i \in I} \lambda_0(i)}, \neq_{\prod_{i \in I} \lambda_0(i)} \right) \in \mathbf{SetIneq}$$

is defined by

$$\Theta \in \prod_{i \in I} \lambda_0(i) :\Leftrightarrow \Theta \in \mathcal{D}\mathcal{O}(I, \lambda_0) \ \& \ \forall (i, j) \in D(I) (\Theta_j =_{\lambda_0(j)} \lambda_{ij}(\Theta_i)),$$

and  $=_{\prod_{i \in I} \lambda_0(i)}$  and  $\neq_{\prod_{i \in I} \lambda_0(i)}$  are the canonical pointwise equality and inequality, respectively, inherited from  $\mathcal{D}\mathcal{O}(I, \lambda_0)$ . If  $\mathbf{X}$  is in **SetIneq** and  $\mathbf{C}^{\mathbf{X}}$  is the constant  $\mathbf{I}$ -family  $\mathbf{X}$ , let

$$\mathbf{X}^I := \left( X^I := \prod_{i \in I} X, =_{X^I}, \neq_{X^I} \right).$$

Clearly, if  $i =_I j$  and  $\Theta \in \prod_{i \in I} \lambda_0(i)$ , then  $(i, \Theta_i) =_{\sum_{i \in I} \lambda_0(i)} (j, \Theta_j)$ . Although the canonical inequality  $=_{\sum_{i \in I} \lambda_0(i)}$  is an inequality on  $(\sum_{i \in I} \lambda_0(i), \neq_{\sum_{i \in I} \lambda_0(i)})$ , with respect to which  $\text{pr}_1^\Lambda$  is a strongly extensional function, we need extra assumptions to make it an apartness relation. The proof of the next proposition makes heavy use of negation, something that we shall avoid in the subsequent sections, where inequalities induced by functions will be considered.

**Proposition 2.7** Let  $\Lambda := (\lambda_0, \lambda_1)$  be an  $\mathbf{I}$ -family of sets with an inequality. If  $\mathbf{I}$  is discrete, and  $\neq_{\lambda_0(i)}$  is an apartness relation on  $\lambda_0(i)$ , for every  $i \in I$ , then  $\neq_{\sum_{i \in I} \lambda_0(i)}$  is an apartness relation. If  $\lambda_0(i)$  is discrete, for every  $i \in I$ , then  $\sum_I \Lambda$  is discrete, and if  $\neq_I$  is tight and  $\neq_{\lambda_0(i)}$  is tight, for every  $i \in I$ , then  $\neq_{\sum_{i \in I} \lambda_0(i)}$  is tight.

**Proof** The condition (Ineq<sub>1</sub>) of Definition 2.1 is trivially satisfied. To show condition (Ineq<sub>4</sub>), we suppose first that  $i \neq_I j$ , hence by the corresponding condition of  $\neq_I$  we get  $(j, y) \neq_{\Sigma_{i \in I} \lambda_0(i)} (i, x)$ . If  $i =_I j$  &  $\lambda_{ij}(x) \neq_{\lambda_0(j)} y$ , we show that  $\lambda_{ji}(y) \neq_{\lambda_0(i)} x$ . By the extensionality of  $\neq_{\lambda_0(j)}$  (Remark 2.2) the inequality  $\lambda_{ij}(x) \neq_{\lambda_0(j)} y$  implies the inequality  $\lambda_{ij}(x) \neq_{\lambda_0(j)} \lambda_{ij}(\lambda_{ji}(y))$ , and since  $\lambda_{ij}$  is strongly extensional, we get  $x \neq_{\lambda_0(i)} \lambda_{ji}(y)$ . To show condition (Ineq<sub>5</sub>), let  $(i, x) \neq_{\Sigma_{i \in I} \lambda_0(i)} (j, y)$ , and let  $(k, z) \in \Sigma_{i \in I} \lambda_0(i)$ . If  $i \neq_I j$ , then by condition (Ineq<sub>5</sub>) of  $\neq_I$  we get  $k \neq_I i$ , or  $k \neq_I j$ , hence  $(k, z) \neq_{\Sigma_{i \in I} \lambda_0(i)} (i, x)$ , or  $(k, z) \neq_{\Sigma_{i \in I} \lambda_0(i)} (j, y)$ . Suppose next  $i =_I j$  &  $\lambda_{ij}(x) \neq_{\lambda_0(j)} y$ . As  $\mathbf{I}$  is discrete,  $k \neq_I i$ , or  $k =_I i =_I j$ . If  $k \neq_I i$ , then what we want to show follows immediately. If  $k =_I i =_I j$ , then by the extensionality of  $\neq_{\lambda_0(j)}$  and the strong extensionality of the transport map  $\lambda_{kj}$  we have that

$$\lambda_{ij}(x) \neq_{\lambda_0(j)} y \Rightarrow \lambda_{kj}(\lambda_{ik}(x)) \neq_{\lambda_0(j)} \lambda_{kj}(\lambda_{jk}(y)) \Rightarrow \lambda_{ik}(x) \neq_{\lambda_0(k)} \lambda_{jk}(y).$$

Hence, by condition (Ineq<sub>5</sub>) of  $\neq_{\lambda_0(k)}$  we get  $\lambda_{ik}(x) \neq_{\lambda_0(k)} z$ , or  $\lambda_{jk}(y) \neq_{\lambda_0(k)} z$ , hence  $(i, x) \neq_{\Sigma_{i \in I} \lambda_0(i)} (k, z)$ , or  $(j, y) \neq_{\Sigma_{i \in I} \lambda_0(i)} (k, z)$ . Let  $\bar{\lambda}_0(i)$  be discrete, for every  $i \in I$ . We show that  $(i, x) =_{\Sigma_{i \in I} \lambda_0(i)} (j, y)$ , namely  $i =_I j$  and  $\lambda_{ij}(x) =_{\lambda_0(i)} y$ , or  $(i, x) \neq_{\Sigma_{i \in I} \lambda_0(i)} (j, y)$ , that is  $i \neq_I j$  or  $i =_I j$  &  $\lambda_{ij}(x) \neq_{\lambda_0(j)} y$ . As  $\mathbf{I}$  is discrete,  $i =_I j$ , or  $i \neq_I j$ . In the first case, and since  $\bar{\lambda}_0(j)$  is discrete, we get  $\lambda_{ij}(x) =_{\lambda_0(j)} y$  or  $\lambda_{ij}(x) \neq_{\lambda_0(j)} y$ , and what we want follows immediately. If  $i \neq_I j$ , we get  $(i, x) \neq_{\Sigma_{i \in I} \lambda_0(i)} (j, y)$ . Finally, we suppose that  $\neq_I$  is tight and  $\neq_{\lambda_0(i)}$  is tight, for every  $i \in I$ . Let  $\neg[(i, x) \neq_{\Sigma_{i \in I} \lambda_0(i)} (j, y)]$ , that is

$$[i \neq_I j \vee (i =_I j \ \& \ \lambda_{ij}(x) \neq_{\lambda_0(j)} y)] \Rightarrow \perp.$$

From this hypothesis we get the conjunction<sup>8</sup>

$$[i \neq_I j \Rightarrow \perp] \ \& \ [(i =_I j \ \& \ \lambda_{ij}(x) \neq_{\lambda_0(j)} y) \Rightarrow \perp].$$

By the tightness of  $\neq_I$  we get  $i =_I j$ . The formula  $(i =_I j \ \& \ \lambda_{ij}(x) \neq_{\lambda_0(j)} y) \Rightarrow \perp$  trivially implies  $(i =_I j) \Rightarrow (\lambda_{ij}(x) \neq_{\lambda_0(j)} y \Rightarrow \perp)$ , and as its premiss  $i =_I j$  is derived by the tightness of  $\neq_I$ , we get  $\lambda_{ij}(x) \neq_{\lambda_0(j)} y \Rightarrow \perp$ . Since  $\neq_{\lambda_0(i)}$  is tight, we conclude that  $\lambda_{ij}(x) =_{\lambda_0(j)} y$ , hence  $(i, x) =_{\Sigma_{i \in I} \lambda_0(i)} (j, y)$ .  $\square$

Notice that the above inequality on the Sigma-set of a family  $\Lambda$  does not give the canonical inequality of the product when the constant family is considered. This will be resolved in section 3 with the use of global families of completely separated sets. An abstract inequality induces a notion of disjoint subsets and a complemented subset is a pair of disjoint subsets, that is a subset with a given ‘‘complement’’. Complemented subsets are first-class citizens of Bishop–Cheng measure theory in [4, 5], a constructive version of Daniell’s approach to measure theory, of Bishop measure theory in [3] (see [34, 43, 47, 48, 13, 20]), and of constructive topologies of open complemented subsets (see [46]).

### 3 Equalities and inequalities induced by functions

A completely positive notion of inequality is implicitly used in the definition of a complemented subset in [3], p. 66. This is the inequality induced by a set of real-valued functions on a given set. This concept, together with the corresponding notion of equality induced by such a set of functions, are the starting point of our study. *Unless otherwise stated, in the rest of the paper*  $(X, =_X), (Y, =_Y), (I, =_I)$  are sets and  $F, G, K$  are extensional subsets of  $\mathbb{F}(X), \mathbb{F}(Y)$  and  $\mathbb{F}(I)$ , respectively<sup>9</sup>. Recall that  $\mathbb{F}(X)$  denotes the set of real-valued functions on  $X$ .

**Definition 3.1** Let  $\mathbf{X}$  be in **SetIneq**. The canonical equality  $x =_{(X, F)} x'$  on  $X$  induced by  $F$  is defined by

$$x =_{(X, F)} x' :\Leftrightarrow \forall f \in F (f(x) =_{\mathbb{R}} f(x')),$$

for every  $x, x' \in X$ , and the canonical inequality  $x \neq_{(X, F)} x'$  on  $X$  induced by  $F$  is defined by

$$x \neq_{(X, F)} x' :\Leftrightarrow \exists f \in F (f(x) \neq_{\mathbb{R}} f(x')),$$

for every  $x, x' \in X$ . We write  $f : x \neq_{(X, F)} x'$  to denote that  $f \in F$  witnesses the inequality  $x \neq_{(X, F)} x'$ . The induced inequality  $x \neq_{(X, F)} x'$  is equal to the given inequality  $\neq_X$  of  $X$ , if

$$x \neq_X x' \Leftrightarrow x \neq_{(X, F)} x',$$

for every  $x, x' \in X$ . The inequality  $\neq_{(X,F)}$  is called tight, if

$$x =_{(X,F)} x' \Rightarrow x =_X x',$$

for every  $x, x' \in X$ , and in this case we call  $F$  a separating set of functions on  $X$ , or we say that  $F$  separates the points of  $X$ . If  $\mathbf{Y}$  in **SetIneq**, and  $\Phi$  is an extensional subset of  $\mathbb{F}(X, Y)$ , the canonical (in)equalities  $(\neq_{(X,\Phi)}) =_{(X,\Phi)}$  on  $X$  induced by  $\Phi$ , and the tightness of  $\neq_{(X,\Phi)}$  are defined in a similar way.

**Remark 3.2** Let  $(X, =_X, \neq_X)$  be a set with an inequality.

- (i) For every  $x, x' \in X$  we have that  $x =_X x' \Rightarrow x =_{(X,F)} x'$ .
- (ii) If  $F$  separates the points of  $X$ , the equality  $=_X$  is equal to the induced equality  $=_{(X,F)}$  on  $X$ .
- (iii) The inequality  $\neq_{(X,F)}$  is tight with respect to the equality  $=_{(X,F)}$  on  $X$ .
- (iv) The canonical inequality  $x \neq_{(X,F)} x'$  on  $X$  induced by  $F$  is an apartness relation.
- (v) If  $\neq_X$  is equal to  $\neq_{(X,F)}$ , then  $\neq_X$  is an apartness relation.
- (vi)  $\neq_{(X,F)}$  is tight, according to Definition 3.1, if and only if it is tight according to Definition 2.1.
- (vii) If each  $f \in F$  is strongly extensional, then  $x \neq_{(X,F)} x' \Rightarrow x \neq_X x'$ , for every  $x, x' \in X$ .

