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Moving energies hide within Noether's first theorem

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Abstract

We show that the moving energies of some well-known nonholonomic systems are hidden among the first integrals that can be obtained by applying Noether's first Theorem to a suitable Lagrangian.

Keywords: Noether symmetries, nonholonomic systems, moving energy

1. Introduction

Nonholonomic systems are mechanical systems with constraints in the velocities that are not derivatives of constraints in the positions. Therefore in nonholonomic systems not all the velocities are permitted. The equations of motion of a nonholonomic system are not of variational type [15]. Nevertheless energy is conserved for time-independent nonholonomic systems with constraints that are linear or homogeneous functions of the velocities [16]. The situation is different and the energy is typically not conserved if the nonholonomic constraint is a generic nonlinear function of the velocities [16]. The case in which the nonholonomic constraints are affine functions of the velocities has been thoroughly studied [9]. The assumptions that guarantee the conservation of the 'generalized' energy in nonholonomic systems with constraints that are affine functions of the velocities are very special [9]. However the so-called

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moving energy, which is a modification of the generalized energy, is conserved under suitable conditions [8, 10, 14].

Although the equations of motion of a nonholonomic system are not variational, the reduced equations may admit a Hamiltonian formulation (see e.g. [3, 5, 6] and references therein). On the other hand the search for a Lagrangian is less investigated. Nevertheless the Lagrangian structure is of great importance, because Noether's first theorem can be directly applied without using the Legendre transform, and therefore even to systems with non-regular Lagrangians[4].

In this work, after briefly recalling what Noether symmetries and moving energies are, we show that the moving energies of three well-known nonholonomic systems are hidden among the first integrals that can be obtained by applying Noether's first theorem to a suitable Lagrangian.

In section 3 we present the case of a homogeneous ball that rolls without sliding on a uniformly rotating horizontal table and find that the reduced equations of motion can be realized as Lagrange equations. Then we apply Noether's first theorem deriving eight first integrals and among them a suitable combination yields the moving energy.

In section 4 we consider the free nonholonomic particle with affine constraint[5], and show that the moving energy is obtained by applying Noether's first theorem to a suitable Lagrangian. A similar result is obtained by adding a potential force to the nonholonomic particle with affine constraint.

In the final section comments and future perspectives are presented.

2. Preliminaries

2.1. Noether's first theorem

Given any system of differential equations on an open subset of $\mathbb{R}^m$, it is a classical question to determine a Lagrangian for it.

For example, it is well-known [23] that if a time-independent Lagrangian $L = L(\mathbf{q}, \dot{\mathbf{q}})$ can be determined for a system of n second-order ordinary differential equations, then L admits the trivial Noether symmetry ∂_t that through Noether's first theorem yields the autonomous first integral:

$$E_L = \sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L, \tag{1}$$

also called *generalized energy* associated to L. Of course, if L is not the so-called natural Lagrangian, i.e. the difference between kinetic energy and potential energy, then E_