**Proof** Cases (i)-(iii) are trivial, and (v) follows (iv). The proof of (iv) relies on the fact that the canonical inequality  $\neq_{\mathbb{R}}$  is an apartness relation. For the proof of (vi), suppose that  $x =_{(X,F)} x' \Rightarrow x =_X x'$ , and let  $\neg(x \neq_{(X,F)} x')$ . Then  $x =_{(X,F)} x'$ , using the tightness of  $\neq_{\mathbb{R}}$ : if  $f \in F$  with  $f(x) \neq_{\mathbb{R}} f(x')$ , then by our hypothesis we get a contradiction, hence  $\neg(f(x) \neq_{\mathbb{R}} f(x'))$ . By the tightness of  $\neq_{\mathbb{R}}$  we get  $f(x) =_{\mathbb{R}} f(x')$ , and we get the required equality  $x =_X x'$ . For the converse implication, let  $\neg(x \neq_{(X,F)} x') \Rightarrow x =_X x'$ , and  $x =_{(X,F)} x'$ . To show  $x =_X x'$ , it suffices to show  $\neg(x \neq_{(X,F)} x')$ . Clearly, the hypothesis  $x \neq_{(X,F)} x'$  together with the assumption  $x =_{(X,F)} x'$  give the required contradiction. The proof of case (vii) is immediate.  $\square$

By Remark 3.2(i), the given equality  $x =_X x'$  is “smaller” than every induced equality  $x =_{(X,F)} x'$ . The inequality  $\neq_{(X,F)}$  is an inequality, but no negation of some sort is used in its definition, and the proof of its extensionality avoids negation, as it relies on the extensionality of  $\neq_{\mathbb{R}}$ . By its extensionality,  $\neq_{(X,F)}$  provides a fully positive definition of the *extensional empty subset* of  $X$  induced by  $F$ , namely

$$\varnothing_{(X,F)} := \{x \in X \mid x \neq_{(X,F)} x\}.$$

With the help of the extensional empty subset a positive definition of the property “a subset is empty”, defined negatively as “it is not inhabited” in [6], p. 8, is possible. Following [47], if  $(A, i_A) \subseteq X$ , we call  $A$  empty, if  $A \subseteq \varnothing_{(X,F)}$  in  $\mathcal{P}(X)$ . Similarly, the tightness of  $\neq_{(X,F)}$  is formulated positively. Notice that the contrapositive of tightness,  $\neg(x =_X x') \Rightarrow \exists f \in F (f(x) \neq_{\mathbb{R}} f(x'))$ , is the standard, classical property of separation of points  $x, x' \in X$  from  $F$ . If  $F$  is a Bishop topology of functions (see [28, 35, 38]), then  $x =_{(X,F)} x'$  and  $x \neq_{(X,F)} x'$  are the canonical equality and inequality of a Bishop space, respectively. The inequality  $a \neq_{\mathbb{R}} b$  is equal to  $\neq_{(\mathbb{R}, \text{Bic}(\mathbb{R}))}$ , where  $\text{Bic}(\mathbb{R})$  is the topology of Bishop-continuous functions of type  $\mathbb{R} \rightarrow \mathbb{R}$  (see [28], Proposition 5.1.2, where even pointwise continuous functions are shown to separate the reals). But we can go even further, noticing that the following extensional subset of  $\mathbb{F}(\mathbb{R})$

$$\{\text{id}_{\mathbb{R}}\} := \{f \in \mathbb{F}(\mathbb{R}) \mid f =_{\mathbb{F}(\mathbb{R})} \text{id}_{\mathbb{R}}\}$$

separates the points of  $\mathbb{R}$ . If  $(Z, d)$  is a metric space, let the set

$$U_0(Z) := \{d_z \in \mathbb{F}(Z) \mid z \in Z\},$$

where  $d_z(z') := d(z, z')$ , for every  $z' \in Z$ . Then  $z \neq_{(Z,d)} z' \Leftrightarrow z \neq_{U_0(Z)} z'$ , for every  $z, z' \in Z$ . The family  $\text{Const}(X)$  of constant real-valued functions on  $X$  is not separating, in general. The following remark is straightforward to show.

**Remark 3.3** Let  $(X, =_X, \neq_X)$ ,  $F, F'$  extensional subsets of  $\mathbb{F}(X)$  with  $F \subseteq F'$ , and  $x, x' \in X$ .

- (i)  $x =_{(X,F')} x' \Rightarrow x =_{(X,F)} x'$ .
- (ii)  $x \neq_{(X,F)} x' \Rightarrow x \neq_{(X,F')} x'$ .
- (iii) If  $\neq_{(X,F)}$  is tight, then  $\neq_{(X,F')}$  is tight.
- (iv)  $\varnothing_{(X,F)} \subseteq \varnothing_{(X,F')}$ .

Consequently,  $=_{(X, \mathbb{F}(X))}$  is the “smallest” equality on  $X$  induced by real-valued functions on  $(X, =_X)$ , and  $\neq_{(X, \mathbb{F}(X))}$  is the largest inequality on  $X$  induced by such functions.

#### 4 Sets completely separated by functions

Next we introduce the category **SetComplSep** of sets completely separated by functions. Roughly speaking, a set with an inequality  $(X, =_X, \neq_X)$  is completely separated, if there is an extensional subset  $F$  of  $\mathbb{F}(X)$  such that  $=_X$  and  $\neq_X$  “are” the ones induced by  $F$ . Consequently,  $\neq_X$  is a tight apartness relation, and this fact is formulated in a completely positive, negation-free framework. In order to avoid quantification over the powerset of  $\mathbb{F}(X)$ , which is a proper class, we take  $F$  to be part of the defining data in Definition 4.1<sup>10</sup>.

**Definition 4.1** A completely separated set is a structure  $(\mathbf{X}; F)$ , where  $\mathbf{X} := (X, =_X, \neq_X)$  is in **SetIneq** and  $F$  is an extensional subset of  $\mathbb{F}(X)$ , such that  $\neq_X$  is equal to  $\neq_{(X, F)}$  and  $\neq_{(X, F)}$  is tight<sup>11</sup>. We call a function  $h: (\mathbf{X}; F) \rightarrow (\mathbf{Y}; G)$  affine, or an affine arrow, if  $g \circ h \in F$ , for every  $g \in G$

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow & \downarrow \\ & F \ni g \circ h & \mathbb{R}, \\ & & g \in G \end{array}$$

that is  $P_F(g \circ h)$ , for every  $g \in \mathbb{F}(Y)$  such that  $P_G(g)$ . Let  $\mathbb{V}_0^f$  be the proper class of completely separated sets, equipped with the equality of  $\mathbb{V}_0^f$ , and let **SetComplSep** be the full subcategory of **SetIneq** of completely separated sets. The category of affine sets **SetAffine** is the subcategory of **SetComplSep** with the same objects and only the affine arrows between them.

By the extensionality of  $F$  we get the extensionality of  $Q(h) := \forall_{g \in G} (g \circ h \in F)$  on  $\mathbb{F}(X, Y)$ , hence by separation we define the set of affine arrows from  $(\mathbf{X}; F)$  to  $(\mathbf{Y}; G)$ , that is

$$\mathbb{A}f((\mathbf{X}; F), (\mathbf{Y}; G)) := \{h \in \mathbb{F}(X, Y) \mid Q(h)\}.$$

Following the previous section,  $(\mathbf{R}; \{\text{id}_{\mathbb{R}}\})$  and  $(\mathbf{Z}; U_0(Z))$  are completely separated. Actually, if  $(\mathbf{X}; F)$  is in **SetComplSep**, every element  $f$  of  $F$  is an affine arrow from  $(\mathbf{X}; F)$  to  $(\mathbf{R}; \{\text{id}_{\mathbb{R}}\})$ . Clearly, an affine arrow between completely separated sets is a strongly extensional function, or an arrow in **SetComplSep**. By Markov’s principle every function from  $\mathbb{R}$  to  $\mathbb{R}$  is strongly extensional, but not necessarily affine. Using intuitionistic logic, and supposing that  $F$  does not include the constant functions on  $X$ , a constant function from  $X$  to  $Y$  is strongly extensional, but not affine.

**Proposition 4.2** Let  $(\mathbf{X}; F) := (X, =_X, \neq_X; F)$  and  $(\mathbf{Y}; G) := (Y, =_Y, \neq_Y; G)$  be in **SetComplSep**, and let  $(A, i_A) \subseteq X$ .

(i) The product  $(\mathbf{X} \times \mathbf{Y}; F \otimes G) := (X \times Y, =_{X \times Y}, \neq_X \otimes \neq_Y; F \otimes G)$  is a completely separated set, where

$$(x, y) \neq_X \otimes \neq_Y (x', y') := \Leftrightarrow x \neq_{(X, F)} x' \vee y \neq_{(Y, G)} y' \Leftrightarrow (x, y) \neq_{(X \times Y, F \otimes G)} (x', y'),$$

$$F \otimes G := \{f \circ \text{pr}_X \mid f \in F\} \cup \{g \circ \text{pr}_Y \mid g \in G\}.$$

The projections  $\text{pr}_X: (\mathbf{X} \times \mathbf{Y}; F \otimes G) \rightarrow (\mathbf{X}; F)$  and  $\text{pr}_Y: (\mathbf{X} \times \mathbf{Y}; F \otimes G) \rightarrow (\mathbf{Y}; G)$  are affine arrows<sup>12</sup>.

(ii) The function set  $(\mathbb{F}(X, Y), =_{\mathbb{F}(X, Y)}, \neq_{\mathbb{F}(X, Y)}; F \rightarrow G)$  is a completely separated set, where

$$h \neq_{\mathbb{F}(X, Y)} h' := \Leftrightarrow \exists_{x \in X} (h(x) \neq_{(Y, G)} h'(x)) \Leftrightarrow h \neq_{(\mathbb{F}(X, Y), F \rightarrow G)} h',$$

$$F \rightarrow G := \{\phi_{x, g} \mid x \in X \ \& \ g \in G\},$$

$$\phi_{x, g}: \mathbb{F}(X, Y) \rightarrow \mathbb{R}, \quad \phi_{x, g}(h) := g(h(x)), \quad h \in \mathbb{F}(X, Y).$$

(iii)  $(A, =_A, \neq_A; F \circ i_A)$  is completely separated, where  $a \neq_A a' := \Leftrightarrow i_A(a) \neq_{(X, F)} i_A(a') \Leftrightarrow a \neq_{F \circ i_A} a'$ ,

$$F \circ i_A := \{f \circ i_A \mid f \in F\},$$

and  $i_A: (A, =_A, \neq_A; F \circ i_A) \rightarrow (X, =_X, \neq_X; F)$  is an affine arrow.



**Proof** We show only (i) and (ii), as the proof of (iii) is immediate. For (i), if  $(x, y), (x', y') \in X \times Y$ , then

$$\begin{aligned} (x, y) \neq_{(X \times Y, F \otimes G)} (x', y') &: \Leftrightarrow \exists h \in F \otimes G (h((x, y)) \neq_{\mathbb{R}} h((x', y'))) \\ &\Leftrightarrow \exists f \in F ((f \circ \text{pr}_X)((x, y)) \neq_{\mathbb{R}} (f \circ \text{pr}_X)((x', y'))) \vee \\ &\quad \exists g \in G ((g \circ \text{pr}_Y)((x, y)) \neq_{\mathbb{R}} (g \circ \text{pr}_Y)((x', y'))) \\ &\Leftrightarrow: x \neq_{(X, F)} x' \vee y \neq_{(Y, G)} y'. \end{aligned}$$

The tightness of  $\neq_X \otimes \neq_Y$  follows from  $(x, y) =_{(X \times Y, F \otimes G)} (x', y') \Leftrightarrow x =_{(X, F)} x' \ \& \ y =_{(Y, G)} y'$  and the tightness of  $\neq_F$  and  $\neq_G$ . The fact that the projections are affine arrows follows immediately.

(ii) If  $h, h' \in \mathbb{F}(X, Y)$ , we have that

$$\begin{aligned} h \neq_{(\mathbb{F}(X, Y), F \rightarrow G)} h' &: \Leftrightarrow \exists x \in X \exists g \in G (\phi_{x, g}(h) \neq_{\mathbb{R}} \phi_{x, g}(h')) \\ &\Leftrightarrow \exists x \in X \exists g \in G (g(h(x)) \neq_{\mathbb{R}} g(h'(x))) \\ &\Leftrightarrow \exists x \in X (h(x) \neq_{(Y, G)} h'(x)) \\ &\Leftrightarrow: h \neq_{\mathbb{F}(X, Y)} h'. \end{aligned}$$

As  $\neq_{(Y, G)}$  is tight, we get

$$\begin{aligned} h =_{(\mathbb{F}(X, Y), F \rightarrow G)} h' &: \Leftrightarrow \forall x \in X \forall g \in G (\phi_{x, g}(h) =_{\mathbb{R}} \phi_{x, g}(h')) \\ &\Leftrightarrow \forall x \in X \forall g \in G (g(h(x)) =_{\mathbb{R}} g(h'(x))) \\ &\Leftrightarrow \forall x \in X (h(x) =_{(Y, G)} h'(x)) \\ &\Rightarrow \forall x \in X (h(x) =_Y h'(x)) \\ &\Leftrightarrow: h =_{\mathbb{F}(X, Y)} h'. \end{aligned} \quad \square$$

Next we define a family of completely separated sets indexed by a completely separated set  $(\mathbf{I}; K)$ .

**Definition 4.3** If  $(\mathbf{I}; K)$  is in **SetComplSep**, a *family of completely separated sets* indexed by  $(\mathbf{I}; K)$  is a structure  $\mathbf{S} := (\lambda_0, \lambda_1; \phi_0, \phi_1)$ , where  $(\lambda_0, \lambda_1)$  is an  $\mathbf{I}$ -family of sets with an inequality,  $\phi_0: I \rightsquigarrow \mathbb{V}_0$  with  $\phi_0(i) := F_i$ , an extensional subset of  $\mathbb{F}(\lambda_0(i))$ , and

$$\phi_1: \bigwedge_{(i, j) \in D(I)} \mathbb{F}(F_i, F_j), \quad \phi_1(i, j) := \phi_{ij}: F_i \rightarrow F_j, \quad (i, j) \in D(I),$$

such that the following conditions hold:

- (a)  $x_i \neq_{\lambda_0(i)} x_i' \Leftrightarrow x_i \neq_{(\lambda_0(i), F_i)} x_i'$ , for every  $x_i, x_i' \in \lambda_0(i)$  and  $i \in I$ .
- (b)  $x_i =_{(\lambda_0(i), F_i)} x_i' \Rightarrow x_i =_{\lambda_0(i)} x_i'$ , for every  $x_i, x_i' \in \lambda_0(i)$  and  $i \in I$ .
- (c) If  $i =_I j$ , then the following triangle commutes

$$\begin{array}{ccc} \lambda_0(j) & \xrightarrow{\lambda_{ji}} & \lambda_0(i) \\ & \searrow F_j \ni \phi_{ij}(f_i) & \downarrow f_i \in F_i \\ & & \mathbb{R}. \end{array}$$

We write  $(\lambda_0(i); F_i)$  in **SetComplSep**, for every  $i \in I$ .

If  $(\mathbf{X}; F)$  and  $(\mathbf{Y}; G)$  are in **SetComplSep**, it is easy to define the constant family  $(\mathbf{X}; F)$  over  $(\mathbf{I}; K)$  and the  $(\mathbf{2}; \mathbb{F}(2))$ -family of  $(\mathbf{X}; F)$  and  $(\mathbf{Y}; G)$ .

**Remark 4.4** Let  $\mathbf{S} := (\lambda_0, \lambda_1; \phi_0, \phi_1)$  be a family of completely separated sets over  $(\mathbf{I}; K)$ .

- (i)  $\phi_{ij}$  is strongly extensional, for every  $(i, j) \in D(I)$ .
- (ii)  $\lambda_{ij}$  is an affine arrow, for every  $(i, j) \in D(I)$ .
- (iii) The pair  $(\phi_0, \phi_1)$  induces an  $\mathbf{I}$ -family of sets with an inequality.
- (iv) The pair  $(\phi_0, \phi_1)$  induces an  $(\mathbf{I}; K)$ -family of completely separated sets.

**Proof** (i) If  $f_i, f'_i \in F_i$ , then we have that

$$\begin{aligned} \phi_{ij}(f_i) \neq_{F_j} \phi_{ij}(f'_i) &:\Leftrightarrow \exists_{y \in \lambda_0(j)} ([\phi_{ij}(f_i)](y) \neq_{\mathbb{R}} [\phi_{ij}(f'_i)](y)) \\ &\Leftrightarrow \exists_{y \in \lambda_0(j)} (f_i(\lambda_{ji}(y)) \neq_{\mathbb{R}} f'_i(\lambda_{ji}(y))) \\ &\Rightarrow \exists_{x \in \lambda_0(i)} (f_i(x) \neq_{\mathbb{R}} f'_i(x)) \\ &\Leftrightarrow: f_i \neq_{F_i} f'_i. \end{aligned}$$

(ii) It follows immediately by condition (c) in Definition 4.3.

(iii) Let  $\bar{\phi}_0(i) := (F_i, =_{F_i}, \neq_{F_i})$ , for every  $i \in I$ . We use (i), and by condition (c) in Definition 4.3 we get  $\phi_{ii}(f_i) = f_i \circ \lambda_{ii} = f_i$ , for every  $f_i \in F_i$ . We show the commutativity of the following triangle

$$\begin{array}{ccc} F_i & & \\ \phi_{ij} \downarrow & \searrow \phi_{ik} & \\ F_j & \xrightarrow{\phi_{jk}} & F_k, \end{array}$$

with the commutativity of the corresponding triangle for  $(\lambda_0, \lambda_1)$  as follows:  $(\phi_{jk} \circ \phi_{ij})(f_i) = (f_i \circ \lambda_{ji}) \circ \lambda_{kj} = f_i \circ (\lambda_{ji} \circ \lambda_{kj}) = f_i \circ \lambda_{ki} = \phi_{ik}(f_i)$ .

(iv) Let the quadruple  $(\phi_0, \phi_1; \theta_0, \theta_1)$ , where  $\theta_0(i) := \widehat{\lambda_0(i)} := \{\widehat{x} \mid x \in \lambda_0(i)\}$ , and  $\widehat{x}: F_i \rightarrow \mathbb{R}$  is given by  $\widehat{x}(f_i) := f_i(x)$ , for every  $x \in \lambda_0(i)$  and  $i \in I$ . and  $\theta_{ij}: \widehat{\lambda_0(i)} \rightarrow \widehat{\lambda_0(j)}$  is given by  $\theta_{ij}(\widehat{x}) := \widehat{\lambda_{ij}(x)}$ , for every  $x \in \lambda_0(i)$  and  $(i, j) \in D(I)$ . It is now straightforward to show conditions (a)–(c) of Definition 4.3.  $\square$

The Sigma-set and the Pi-set of an  $(\mathbf{I}; K)$ -family  $\mathbf{S}$  of completely separated sets are defined as in Definition 2.6. According to the next proposition, the Pi-set of such a family is in **SetComplSep**.

**Proposition 4.5** If  $\mathbf{S} := (\lambda_0, \lambda_1; \phi_0, \phi_1)$  is an  $(\mathbf{I}; K)$ -family of completely separated sets, then

$$\prod_I \mathbf{S} := \left( \prod_{i \in I} \lambda_0(i), =_{\prod_{i \in I} \lambda_0(i)}, \neq_{\prod_{i \in I} \lambda_0(i)}; \bigotimes_{i \in I} F_i \right) \in \mathbf{SetComplSep},$$

where

$$\bigotimes_{i \in I} F_i := \left\{ f_i \circ \text{pr}_i^\Lambda \mid f_i \in F_i, i \in I \right\}.$$

**Proof** Proceeding as in the proof of Proposition 4.2(i), we get

$$\Theta \neq_{\prod_{i \in I} \lambda_0(i)} \Theta' :\Leftrightarrow \exists_{i \in I} (\Theta_i \neq_{\lambda_0(i)} \Theta'_i) \Leftrightarrow \Theta \neq_{(\prod_{i \in I} \lambda_0(i), \bigotimes_{i \in I} F_i)} \Theta'. \quad \square$$

Similarly, the operation  $\text{pr}_i^\Lambda: \prod_{i \in I} \lambda_0(i) \rightsquigarrow \lambda_0(i)$ , defined by  $\Theta \mapsto \Theta_i$ , is an affine arrow, for every  $i \in I$ . If  $(\mathbf{I}; K)$  is discrete, and  $\mathbf{S}$  is an  $(\mathbf{I}; K)$ -family of completely separated sets, then working as in the proof of Proposition 2.7 we get an apartness relation on  $\sum_{i \in I} \lambda_0(i)$ , with respect to which  $\text{pr}_i^\Lambda$  is strongly extensional. Next we examine whether this inequality is induced by a set of functions.

**Proposition 4.6** Let  $(\mathbf{I}; K)$  be discrete, and  $\mathbf{S} := (\lambda_0, \lambda_1; \phi_0, \phi_1)$  an  $(\mathbf{I}; K)$ -family of completely separated sets. Let the following extensional subsets of  $\mathbb{F}(\sum_{i \in I} \lambda_0(i))$ :

$$\begin{aligned} \widehat{K} &:= \{\widehat{k} \mid k \in K\}, \\ \widehat{k}: \sum_{i \in I} \lambda_0(i) &\rightarrow \mathbb{R}, \quad \widehat{k}((i, x)) := k(i), \quad (i, x) \in \sum_{i \in I} \lambda_0(i), \\ \widehat{H} &:= \left\{ \widehat{\Phi} \mid \Phi \in \prod_{i \in I} F_i \right\}, \\ \widehat{\Phi}: \sum_{i \in I} \lambda_0(i) &\rightarrow \mathbb{R}, \quad \widehat{\Phi}((i, x)) := \Phi_i(x), \quad (i, x) \in \sum_{i \in I} \lambda_0(i). \end{aligned}$$

(i) For every  $(i, x), (j, y) \in \sum_{i \in I} \lambda_0(i)$ , we have that

$$(i, x) \neq_{\widehat{K} \cup \widehat{H}} (j, y) \Rightarrow (i, x) \neq_{\sum_{i \in I} \lambda_0(i)} (j, y).$$

(ii) If for every  $j \in I$  and every  $h \in F_j$  there is  $\Phi \in \prod_{i \in I} F_i$ , such that  $\Phi_j =_{\mathbb{F}(\lambda_0(j))} h$ , then for every  $(i, x), (j, y) \in \sum_{i \in I} \lambda_0(i)$ , we have that

$$(i, x) \neq_{\sum_{i \in I} \lambda_0(i)} (j, y) \Rightarrow (i, x) \neq_{\widehat{K} \cup \widehat{H}} (j, y).$$

**Proof** (i) If there is  $k \in K$  such that  $\widehat{k}((i, x)) := k(i) \neq_{\mathbb{R}} k(j) =: \widehat{k}((j, y))$ , then  $i \neq_K j$ , and the required inequality follows. Let  $\Phi \in \prod_{i \in I} F_i$  such that

$$\widehat{\Phi}(i, x) := \Phi_i(x) \neq_{\mathbb{R}} \Phi_j(y) =: \widehat{\Phi}(j, y).$$

If  $i \neq_K j$ , then we are reduced to the previous case. If  $i =_I j$ , then

$$\Phi_i(x) = [\phi_{ji}(\Phi_j)](x) = \Phi_j(\lambda_{ij}(x)) \neq \Phi_j(y),$$

hence  $\lambda_{ij}(x) \neq_{\lambda_0(j)} y$ , as by the definition of a family of completely separated sets the elements of  $F_j$ , for every  $j \in I$ , are strongly extensional functions. The required inequality follows then immediately.

(ii) If  $i \neq_K j$ , the implication follows trivially. If  $i =_I j$  and  $\lambda_{ij}(x) \neq_{\lambda_0(j)} y$ , there is  $h \in F_j$  such that  $h(\lambda_{ij}(x)) \neq_{\mathbb{R}} h(y)$ . Let  $\Phi \in \prod_{i \in I} F_i$ , such that  $\Phi_j = h$ . As  $\Phi_i = \phi_{ji}(\Phi_j) = \Phi_j \circ \lambda_{ij}$ , we get

$$\widehat{\Phi}(i, x) := \Phi_i(x) =_{\mathbb{R}} \Phi_j(\lambda_{ij}(x)) =_{\mathbb{R}} h(\lambda_{ij}(x)) \neq_{\mathbb{R}} h(y) =_{\mathbb{R}} \Phi_j(y) =: \widehat{\Phi}(j, y). \quad \square$$

By Proposition 4.6 to get that the Sigma-set of  $\mathbf{S}$  is completely separated, we need a discrete, completely separated index-set  $(\mathbf{I}; K)$ , and to suppose that for every  $j \in I$ , every element of  $F_j$  is “extended” to an element of  $\prod_{i \in I} F_i$ . To overcome these difficulties, we introduce the notion of a *global* family of completely separated sets. In contrast to Definition 2.5, where the transport maps  $\lambda_{ij}$  are defined on pairs  $(i, j)$  with  $i =_I j$ , in a global family the transport maps are defined on every pair  $(i, j)$ .

**Definition 4.7** If  $\mathbf{I}$  is in **SetIneq**, a global  $\mathbf{I}$ -family of sets with an inequality is a pair  $\Lambda^* := (\lambda_0, \lambda_1^*)$ , where  $\lambda_0: I \rightsquigarrow \mathbb{V}_0^{\neq}$ , and

$$\lambda_1^*: \bigwedge_{(i, j) \in I \times I} \mathbb{F}^{\neq}(\lambda_0(i), \lambda_0(j)), \quad \lambda_1^*(i, j) := \lambda_{ij}^*, \quad (i, j) \in I \times I,$$

such that the *global transport maps*  $\lambda_{ij}^*$  of  $\Lambda^*$  satisfy condition (a) of Definition 2.5 and condition (b) of Definition 2.5, for every  $i, j \in I$  with  $i =_I j$ , and every  $k \in I$ . If  $(\mathbf{I}; K)$  is in **SetComplSep**, an  $(\mathbf{I}; K)$ -family of completely separated sets is a structure  $\mathbf{S}^* := (\lambda_0, \lambda_1^*; \phi_0, \phi_1^*)$ , where  $(\lambda_0, \lambda_1^*)$  is a global  $\mathbf{I}$ -family of sets with an inequality,  $\phi_0$  is as in Definition 4.3,

$$\phi_1^*: \bigwedge_{(i, j) \in I \times I} \mathbb{F}(F_i, F_j), \quad \phi_1^*(i, j) := \phi_{ij}^*: F_i \rightarrow F_j, \quad (i, j) \in I \times I,$$

and the global maps  $\phi_{ij}^*$  satisfy conditions (a, b) of Definition 4.3 and condition (c) of Definition 4.3, for every  $i, j \in I$ . The equality and inequality of the Sigma-set of  $\Lambda^*(\mathbf{S}^*)$  are given by

$$(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow i =_I j \ \& \ \lambda_{ij}^*(x) =_{\lambda_0(j)} y,$$

$$(i, x) \neq_{\sum_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow i \neq_{(I, K)} j \ \vee \ \lambda_{ij}^*(x) \neq_{\lambda_0(j)} y.$$

The Pi-set of  $\Lambda^*(\mathbf{S}^*)$  is defined as in Definition 2.6.

The constant global  $\mathbf{I}$ -family of sets with an inequality  $\mathbf{X}$  is defined as the constant  $\mathbf{I}$ -family  $\mathbf{X}$ , with  $\lambda_{ij}^* := \text{id}_X$ , for every  $(i, j) \in I \times I$ . If this constant global family is considered, then the inequality of its Sigma-set is reduced to the inequality of the product set. The constant global  $(\mathbf{I}; K)$ -family of completely separated sets  $(\mathbf{X}; F)$  is defined as the constant  $(\mathbf{I}; K)$ -family  $(\mathbf{X}; F)$ , with  $\phi_{ij}^* := \text{id}_F$ , for every  $(i, j) \in I \times I$ . To define the global 2-family  $\Lambda^2(\mathbf{X}, \mathbf{Y})$  of  $\mathbf{X}$  and  $\mathbf{Y}$ , we add any two strongly extensional functions  $\lambda_{01}^*: X \rightarrow Y$  and  $\lambda_{10}^*: Y \rightarrow X$ , as the following triangles trivially commute

$$\begin{array}{ccc} X & & Y \\ \text{id}_X \downarrow & \searrow \lambda_{01}^* & \downarrow \text{id}_Y \\ X & \xrightarrow{\lambda_{01}^*} & Y \\ & & \downarrow \lambda_{10}^* \\ & & Y \xrightarrow{\lambda_{10}^*} X \end{array}$$

To define the global  $(\mathbf{2}; \mathbb{F}(2))$ -family of  $(\mathbf{X}; F)$  and  $(\mathbf{Y}; G)$ , we also add maps  $\phi_{01}^* : F \rightarrow G$  and  $\phi_{10}^* : G \rightarrow F$ , such that the following triangles commute

$$\begin{array}{ccc} X & \xrightarrow{\lambda_{01}^*} & Y \\ F \ni \phi_{10}^*(g) \searrow & & \downarrow g \in G \\ & & \mathbb{R} \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{\lambda_{10}^*} & X \\ G \ni \phi_{01}^*(f) \searrow & & \downarrow f \in F \\ & & \mathbb{R} \end{array}$$

and moreover,  $g \circ \lambda_{01}^* \in F$  and  $f \circ \lambda_{10}^* \in G$ , for every  $g \in G$  and  $f \in F$ , respectively. Next we show that the Sigma-set of a global  $(\mathbf{I}; K)$ -family of completely separated sets is also completely separated.

**Theorem 4.8** *Let  $(\mathbf{I}; K)$  be completely separated and let  $\mathbf{S}^* := (\lambda_0, \lambda_1^*; \phi_0, \phi_1^*)$  be a global  $(\mathbf{I}; K)$ -family of completely separated sets. Let the extensional subsets  $\widehat{K}$  and  $\widehat{H}$  of  $\mathbb{F}(\sum_{i \in I} \lambda_0(i))$  that were defined in Proposition 4.6. If the Sigma-set of  $\Lambda^*$  is equipped with the equality and inequality given in Definition 4.7, then for every  $(i, x), (j, y) \in \sum_{i \in I} \lambda_0(i)$  we have that*

$$(i, x) \neq_{\widehat{K} \cup \widehat{H}} (j, y) \Leftrightarrow (i, x) \neq_{\sum_{i \in I} \lambda_0(i)} (j, y),$$

and hence

$$\sum \mathbf{S}^* := \left( \sum_{i \in I} \lambda_0(i), =_{\sum_{i \in I} \lambda_0(i)}, \neq_{\sum_{i \in I} \lambda_0(i)} ; \widehat{K} \cup \widehat{H} \right) \in \mathbf{SetCompSep}.$$

**Proof** To show  $(i, x) \neq_{\widehat{K} \cup \widehat{H}} (j, y) \Rightarrow (i, x) \neq_{\sum_{i \in I} \lambda_0(i)} (j, y)$ , we work exactly as in the proof of Proposition 4.6(i). For the converse implication, we work as in the proof of Proposition 4.6(ii). It suffices to show that, if  $h \in F_j$  with  $h(\lambda_{ij}^*(x)) \neq_{\mathbb{R}} h(y)$ , then there is  $\Phi^h \in \prod_{i \in I} F_i$  with  $\Phi_j^h = h$ . If  $i \in I$ , we define  $\Phi_i^h := h \circ \lambda_{ij}^*$

$$\begin{array}{ccc} \lambda_0(i) & \xrightarrow{\lambda_{ij}^*} & \lambda_0(j) \\ F_i \ni \Phi_i^h \searrow & & \downarrow h \in F_j \\ & & \mathbb{R} \end{array}$$

By the definition of a global  $(\mathbf{I}; K)$ -family of completely separated sets we get  $\Phi_i^h \in F_i$ , for every  $i \in I$ . Moreover,  $\Phi_j^h = h \circ \lambda_{jj}^* = h \circ \text{id}_{\lambda_0(j)} = h$ . To show that  $\Phi^h \in \prod_{i \in I} F_i$ , let  $i, k \in I$  such that  $i =_{(I, K)} k$ . By the new condition (b) satisfied by  $\Lambda^*$  we have that

$$\phi_{ik}^*(\Phi_i^h) = \phi_{ik}^*(h \circ \lambda_{ij}^*) = h \circ \lambda_{ij}^* \circ \lambda_{ki}^* = h \circ \lambda_{kj}^* = \Phi_k^h.$$

Let  $(i, x) =_{\widehat{K} \cup \widehat{H}} (j, y)$ . By the tightness of  $\neq_{(I, K)}$  we have that

$$\forall_{k \in K} (\widehat{k}((i, x)) =_{\mathbb{R}} \widehat{k}((j, y))) \Leftrightarrow \forall_{k \in K} (k(i) =_{\mathbb{R}} k(j)) \Rightarrow i =_I j.$$

If  $\Phi \in \prod_{i \in I} F_i$ , then

$$\widehat{\Phi}((i, x)) =_{\mathbb{R}} \widehat{\Phi}((j, y)) \Leftrightarrow \Phi_i(x) =_{\mathbb{R}} \Phi_j(y) \Leftrightarrow \Phi_j(y) =_{\mathbb{R}} [\phi_{ij}^*(\Phi_j)](x) = \Phi_j(\lambda_{ij}^*(x)).$$

Hence, for every  $h \in F_j$  we have that  $h(y) =: \Phi_j^h(y) =_{\mathbb{R}} \Phi_j^h(\lambda_{ij}^*(x)) := h(\lambda_{ij}^*(x))$ . By the tightness of  $\neq_{F_j}$  we get the required equality  $y =_{\lambda_0(j)} \lambda_{ij}^*(x)$ .  $\square$

The importance of the previous theorem lies on the fact that there is no obvious way to show that the canonical inequality of the Sigma-set of a global family  $\mathbf{S}^*$  of completely separated sets is an apartness relation using only its definition. The proof of Proposition 2.7 cannot be carried out, as the transport maps are between any two sets of the given family. By showing though, that this inequality is equivalent to the inequality induced by  $\widehat{K} \cup \widehat{H}$ , then by Remark 3.2(v) this inequality is also an apartness relation on the Sigma-set of  $\mathbf{S}^*$ ! The reason behind this, seemingly unexpected, result is the extra information provided by the data associated to the notion of a global family of completely separated sets. Through the notion of a global family  $\mathbf{S}^*$  of completely separated sets we can also define when a dependent function over  $\Lambda^*(\mathbf{S}^*)$  is strongly extensional in a way that generalises the strong extensionality of (non-dependent) functions.

**Definition 4.9** If  $\Lambda^*(\mathbf{S}^*)$  is a global  $\mathbf{I}$ -family of sets with an inequality, we call a dependent function  $\Phi$  in the Pi-set of  $\Lambda^*(\mathbf{S}^*)$  strongly extensional, if

$$\forall i, j \in I (\lambda_{ij}^*(\Phi_i) \neq_{\lambda_0(j)} \Phi_j \Rightarrow i \neq_{(I, K)} j).$$

Notice that if  $\lambda_{ij}^*(\Phi_i) \neq_{\lambda_0(j)} \Phi_j$ , then  $\neg(i =_I j)$ , as if  $i =_I j$ , then  $\lambda_{ij}^*(\Phi_i) =_{\lambda_0(j)} \Phi_j$ . The above definition of strong extensionality implies the stronger inequality  $i \neq_I j$ , and clearly generalises the strong extensionality of non-dependent functions, if a constant global family of completely separated sets is considered. In analogy to the strong extensionality of the first projection, we can show that the second projection  $\text{pr}_2^{\Lambda^*}$  of the Sigma-set of  $\Lambda^*(\mathbf{S}^*)$ , defined as in Definition 2.6, is a strongly extensional dependent function.

**Corollary 4.10** Let  $\Lambda^*$  be a global  $\mathbf{I}$ -family of sets with an inequality.

(i) If  $\sigma_0: \sum_{i \in I} \lambda_0(i) \rightsquigarrow \mathbb{V}_0^\#$  is given by  $\sigma_0(i, x) := \lambda_0(i)$ , for every  $(i, x) \in \sum_{i \in I} \lambda_0(i)$ , and  $\sigma_{(i, x), (j, y)}^* := \lambda_{ij}^*$ , for every  $(i, x), (j, y) \in \sum_{i \in I} \lambda_0(i)$ , then  $\Sigma^* := (\sigma_0, \sigma_1^*)$  is a global  $\mathbf{I}$ -family of sets with an inequality.

(ii) The second projection  $\text{pr}_2^{\Lambda^*}$  is a strongly extensional dependent function over  $\Sigma^*$ .

**Proof** The proof of (i) is immediate. For the proof of (ii), that is

$$\text{pr}_2^{\Lambda^*} \in \prod_{(i, x) \in \sum_{i \in I} \lambda_0(i)} \sigma_0(i, x) := \prod_{(i, x) \in \sum_{i \in I} \lambda_0(i)} \lambda_0(i),$$

let  $(i, x), (j, y) \in \sum_{i \in I} \lambda_0(i)$  such that  $(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y)$ , that is  $i =_{(I, K)} j$  and  $\lambda_{ij}^*(x) =_{\lambda_0(j)} y$ . Then,

$$\text{pr}_2^{\Lambda^*}(j, y) := y =_{\lambda_0(j)} \lambda_{ij}^*(x) := \sigma_{(i, x), (j, y)}^*(\text{pr}_2^{\Lambda^*}(i, x)).$$

If  $(i, x), (j, y)$  are arbitrary elements of  $\sum_{i \in I} \lambda_0(i)$ , such that

$$\sigma_{(i, x), (j, y)}^*(\text{pr}_2^{\Lambda^*}((i, x)) \neq_{\sigma_0((j, y))} \text{pr}_2^{\Lambda^*}((j, y)) \Leftrightarrow \lambda_{ij}^*(x) \neq_{\lambda_0(j)} y,$$

then the inequality  $(i, x) \neq_{\sum_{i \in I} \lambda_0(i)} (j, y)$  follows immediately.  $\square$

## 5 The free completely separated set

Next we define the free completely separated set on a given set  $(X, =_X)$ . If  $(X, =_X)$  is in **Set**, then by Remark 3.2(iii)  $(X, =_{(X, F)}, \neq_{(X, F)}; F)$  is in **SetComplSep**. As  $\mathbb{F}(X)$  is an extensional subset of itself

$$\mathbb{F}(X) = \{f \in \mathbb{F}(X) \mid f =_{\mathbb{F}(X)} f\},$$

the structure  $(X, =_{(X, \mathbb{F}(X))}, \neq_{(X, \mathbb{F}(X))}; \mathbb{F}(X))$  is in **SetComplSep**.

**Definition 5.1** If  $(X, =_X)$  is in **Set**, the free completely separated set on  $(X, =_X)$  is the structure  $\varepsilon\mathbf{X} := (X, =_{(X, \mathbb{F}(X))}, \neq_{(X, \mathbb{F}(X))}; \mathbb{F}(X))$ .

**Theorem 5.2** (i)  $\varepsilon\mathbf{X}$  has the universal property of the free completely separated set on  $(X, =_X)$ , namely there is a function  $i_X: X \rightarrow |\varepsilon\mathbf{X}|$  such that for every completely separated set  $\mathbf{Y}$  and function  $h: X \rightarrow |\mathbf{Y}|$  there is a unique strongly extensional function  $\varepsilon h: |\varepsilon\mathbf{X}| \rightarrow |\mathbf{Y}|$  such the following triangle commutes

$$\begin{array}{ccc}
X & \xrightarrow{i_X} & |\varepsilon\mathbf{X}| \\
& \searrow h & \downarrow \varepsilon h \\
& & |\mathbf{Y}|.
\end{array}$$

(ii) Every completely separated set is the quotient of the free completely separated set over it.

(iii) Let the functor  $\text{Free}: \mathbf{Set} \rightarrow \mathbf{SetCompSep}$ , defined by

$$\text{Free}(X, =_X) := \varepsilon\mathbf{X}, \quad \text{Free}(f: X \rightarrow Y) := \varepsilon f: |\varepsilon\mathbf{X}| \rightarrow |\varepsilon\mathbf{Y}|$$

$$\begin{array}{ccc}
X & \xrightarrow{i_X} & |\varepsilon\mathbf{X}| \\
f \downarrow & \searrow & \downarrow \varepsilon f := \varepsilon(i_Y \circ f) \\
Y & \xrightarrow{i_Y} & |\varepsilon\mathbf{Y}|,
\end{array}$$

and let the forgetful functor  $\text{Frg}: \mathbf{SetCompSep} \rightarrow \mathbf{Set}$ , defined by

$$\text{Frg}(X, =_X, \neq_X; F) := (X, =_X), \quad \text{Frg}(h: \mathbf{X} \rightarrow \mathbf{Y}) := h.$$

Then,  $\text{Free}$  is left adjoint to  $\text{Frg}$ .

**Proof** (i) Let  $i_X := \text{id}_X$  and  $\varepsilon h := h$ . That  $\text{id}_X$  is a function follows from Remark 3.2(i). To show that  $h: |\varepsilon\mathbf{X}| \rightsquigarrow |\mathbf{Y}|$  is a function, we show that every operation from  $(X, =_{(X, \mathbb{F}(X))})$  to  $\mathbf{Y} := (Y, =_Y, \neq_Y; G)$  is a function; if  $x, x' \in X$  such that  $x =_{(X, \mathbb{F}(X))} x'$ , then  $h(x) =_Y h(x') \Leftrightarrow h(x) =_{(Y, G)} h(x')$ , and if  $g \in G$ , then  $g \circ h \in \mathbb{F}(X)$  and the required equality follows from the hypothesis  $x =_{(X, \mathbb{F}(X))} x'$ . That  $h: |\varepsilon\mathbf{X}| \rightsquigarrow |\mathbf{Y}|$  is strongly extensional, is shown as follows: if  $x, x' \in X$ , and if  $g \in G$  such that  $g: h(x) =_{(Y, G)} h(x')$ , then  $F \ni g \circ h: x \neq_{(X, \mathbb{F}(X))} x'$ . The commutativity of the triangle and the uniqueness of  $\varepsilon h$  follow immediately.

(ii) If  $\mathbf{X} := (X, =_X, \neq_X; F)$ , then the identity  $\text{id}_X$  from  $\varepsilon\mathbf{X}$  to  $\mathbf{X}$  is the required surjection.

(iii) Clearly,  $\text{Free}$  and  $\text{Frg}$  are functors. Let  $i_{X, \mathbf{Y}}: \text{Hom}(\varepsilon\mathbf{X}, \mathbf{Y}) \rightarrow \text{Hom}(X, \text{Frg}(\mathbf{Y}))$  defined by  $i_{X, \mathbf{Y}}(h) := h$ , for every  $h \in \text{Hom}(\varepsilon\mathbf{X}, \mathbf{Y})$ . To show that  $i_{X, \mathbf{Y}}$  is well-defined, let  $x, x' \in X$  with  $x =_X x'$ , hence  $x =_{(X, \mathbb{F}(X))} x'$ . By our hypothesis on  $h$ , we get  $h(x) =_Y h(x')$ . The fact that  $i_{X, \mathbf{Y}}$  is a function, is trivial to show. Let  $j_{X, \mathbf{Y}}: \text{Hom}(X, \text{Frg}(\mathbf{Y})) \rightarrow \text{Hom}(\varepsilon\mathbf{X}, \mathbf{Y})$  defined by  $j_{X, \mathbf{Y}}(h) := \varepsilon h$ , for every  $h \in \text{Hom}(X, \text{Frg}(\mathbf{Y}))$ , where  $\varepsilon h$  is determined by the universal property of  $\varepsilon\mathbf{X}$  as follows:

$$\begin{array}{ccc}
X & \xrightarrow{i_X} & |\varepsilon\mathbf{X}| \\
h \downarrow & \searrow & \downarrow \varepsilon h := \varepsilon(\text{id}_Y \circ h) \\
Y & \xrightarrow{\text{id}_Y} & |\mathbf{Y}|.
\end{array}$$

Clearly,  $j_{X, \mathbf{Y}}$  is a well-defined function, and  $(i_{X, \mathbf{Y}}, j_{X, \mathbf{Y}})$  witness the equality of the two Hom-sets. If  $\phi: X' \rightarrow X$ , the commutativity of the rectangle

$$\begin{array}{ccc}
\text{Hom}(\varepsilon\mathbf{X}, \mathbf{Y}) & \xrightarrow{i_{X, \mathbf{Y}}} & \text{Hom}(X, \text{Frg}(\mathbf{Y})) \\
\text{Free}(\phi)^* \downarrow & & \downarrow \phi^* \\
\text{Hom}(\varepsilon\mathbf{X}', \mathbf{Y}) & \xrightarrow{i_{X', \mathbf{Y}}} & \text{Hom}(X', \text{Frg}(\mathbf{Y})),
\end{array}$$

where  $\phi^*(h) := h \circ \phi =: \text{Free}(\phi)^*(h)$ , follows easily. Similarly, if  $\theta: \mathbf{Y} \rightarrow \mathbf{Y}'$ , the rectangle

$$\begin{array}{ccc}
\mathrm{Hom}(\varepsilon\mathbf{X}, \mathbf{Y}) & \xrightarrow{i_{X,\mathbf{Y}}} & \mathrm{Hom}(X, \mathrm{Frg}(\mathbf{Y})) \\
\theta_* \downarrow & & \downarrow \mathrm{Frg}(\theta)_* \\
\mathrm{Hom}(\varepsilon\mathbf{X}, \mathbf{Y}') & \xrightarrow{i_{X,\mathbf{Y}'}} & \mathrm{Hom}(X, \mathrm{Frg}(\mathbf{Y}'))
\end{array}$$

is commutative, where  $\theta_*(h) := \theta \circ h =: \mathrm{Frg}(\theta)_*(h)$ .  $\square$

Theorem 5.2(i) is the BST-analogue to the (intentional) type-theoretic fact that the setoid  $(X, =_X)$ , where  $=_X: X \rightarrow X \rightarrow \mathcal{U}$  is the equality type-family on the type  $X: \mathcal{U}$ , is the free setoid on  $X$  (see [8], p. 74). Its simple proof rests on the elimination axiom for  $=_X$ , which is a proof-relevant formulation of the least-reflexive property of  $=_X$  within intensional MLTT, a property that cannot be translated directly in BST (see also [37], p. 2 and p. 7). As we have noticed after Remark 3.3, the induced equality  $=_{(X, \mathbb{F}(X))}$  is in BST the “smallest” equality on  $X$  induced by real-valued functions on  $(X, =_X)$ .

## 6 A set-theoretic Stone-Čech theorem

A completely regular topological space  $(X, \mathcal{T})$  is one in which any pair  $(x, B)$ , where  $B$  is closed and  $x \notin B$ , is separated by some  $f \in C(X, [0, 1])$ . The ring of real-valued, continuous functions  $C(X)$  of a completely regular and  $T_1$ -space  $X$ , also known as a *Tychonoff space*, separates the points of  $X$ , that is

$$\forall_{x, x' \in X} (\forall_{f \in C(X)} (f(x) = f(x')) \Rightarrow x = x').$$

The “sufficiency” of the completely regular topological spaces in the theory of  $C(X)$  is provided by the Stone-Čech theorem, according to which, for every topological space  $X$  there exists a completely regular space  $\rho X$  and a continuous mapping  $\tau_X: X \rightarrow \rho X$  such that the induced function  $f \mapsto \tau_X^*(f)$ , where  $\tau_X^*(f) = f \circ \tau_X$ , is a ring isomorphism between  $C(\rho X)$  and  $C(X)$  (see [12], p. 41).

$$\begin{array}{ccc}
X & \xrightarrow{\tau_X} & \rho X \\
\downarrow \tau_X^*(f) & & \downarrow f \in C(\rho X) \\
C(X) \ni \tau_X^*(f) & & \mathbb{R}
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\tau_X} & \rho X \\
\downarrow h & & \downarrow \rho h \in C(\rho X, Y) \\
C(X, Y) \ni h & & Y
\end{array}$$

Consequently, a functor  $\rho: \mathbf{Top} \rightarrow \mathbf{crTop}$  from the category of topological spaces  $\mathbf{Top}$  to its subcategory of completely regular topological spaces  $\mathbf{crTop}$  is defined, which is a reflector, that is for every continuous function  $h$  from  $X$  to a completely regular space  $Y$  there is a unique continuous function  $\rho h: \rho X \rightarrow Y$  such that the above right triangle commutes (see [14], p. 6). Next we present an abstract version of the Stone-Čech theorem for completely separated sets, which expresses the corresponding “sufficiency” of completely separated sets. Topological spaces are replaced by function spaces and completely regular topological spaces by completely separated sets<sup>13</sup>. Although no topological notions are involved in the formulation and the proof of Theorem 6.2, the structure of our constructive proof is very similar to the structure of the classical proof of the topological Stone-Čech theorem (see [52], p. 5–6). This is the reason why we chose to name it like that. The similarity is even more evident, if one compares the proof of Theorem 6.2 with the constructive proof of the Stone-Čech theorem for Bishop spaces (see [29], p. 307). The category of function spaces was introduced by Ishihara in [15] without restricting though, to extensional subsets of  $\mathbb{F}(X)$ .

**Definition 6.1** A function space is a structure  $(X, =_X; F)$ , where  $(X, =_X)$  is a set and  $F$  is an extensional subset of  $\mathbb{F}(X)$ . If  $=_X$  is clear from the context, we also write  $(X, F)$ . In the category  $\mathbf{FunSpace}$  of function spaces the arrows are the affine maps, defined in Definition 4.1.

**Theorem 6.2 (Stone-Čech theorem for function spaces and completely separated sets)** *Let  $(X, =_X; F)$  be a function space.*

(i) *There is a completely separated set  $\rho_F \mathbf{X} := (X, =_{(X, F)}, \neq_{(X, F)}; \rho F)$  and a function  $\tau_X: X \rightarrow |\rho_F \mathbf{X}|$ ,*

such that  $(\tau_X^*, \rho_X): F =_{\nu_0} \rho F$ , where  $\tau_X^*: \rho F \rightarrow F$  is defined by  $\tau_X^*(g) := g \circ \tau_X$ , for every  $g \in \rho F$ , and  $\rho_X: F \rightarrow \rho F$  is defined by  $\rho_X(f) := f$ , for every  $f \in F$ . Moreover, every function  $f$  in  $\rho F$  is strongly extensional.

$$\begin{array}{ccc}
 X & \xrightarrow{\tau_X} & |\rho_F \mathbf{X}| \\
 F \ni \tau_X^*(f) \searrow & & \downarrow f \in \rho F \\
 & & \mathbb{R}
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \xrightarrow{\tau_X} & |\rho_F \mathbf{X}| \\
 \mathbb{A}f((X;F), (Y;G)) \ni h \searrow & & \downarrow \rho h \in \mathbb{A}f((X;\rho F), (Y;G)) \\
 & & |\mathbf{Y}|
 \end{array}$$

(ii) If  $\mathbf{Y} := (Y, =_Y, \neq_Y; G)$  is in **SetComplSep** and  $h: (X, =_X; F) \rightarrow (Y, =_Y; G)$  is an affine map, there is a unique affine map  $\rho h: |\rho_F \mathbf{X}| \rightarrow |\mathbf{Y}|$ , such that the above right triangle commutes.

**Proof** (i) Let the assignment routine  $\tau_X: X \rightsquigarrow X$  given by the identity rule  $\tau_X(x) := x$ , for every  $x \in X$ . If  $X$  is equipped with the equality  $=_{(X,F)}$ , let  $\rho F$  be the following extensional subset of  $\mathbb{F}(X)$

$$\rho F := \{f: (X, =_F) \rightarrow (\mathbb{R}, =_{\mathbb{R}}) \mid f \circ \tau_X \in F\}.$$

Clearly, the extensionality of  $F$  implies the extensionality of  $\rho F$ , and  $\rho_F \mathbf{X}$  is in **SetComplSep** as

$$\begin{aligned}
 x =_{(X,\rho F)} x' &\Leftrightarrow \forall f \in \rho F (f(x) =_{\mathbb{R}} f(x')) \\
 &\Leftrightarrow \forall f \in \rho F ((f \circ \tau_X)(x) =_{\mathbb{R}} (f \circ \tau_X)(x')) \\
 &\Leftrightarrow \forall f \in F (f(x) =_{\mathbb{R}} f(x')) \\
 &\Leftrightarrow x =_{(X,F)} x'.
 \end{aligned}$$

In the non-definitional equivalence above we use the fact that if  $f \in \rho F$ , then  $f \circ \tau_X = f \in F$ , and if  $f \in F$ , then  $f \circ \tau_X = f \in F$ , and hence  $f \in \rho F$ . Similarly we show that  $x \neq_{(X,\rho F)} x' \Leftrightarrow x \neq_{(X,F)} x'$ . The facts that  $\tau_X$  is a function and the pair of functions  $(\tau_X^*, \rho_X)$  witness the equality  $F =_{\nu_0} \rho F$  follow immediately. The strong extensionality of each  $f \in \rho F$  follows from the following simple argument; if  $f \in \rho F$  and  $x, x' \in X$ , such that  $f(x) \neq_{\mathbb{R}} f(x')$ , then  $f: x \neq_{X,\rho F} x'$ , which implies that  $f \circ \tau_X: x \neq_{(X,F)} x'$ . and  $\neq_{(X,F)}$ .

(ii) Let  $\rho h := h$ . To show that  $\rho h$  is a function, let  $x =_{(X,F)} x' \Leftrightarrow x =_{(X,\rho F)} x'$ . If  $g \in G$ , then  $g(\rho h(x)) := g(h(x)) =_{\mathbb{R}} g(h(x')) =: g(\rho h(x'))$ , as  $g \circ h \in F$ , hence  $g \circ h \in \rho F$ . Consequently,  $h(x) =_Y h(x')$ . The fact that  $\rho h \in \mathbb{A}f((X;\rho F), (Y;G))$  follows immediately. The uniqueness of  $\rho h$  follows trivially from the definition of  $\tau_X$  and the required commutativity of the above right triangle.  $\square$

From a given set  $(X, =_X)$  and an extensional subset  $F$  of  $\mathbb{F}(X)$  we constructed a completely separated set  $\rho_F \mathbf{X}$  with the same carrier set  $X$  and a larger equality  $=_{(X,\rho F)} \Leftrightarrow =_{(X,F)}$ , such that all functions that separate the points of  $(X, =_{(X,\rho F)})$  that is, the elements of  $\rho F$ , are strongly extensional, as we showed in the end of case (i) of Theorem 6.2. Of course, the inequality  $\neq_{(X,F)}$  is tight with respect to the equality  $=_{(X,F)}$ . In the proof of the Stone-Ćech theorem for topological spaces and completely regular spaces  $\rho X$  is the set of equivalence classes of  $X$  with respect to the equivalence relation  $x \sim x' :\Leftrightarrow \forall f \in C(X) (f(x) = f(x'))$ . Following [19], p. 38, the quotient of a set over an equivalence relation is treated here as the same totality with the equivalence relation as its new equality. Clearly, if  $\mathbf{X}$  is in **SetComplSep**, the above construction on  $(X, =_F)$  induces  $\mathbf{X}$  again.

Let the functor  $\rho: \mathbf{FunSpace} \rightarrow \mathbf{SetAffine}$ , defined by  $\rho(X, =_X; F) := \rho_F \mathbf{X}$  and

$$\rho(h: (X, =_X; F) \rightarrow (Y, =_Y; G)) := \rho h: \rho_F \mathbf{X} \rightarrow \rho_F \mathbf{Y},$$

where, according to Theorem 6.2(ii),  $\rho h$  is the unique function that makes the following rectangle commutative

$$\begin{array}{ccc}
 X & \xrightarrow{\tau_X} & |\rho_F \mathbf{X}| \\
 h \downarrow & \searrow & \downarrow \rho h \\
 Y & \xrightarrow{\tau_Y} & |\rho_F \mathbf{Y}|
 \end{array}$$



The fact that  $\rho$  is a functor follows immediately by Theorem 6.2(ii). If  $\text{Emb} : \mathbf{SetAffine} \rightarrow \mathbf{FunSpace}$  is the corresponding embedding functor, defined by  $\text{Emb}(\mathbf{X}) := (X, =_X; F)$  and  $\text{Emb}(h : (\mathbf{X}; F) \rightarrow (\mathbf{Y}; G)) := h$ , then  $\rho$  is left adjoint to  $\text{Emb}$ . By Theorem 6.2(ii) again it follows that  $\mathbf{SetAffine}$  is reflective in  $\mathbf{FunSpace}$ , or equivalently, that  $\rho$  is left adjoint to  $\text{Emb}$ . Actually, one can show that  $\text{Emb}$  is left adjoint to  $\rho$ . Because of the relations  $\rho \dashv \text{Emb} \dashv \rho$ , we have that  $\rho$  preserves all limits and colimits. For example, as  $(X \times Y, =_{X \times Y}; F \otimes G)$  is the product of  $(X, =_X; F)$  and  $(Y, =_Y; G)$  in  $\mathbf{FunSpace}$ , we have that  $\rho_{F \otimes G}(\mathbf{X} \times \mathbf{Y}) = \rho_F(\mathbf{X}) \times \rho_G(\mathbf{Y})$ . The reason behind the relations  $\rho \dashv \text{Emb} \dashv \rho$  is<sup>14</sup> that the pair of functors  $(\rho, \text{Emb})$  is an equivalence of categories.

**Proposition 6.3** The functors  $(\rho, \text{Emb})$  form an equivalence of the categories  $\mathbf{FunSpace}$  and  $\mathbf{SetAffine}$ .

**Proof** We first define a natural transformation  $\alpha : 1_{\mathbf{FunSpace}} \Rightarrow \text{Emb} \circ \rho$ . If  $(X, F) \in \mathbf{FunSpace}$ , let  $\alpha_{(X, F)} : (X, F) \rightarrow (X, =_{(X, F)}, \neq_{(X, F)}; \rho F)$  be the identity on  $X$ . It is straightforward to show that this is an affine arrow. By the definition of  $\rho h$  above, we have that  $\alpha$  is indeed a natural transformation. In order to show that  $\alpha$  is an iso in the functor category  $\mathbf{FunSpace}^{\mathbf{FunSpace}}$ , let  $\alpha' : \text{Emb} \circ \rho \Rightarrow 1_{\mathbf{FunSpace}}$ , where  $\alpha'_{(X, F)} : (X, =_{(X, F)}, \neq_{(X, F)}; \rho F) \Rightarrow (X, F)$  is defined as the identity on  $X$  again. By the definition of  $\rho F$  this is an affine arrow, and it is immediate to show that  $\alpha'$  is a natural transformation. The fact that the compositions  $\alpha \circ \alpha'$  and  $\alpha' \circ \alpha$  are equal to the corresponding units follows easily. The natural transformation  $\beta : 1_{\mathbf{SetAffine}} \Rightarrow \rho \circ \text{Emb}$  and its inverse  $\beta' : \rho \circ \text{Emb} \Rightarrow 1_{\mathbf{SetAffine}}$  are defined similarly through the identity functions, and the proof that  $\beta$  is an iso in the functor category  $\mathbf{SetAffine}^{\mathbf{SetAffine}}$  is trivial.  $\square$

## 7 A set-theoretic Tychonoff embedding theorem

According to the classical Tychonoff embedding theorem, a  $T_1$  topological space is completely regular if and only if it is topologically embedded into a product of  $[0, 1]$  equipped with its standard topology. Here we give a purely set-theoretic formulation of this result within BST. As the separating functions considered here take values in  $\mathbb{R}$ , the corresponding product will be a (dependent) product of  $\mathbb{R}$ . As in the previous section, we replace complete regularity by complete separation. Although no topological notions are involved in the formulation and the proof of Theorem 7.5, the structure of our constructive proof is very similar to the structure of the classical proof of the topological Tychonoff embedding theorem (see [52], p. 4–5). This is the reason why we chose to name it like that. The similarity is even more evident, if one compares the proof of Theorem 7.5 with the constructive proof of the Tychonoff embedding theorem for Bishop spaces (see [29], p. 308).

**Lemma 7.1** Let  $\mathbf{X}, \mathbf{I}$  be in  $\mathbf{SetIneq}$  and  $\Lambda := (\lambda_0, \lambda_1)$  a family of sets with an inequality over  $\mathbf{I}$ . If  $M := (\mu_0, \mu_1)$ , where  $\mu_0 : I \rightsquigarrow \mathbb{V}_0^\neq$  is defined by  $\bar{\mu}_0(i) := (\mathbb{F}(X, \lambda_0(i)), =_{\mathbb{F}(X, \lambda_0(i))}, \neq_{\mathbb{F}(X, \lambda_0(i))})$  and

$$\mu_1 : \bigwedge_{(i, j) \in D(I)} \mathbb{F}^\neq(\mathbb{F}(X, \lambda_0(i)), \mathbb{F}(X, \lambda_0(j))), \quad \mu_1(i, j) := \mu_{ij}, \quad (i, j) \in D(I),$$

$$\mu_{ij} : \mathbb{F}(X, \lambda_0(i)) \rightarrow \mathbb{F}(X, \lambda_0(j)), \quad \mu_{ij}(\phi) := \lambda_{ij} \circ \phi, \quad \phi \in \mathbb{F}(X, \lambda_0(i))$$

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \lambda_0(i) \\ & \searrow \mu_{ij}(\phi) & \downarrow \lambda_{ij} \\ & & \lambda_0(j) \end{array}$$

then  $M$  is a family of sets with an inequality over  $\mathbf{I}$ .

**Proof** If  $i \in I$ , then  $\mu_{ii}(\phi) := \lambda_{ii} \circ \phi = \text{id}_{\lambda_0(i)} \circ \phi = \phi$ , for every  $\phi \in \mathbb{F}(X, \lambda_0(i))$ . If  $i =_I j =_I k$ , then

$$\begin{array}{ccc}
\mathbb{F}(X, \lambda_0(i)) & & \\
\downarrow \mu_{ij} & \searrow \mu_{ik} & \\
\mathbb{F}(X, \lambda_0(j)) & \xrightarrow{\mu_{jk}} & \mathbb{F}(X, \lambda_0(k)).
\end{array}$$

$(\mu_{jk} \circ \mu_{ij})(\phi) := \mu_{jk}(\lambda_{ij} \circ \phi) := \lambda_{jk} \circ (\lambda_{ij} \circ \phi) = \lambda_{ij} \circ \phi =: \mu_{ik}(\phi)$ , for every  $\phi \in \mathbb{F}(X, \lambda_0(i))$ . To show that  $\mu_{ij}$  is strongly extensional, let  $\phi, \phi' \in \mathbb{F}(X, \lambda_0(i))$ . As  $\lambda_{ij}$  is strongly extensional, we have that

$$\begin{aligned}
\mu_{ij}(\phi) \neq_{\mathbb{F}(X, \lambda_0(i))} \mu_{ij}(\phi') &:\Leftrightarrow \lambda_{ij} \circ \phi \neq_{\mathbb{F}(X, \lambda_0(i))} \lambda_{ij} \circ \phi' \\
&:\Leftrightarrow \exists x \in X (\lambda_{ij}(\phi(x)) \neq_{\lambda_0(j)} \lambda_{ij}(\phi'(x))) \\
&\Rightarrow \exists x \in X (\phi(x) \neq_{\lambda_0(i)} \phi'(x)) \\
&:\Leftrightarrow \phi \neq_{\mathbb{F}(X, \lambda_0(i))} \phi'. \quad \square
\end{aligned}$$

**Theorem 7.2 (Embedding lemma for sets with an inequality)** *Let  $\mathbf{X}, \mathbf{I}$  be in **SetIneq** and  $\Lambda := (\lambda_0, \lambda_1)$  a family of sets with an inequality over  $\mathbf{I}$ . Let also  $H \in \prod_{i \in I} \mu_0(i)$ , where  $M := (\mu_0, \mu_1)$  is the family of sets with inequality over  $\mathbf{I}$  defined in Lemma 7.1. If*

$$\prod_I \Lambda := \left( \prod_{i \in I} \lambda_0(i), =_{\prod_{i \in I} \lambda_0(i)}, \neq_{\prod_{i \in I} \lambda_0(i)} \right),$$

let the assignment routine

$$\begin{aligned}
e^H : X &\rightsquigarrow \prod_{i \in I} \lambda_0(i), \quad x \mapsto e^H(x), \\
[e^H(x)]_i &:= H_i(x), \quad i \in I.
\end{aligned}$$

(i)  $e^H$  is a well-defined function.

(ii) Let the extensional subset  $\Phi := \{e^H\}$  of  $\mathbb{F}(X, \prod_{i \in I} \lambda_0(i))$ . If the induced inequality  $\neq_{(X, \Phi)}$  on  $X$  is tight, then  $e^H$  is an embedding, and if  $H_i$  is strongly extensional, for every  $i \in I$ , then  $e^H$  is strongly extensional.

**Proof** (i) First we show that  $e^H$  is well-defined, that is  $e^H \in \prod_{i \in I} \lambda_0(i)$ . If  $i =_I j$ , then

$$[e^H(x)]_j := H_j(x) =_{\lambda_0(j)} [(\mu_{ij}(H_i))](x) := [\lambda_{ij} \circ H_i](x) := \lambda_{ij}(H_i(x)) =: \lambda_{ij}([e^H(x)]_i).$$

If  $x =_X x'$ , then  $e^H(x) =_{\prod_{i \in I} \lambda_0(i)} e^H(x')$ , as for every  $i \in I$  we have that  $H_i(x) =_{\lambda_0(i)} H_i(x')$ .

(ii) The tightness of  $\neq_{(X, \Phi)}$  means that  $x =_{(X, \Phi)} x' \Rightarrow x =_X x'$ , for every  $x, x' \in X$ , where

$$\begin{aligned}
x =_{(X, \Phi)} x' &:\Leftrightarrow e^H(x) =_{\prod_{i \in I} \lambda_0(i)} e^H(x') \\
&:\Leftrightarrow \forall i \in I ([e^H(x)]_i =_{\lambda_0(i)} [e^H(x')]_i) \\
&:\Leftrightarrow \forall i \in I (H_i(x) =_{\lambda_0(i)} H_i(x')).
\end{aligned}$$

Hence,  $e^H(x) =_{\prod_{i \in I} \lambda_0(i)} e^H(x') \Rightarrow x =_X x'$  follows immediately. If each  $H_i$  is strongly extensional, then

$$\begin{aligned}
e^H(x) \neq_{\prod_{i \in I} \lambda_0(i)} e^H(x') &:\Leftrightarrow \exists i \in I ([e^H(x)]_i \neq_{\lambda_0(i)} [e^H(x')]_i) \\
&:\Leftrightarrow \exists i \in I (H_i(x) \neq_{\lambda_0(i)} H_i(x')) \\
&\Rightarrow x \neq_X x'. \quad \square
\end{aligned}$$

**Definition 7.3** If  $(X, =_X; F)$  is a function space, its dual completely separated set is the structure  $\mathbf{X}^* := (F, =_F, \neq_F; \widehat{X})$ , where  $=_F$  and  $\neq_F$  are induced by  $\mathbb{F}(X)$ , and

$$\widehat{X} := \{\widehat{x} \mid x \in X\}, \quad \widehat{x} : F \rightarrow \mathbb{R}, \quad \widehat{x}(f) := f(x); \quad f \in F, x \in X.$$

The contravariant functor  $\text{Dual} : \mathbf{FunSpace}^{\text{op}} \rightarrow \mathbf{SetAffine}$  is defined by  $\text{Dual}(X, =_X; F) := \mathbf{X}^*$  and  $\text{Dual}(h : (X, =_X; F) \rightarrow (Y, =_Y; G)) := h^* : \mathbf{Y}^* \rightarrow \mathbf{X}^*$ , where  $h^*(g) := g \circ h \in F$ , for every  $g \in G$ .

Clearly, one can “identify”  $(X, =_X; F)$  with  $(\widehat{X}, =_{\widehat{X}}; \widehat{F})$ , where  $\widehat{F} := \{\widehat{f} \mid f \in F\}$  and  $\widehat{f}(\widehat{x}) := \widehat{x}(\widehat{f}) := f(x)$ , for every  $f \in F$  and  $x \in X$ , as the maps  $x \mapsto \widehat{x}$  and  $\widehat{x} \mapsto x$  form an iso in the category **FunSpace**.

**Remark 7.4** Let  $(X, =_X; F)$  be a function space.

(i) Let  $\lambda_0(f) := \mathbf{R}$ , for every  $f \in F$ , and  $\lambda_{fg} := \text{id}_{\mathbb{R}}$ , for every  $f, g \in F$  such that  $f =_F g$ . Moreover, let  $\phi_0(f) := \{\text{id}_{\mathbb{R}}\}$ , for every  $f \in F$  and, if  $f =_F g$ , let  $\phi_{fg}: \{\text{id}_{\mathbb{R}}\} \rightarrow \{\text{id}_{\mathbb{R}}\}$  be given by the identity rule. Then the structure  $\mathbf{S} := (\lambda_0, \lambda_1; \phi_0, \phi_1)$  is a family of completely separated sets over the dual completely separated set  $\mathbf{X}^* := (\mathbf{F}; \widehat{X})$  of the function space  $(X, =_X; F)$ .

(ii) If  $\mathbf{R}^F := (\mathbb{R}^F, =_{\mathbb{R}^F}, \neq_{\mathbb{R}^F}; \otimes_{f \in F} \{\text{id}_{\mathbb{R}}\})$ , where

$$\otimes_{f \in F} \{\text{id}_{\mathbb{R}}\} := \{\text{id}_{\mathbb{R}} \circ \text{pr}_f^\Lambda = \text{pr}_f^\Lambda \mid f \in F\},$$

then  $\mathbf{R}^F \in \text{SetComplSep}$ .

**Proof** The proof of case (i) is straightforward and case (ii) follows from case (i) and Proposition 4.5.  $\square$

Clearly, if  $(X, =_X)$  is in **Set**, and  $\neq_{(X,F)}$  is tight, then  $(X, =_X, \neq_{(X,F)}; F)$  is in **SetComplSep**.

**Theorem 7.5 (Tychonoff embedding theorem for function spaces and completely separated sets)** Let  $(X, =_X; F)$  be a function space.

(i) If the induced inequality  $\neq_{(X,F)}$  on  $X$  is tight, then there is an affine embedding (injection) of the completely separated set  $(X, =_X, \neq_{(X,F)}; F)$  into the completely separated set  $\mathbf{R}^F$ .

(ii) If  $e: (X, =_X; F) \rightarrow (\mathbb{R}^F, =_{\mathbb{R}^F}; \otimes_{f \in F} \{\text{id}_{\mathbb{R}}\})$  is an affine embedding, then the induced inequality  $\neq_{(X,F)}$  on  $X$  is tight, and hence  $(X, =_X, \neq_{(X,F)}; F)$  is in **SetComplSep**.

**Proof** (i) Let  $M := (\mu_0, \mu_1)$  be the family of sets over  $\mathbf{F}$  from Lemma 7.1 that corresponds to the constant  $\mathbf{F}$ -family of sets with an inequality  $(\lambda_0, \lambda_1)$  in Remark 7.4(i). Let the dependent function

$$H(F): \prod_{f \in F} \mu_0(f) := \prod_{f \in F} \mathbb{F}(X, \mathbb{R}),$$

$$H(F)_f := f, \quad f \in F.$$

If  $f =_F g$ , then  $H_g^F := g =_F f := H(F)_f := \text{id}_{\mathbb{R}} \circ H(F)_f := \lambda_{fg} \circ H(F)_f := \mu_{fg}(H(F)_f)$ , hence  $H(F)$  is well defined. By Theorem 7.2(i) we get the function

$$e^{H(F)}: X \rightarrow \prod_{f \in F} \mathbb{R} := \mathbb{R}^F, \quad x \mapsto e^{H(F)}(x),$$

$$[e^{H(F)}(x)]_f := H(F)_f(x) := f(x), \quad f \in F, x \in X.$$

Next we show that the inequality  $\neq_{(X,\Phi)}$  on  $X$ , where  $\Phi := \{e^{H(F)}\}$ , is tight. If  $x, x' \in X$ , then

$$\begin{aligned} x =_{(X,\Phi)} x' &: \Leftrightarrow e^{H(F)}(x) =_{\mathbb{R}^F} e^{H(F)}(x') \\ &: \Leftrightarrow \forall f \in F ([e^{H(F)}(x)]_f =_{\mathbb{R}} [e^{H(F)}(x')]_f) \\ &: \Leftrightarrow \forall f \in F (f(x) =_{\mathbb{R}} f(x')) \\ &: \Leftrightarrow x =_F x' \\ &\Rightarrow x =_X x', \end{aligned}$$

where the last implication follows from the tightness of  $\neq_{(X,F)}$ . To show that  $e^{H(F)}$  is affine, it suffices to show that  $\text{pr}_f^\Lambda \circ e^{H(F)} \in F$ , for every  $f \in F$ . By the definition of  $e^{H(F)}$  though, we have that  $\text{pr}_f^\Lambda \circ e^{H(F)} =_F f \in F$ .

(ii) If  $x, x' \in X$  such that  $x =_{(X,F)} x'$ . we show that  $x =_X x'$ . As  $e$  is an embedding, it suffices to show that  $e(x) =_{\mathbb{R}^F} e(x') \Leftrightarrow e(x) =_{\otimes_{f \in F} \{\text{id}_{\mathbb{R}}\}} e(x') \Leftrightarrow \forall f \in F (\text{pr}_f^\Lambda(e(x)) =_{\mathbb{R}} \text{pr}_f^\Lambda(e(x')))$ . As  $e$  is affine, the composition  $\text{pr}_f^\Lambda \circ e$  is in  $F$ , for every  $f \in F$ , and we use the hypothesis  $x =_{(X,F)} x'$ .  $\square$

Notice that in the above proof we avoided negation completely. Theorem 7.5 offers a criterion for the generation of a completely separated set  $(X, =_X, \neq_{(X,F)}; F)$  from a given function space  $(X, =_X; F)$ .

## 8 Concluding remarks

In this paper we begun the study of completely separated sets within BST, realising both the attitude within CM of using positive definitions, instead of negative ones, and the attitude of employing functions, instead of sets, whenever that is possible. Completely separated sets are equipped with a positive notion of an inequality, as their given inequality is equivalent to the inequality induced by a set of real-valued functions. Showing that a set with an inequality  $(X, =_X, \neq_X)$  is completely separated, by finding an extensional subset  $F$  of  $\mathbb{F}(X)$  such that  $=_X$  is equivalent to  $=_{(X,F)}$  and  $\neq_X$  is equivalent to  $\neq_{(X,F)}$ , is a method of proving that  $\neq_X$  is an apartness, and hence an extentional relation. As we saw in the case of Theorem 4.8, to prove directly that  $\neq_X$  is an apartness relation can be non-trivial.

The various distinctions that are made within CM facilitate the definition of interesting, new categories of sets, such as the category **SetIneq** of sets with an inequality, and the categories **SetComplSep** and **SetAffine** of completely separated and affine sets, respectively. If  $(\mathbf{C} \subseteq \mathbf{D})$   $\mathbf{C} \subseteq \mathbf{D}$  denotes that  $\mathbf{C}$  is a (full) subcategory of  $\mathbf{D}$ , we have that

$$\mathbf{SetAffine} \subseteq \mathbf{SetComplSep} \subseteq \mathbf{SetIneq}.$$

We defined the corresponding notions of families of sets in these categories and their Pi- and Sigma-sets, extending<sup>15</sup> in this way our previous work [37]. We also introduced the notions of a global family of sets with an inequality over an index-set with an inequality and of a global family of completely separated sets over an index-completely separated set, in order to describe the Sigma-set of the latter. Global families helped us to formulate a notion of strong extensionality for dependent functions that generalises the strong extensionality of non-dependent functions. The free completely separated set on a given set gave us a way to translate into BST the freeness of the equality setoid on a given type within intentional MLTT. We also provided purely set-theoretic versions of the classical Stone-Ćech theorem and the Tychonoff embedding theorem for completely regular spaces, by replacing topological spaces with function spaces and completely regular spaces with completely separated sets

$$\frac{\mathbf{SetComplSep}}{\mathbf{FunSpace}} \approx \frac{\mathbf{crTop}}{\mathbf{Top}}.$$

Although we cannot show constructively that all real-valued functions on a function space  $(X, =_X; F)$  are strongly extensional, by Theorem 6.2 we can always find a new equality on  $X$  such that all elements of  $F$  become strongly extensional! The positive formulation of our basic concepts also helped us to avoid negation in the proofs of Theorems 6.2 and 7.5. If the set  $F$  in a function space is closed under the minimum operation, namely  $P_F(f) \ \& \ P_F(g) \Rightarrow P_F(f \wedge g)$ , where  $f \wedge g := \min\{f, g\}$ , and if for every  $x \in X$  there is some  $f \in F$  such that  $f(x) > 0$ , then the family  $(U(f))_{f \in F}$ , where

$$U(f) = \{x \in X \mid f(x) > 0\},$$

is a base for a topology of open sets on  $X$ . The tightness of the inequality  $\neq_{(X,F)}$  is equivalent then to the topological fact that each singleton  $\{x\}$  in  $X$  is a closed set (see Proposition 3.5 in [39]). Theorems 6.2 and 7.5 can also be seen as purely set-theoretic versions of the constructive Stone-Ćech theorem and the Tychonoff embedding theorem for Bishop spaces in [28, 29].

The categories **Set**, **SetIneq**, **SetComplSep** and **SetAffine** need to be studied further, and to be connected with the elaboration of abstract category theory within BST in [44]. The theory of (global) families of completely separated sets is expected to be developed in line of the general theory of families of sets in [37]. The arrows between these families, the corresponding distributivity of the Pi-set over the Sigma-set, and the families of subsets of a completely separated set, are some of the topics to be studied along the lines of [37]. Inequalities are easier to study in BST, rather than in (intensional) MLTT. Also, the notion of a global family of completely separated sets over an index-completely separated set seems to have no (intentional) type-theoretic analogue. The question whether the above categories of sets with an inequality, and the notions of their indexed families studied here, can be grasped by intensional MLTT, seems to us both interesting and hard.

## Notes

1. An approach to CM without countable choice was spearheaded by Richman [49].
2. For the relation between BMT and BCMT see [37, 47, 48].
3. The type-theoretic interpretation of Bishop's set theory into the theory of setoids (see especially the work of Palmgren [22, 24, 23, 25, 26, 27]) has become nowadays the standard way to understand Bishop sets. For an analysis of the relation between intensional MLTT and Bishop's theory of sets see [37], Chapter 1. Other formal systems for BISH are Myhill's Constructive Set Theory (CST), introduced in [21], and Aczel's system CZF (see [1]). Palmgren's work [24] and Coquand's work [9] are categorical approaches to Bishop sets.
4. All notions of families of sets defined here are not studied in [37].
5. If  $(X, =_X)$  and  $(Y, =_Y)$  are sets, an assignment routine  $f: X \rightsquigarrow Y$  is a function, if it preserves the corresponding equalities. The notion of a (non-dependent) assignment routine, that is of a routine that corresponds an element  $f(x)$  of  $Y$  to each element  $x$  of  $X$ , is primitive in BST.
6. One needs the extensionality of  $\neq_Y$  to show that the strong extensionality of  $f \in \mathbb{F}(X, Y)$  is an extensional property on  $\mathbb{F}(X, Y)$ , and define then  $\mathbb{F}^\neq(\mathbf{X}, \mathbf{Y})$  using the separation scheme. As  $(\mathbf{Ineq}_2)$  is not considered here part of the definition of an inequality, we introduce  $\mathbb{F}^\neq(\mathbf{X}, \mathbf{Y})$  independently from  $\mathbb{F}(X, Y)$ .
7. The global projection operations  $\mathbf{pr}_1$  and  $\mathbf{pr}_2$  are primitive operations in BST.
8. Here we use the logical implication  $((\phi \vee \psi) \Rightarrow \perp) \Rightarrow [(\phi \Rightarrow \perp) \& (\psi \Rightarrow \perp)]$ .
9. That is there is an extensional property  $P_F$  on  $\mathbb{F}(X)$  such that

$$F := \{f \in \mathbb{F}(X) \mid P_F(f)\},$$

where the extensionality of  $P_F$  is the property  $f =_{\mathbb{F}(X)} g \& P_F(f) \Rightarrow P_F(g)$ , for every  $f, g \in \mathbb{F}(X)$ . The use of extensional subsets, instead of arbitrary subsets given by their embeddings, is crucial to the definition of affine arrows in Definition 4.1, as the expression  $f \in F$ , which is in principle a judgment, in Martin-Löf's sense, is replaced by the formula  $P_F(f)$  (see also [42]).

10. In [47] an  $f$ -inequality  $\neq_X$  was defined impredicatively by the existence of an extensional subset  $F$  of  $\mathbb{F}(X)$ , such that  $\neq_X$  is equivalent to  $\neq_{(X, F)}$ .
11. Hence,  $=_X$  is also equal to the induced equality  $=_{(X, F)}$  on  $X$ . Moreover,  $\neq_X$  is tight, but as the tightness of  $\neq_X$  is negativistic, we prefer the positive formulation of the tightness of  $\neq_{(X, F)}$  in the definition of a completely separated set.
12. Hence,  $(\mathbf{X} \times \mathbf{Y}; F \otimes G)$  is the product in **SetAffine**.
13. The adverb "completely" in the term completely separated set comes from this analogy.
14. We would like to thank one of the anonymous referees for suggesting Proposition 6.3.
15. We also added two more open-ended universes in BST, the universe  $\mathbb{V}_0^\neq$  of predicative sets with an inequality and the universe of completely separated sets  $\mathbb{V}_0^f$ . Clearly, both of them are proper classes.

## References

- [1] Aczel, P., and M. Rathjen, *Constructive Set Theory*, book draft, 2010. 21

- [2] Awodey, S., *Category Theory*, Oxford University Press, 2010. [3](#)
- [3] Bishop, E., *Foundations of Constructive Analysis*, McGraw-Hill, 1967. [1](#), [2](#), [6](#)
- [4] Bishop, E., and H. Cheng, *Constructive Measure Theory*, Mem. Amer. Math. Soc. 116, 1972. [1](#), [2](#), [6](#)
- [5] Bishop, E., and D. S. Bridges, *Constructive Analysis*, Springer-Verlag, Heidelberg-Berlin-New York, 1985. [1](#), [2](#), [3](#), [4](#), [6](#)
- [6] Bridges, D. S., and F. Richman, *Varieties of Constructive Mathematics*, Cambridge University Press, 1987. [7](#)
- [7] Bridges, D. S., “Reflections on function spaces,” *Annals of Pure and Applied Logic*, vol. 163 (2012), pp. 101–110. [Zbl 1251.03093](#). [MR 2855322](#). [1](#)
- [8] Coquand, T., and P. Dybjer, and E. Palmgren, and A. Setzer, *Type-theoretic Foundations of Constructive Mathematics*, book draft, 2005. [15](#)
- [9] Coquand, T., “Universe of Bishop sets,” manuscript, 2017. [21](#)
- [10] Diener, H., “Constructive Reverse Mathematics,” arXiv:1804.05495, 2020. [4](#)
- [11] Feferman, S., “Constructive theories of functions and classes,” in Boffa et al. (Eds.) *Logic Colloquium 78*, North-Holland, 1979, pp. 159–224. [Zbl 0441.03022](#). [MR 0567670](#). [2](#)
- [12] Gillman, L., and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, 1960. [15](#)
- [13] Grubmüller, F. L., and I. Petrakis, “A Predicative Approach to the Constructive Integration Theory of Locally Compact Metric Spaces,” submitted (2023). [6](#)
- [14] Herrlich, H., *Topologische Reflexionen und Coreflexionen*, Lecture Notes in Mathematics, Springer, 1968. [15](#)
- [15] Ishhara, H., “Relating Bishop’s function spaces to neighborhood spaces,” *Annals of Pure and Applied Logic*, vol. 164 (2013), pp. 482–490. [Zbl 1275.03163](#). [MR 3015322](#). [1](#), [15](#)
- [16] Köpp, N., and I. Petrakis, “Strong negation in the theory of computable functionals TCF,” arXiv:2210.05491v4 (2023), pp. 1–26. [2](#)
- [17] Martin-Löf, P., “An intuitionistic theory of types: predicative part,” in Rose et al. (Eds.) *Logic Colloquium 73*, North-Holland, 1975, pp. 73–118. [2](#)
- [18] Martin-Löf, P., “An intuitionistic theory of types,” in Sambin et al. (Eds.) *Twenty-five years of constructive type theory*, Oxford University Press, 1998, pp. 127–172. [2](#)
- [19] Mines, R., and F. Richman, and W. Ruitenburg, *A Course in Constructive Algebra*, Springer, 1988. [3](#), [16](#)
- [20] Misselbeck-Wessel, D., and I. Petrakis, “Complemented subsets and Boolean-valued, partial functions,” *Computability*, (2024), pp. 1–33. DOI: 10.3233/COM-230462 [6](#)
- [21] Myhill, J., “Constructive Set Theory,” *J. Symbolic Logic*, vol. 40 (1975), pp. 347–382. [Zbl 0314.02045](#). [MR 0381941](#). [21](#)
- [22] Palmgren, E., *Bishop’s set theory*, Slides from TYPES Summer School 2005, Gothenburg, 2005. [21](#)
- [23] Palmgren, E., “Proof-relevance of families of setoids and identity in type theory,” *Arch. Math. Logic*, vol. 51 (2012), pp. 35–47. [Zbl 1241.03005](#). [MR 2864396](#). [21](#)
- [24] Palmgren, E., “Constructivist and structuralist foundations: Bishop’s and Lawvere’s theories of sets,” *Annals of Pure and Applied Logic*, vol. 163 (2012), pp. 1384–1399. [Zbl 1257.03095](#). [MR 2944492](#). [21](#)
- [25] Palmgren, E., *Bishop-style constructive mathematics in type theory - A tutorial*, Slides, 2013. [21](#)
- [26] Palmgren, E., *Lecture Notes on Type Theory*, book-draft, 2014. [21](#)
- [27] Palmgren, E., and O. Wilander, “Constructing categories and setoids of setoids in type theory,” *Logical Methods in Computer Science*, vol. 10 (2014), Issue 3, paper 25. [Zbl 1341.03012](#). [MR 3263877](#). [21](#)

- [28] Petrakis, I., *Constructive Topology of Bishop Spaces*, PhD Thesis, LMU Munich, 2015. [1](#), [7](#), [20](#)
- [29] Petrakis, I., “Completely Regular Bishop Spaces,” in Beckmann et al. (Eds.) *Evolving Computability*, LNCS 9136, (2015), pp. 302–312. [Zbl 3382370](#). [MR 3382370](#). [1](#), [15](#), [17](#), [20](#)
- [30] Petrakis, I., “The Urysohn Extension Theorem for Bishop Spaces,” in Artemov et al. (Eds.) *Symposium on Logical Foundations in Computer Science*, LNCS 9537, (2016), pp. 299–316. [Zbl 1477.03254](#). [MR 3502848](#). [1](#)
- [31] Petrakis, I., “A constructive function-theoretic approach to topological compactness,” LICS 2016, pp. 605–614. [Zbl 1394.03076](#). [MR 3776780](#). [1](#)
- [32] Petrakis, I., “Constructive uniformities of pseudometrics and Bishop topologies,” *Journal of Logic and Analysis*, 11:FT2, (2019), pp. 1–44. [Zbl 1436.03313](#). [MR 3978733](#). [1](#)
- [33] Petrakis, I., “Dependent sums and Dependent Products in Bishop’s Set Theory,” in Dybjer et al. (Eds.) *TYPES 2018*, LIPIcs, Vol. 130, Article No. 3, (2019) pp. 1–21. [Zbl 1528.03241](#). [MR 4033392](#). [2](#)
- [34] Petrakis, I., “Borel and Baire sets in Bishop Spaces,” in Manea et al. (Eds.) *Computing with Foresight and Industry*, LNCS 11558, (2019), pp. 240–252. [Zbl 1434.03142](#). [MR 3981886](#). [1](#), [6](#)
- [35] Petrakis, I., “Embeddings of Bishop spaces,” *Journal of Logic and Computation*, exaa015, (2020), pp. 1–31. <https://doi.org/10.1093/logcom/exaa01511:FT2>, [Zbl 1515.03224](#). [MR 4082955](#). [1](#), [7](#)
- [36] Petrakis, I., “Functions of Baire-class one over a Bishop topology,” in Anselmo et al. (Eds.) *Beyond the Horizon of Computability*, LNCS 12098, (2020), pp. 215–227. [Zbl 7633510](#). [MR 4139538](#). [1](#)
- [37] Petrakis, I., *Families of Sets in Bishop Set Theory*, Habilitationsschrift, LMU Munich, 2020, available at <https://www.mathematik.uni-muenchen.de/~petrakis/content/Theses.php> [1](#), [2](#), [3](#), [4](#), [15](#), [20](#), [21](#)
- [38] Petrakis, I., “Direct spectra of Bishop spaces and their limits,” *Logical Methods in Computer Science*, Vol. 17 (2021), pp. 4:1–4:50. [Zbl 1473.03038](#). [MR 4245805](#). [1](#), [2](#), [7](#)
- [39] Petrakis, I., “Closed subsets in Bishop topological groups,” *Theoretical Computer Science*, 935 (2022), pp. 128–143. [Zbl 7601542](#). [MR 4495827](#). [1](#), [20](#)
- [40] Petrakis, I., “Proof-relevance in Bishop-style constructive mathematics,” *Mathematical Structures in Computer Science*, Volume 32, Issue 1, (2022), pp. 1–43. [Zbl 7630468](#). [MR 4502018](#). [2](#)
- [41] Petrakis, I., “Strong negation in constructive mathematics,” in preparation (2024). [2](#), [3](#)
- [42] Petrakis, I., “On the definition of Bishop sets,” in preparation (2024). [21](#)
- [43] Petrakis, I., “From Daniell spaces to the integration spaces of Bishop and Cheng,” in preparation (2024). [6](#)
- [44] Petrakis, I., “Category Theory in Bishop Set Theory,” in preparation (2024). [3](#), [20](#)
- [45] Petrakis, I., “Bases of pseudocompact Bishop spaces,” in Bridges et al. (Eds.) *Handbook of Constructive Mathematics*, Cambridge University Press, (2023), pp. 359–394. [Zbl 7725652](#). [MR 4571475](#). [1](#)
- [46] Petrakis, I., “Topologies of open complemented subsets,” arXiv:2312.17095v1 (2023), pp. 1–45. [6](#)
- [47] Petrakis, I., and D. Wessel, “Algebras of complemented subsets,” in Berger et al. (Eds.) *Revolutions and Revelations in Computability*, LNCS 13359, (2022), pp. 246–258. [Zbl 7627933](#). [MR 4501267](#). [6](#), [7](#), [21](#)
- [48] Petrakis, I., and M. Zeuner, “Pre-measure spaces and pre-integration spaces in predicative Bishop–Cheng measure theory,” arXiv:2207.08684 (2022), pp. 1–37. [6](#), [21](#)
- [49] Richman, F., “Constructive mathematics without choice,” in Schuster et al. (Eds.) *Reuniting the Antipodes Constructive and Nonstandard Views of the Continuum*, Dordrecht: Kluwer, (2001), pp. 199–205. [Zbl 1251.03093](#). [MR 0567670](#). [21](#)
- [50] Ruitenburg, W., “Inequality in Constructive Mathematics,” *Notre Dame Journal of Formal Logic*, Vol. 32 (4) (1991), pp. 533–553. [Zbl 0746.03050](#). [MR 1146609](#). [2](#)

- [51] Shulman, M., “Affine Logic for Constructive Mathematics,” *The Bulletin of Symbolic Logic* Volume 28, Number 3 (2022), pp. 327–386. [Zbl 7606771](#). [MR 4498977](#). 3
- [52] Walker, R. C., *The Stone-Čech Compactification*, Springer-Verlag, 1974. 15, 17

Department of Computer Science  
University of Verona  
Strada le Grazie 15  
Verona 37134  
ITALY

[iosif.petrakis@univr.it](mailto:iosif.petrakis@univr.it)

<https://www.di.univr.it/?ent=persona&id=42498&lang=en#tab-presentazione>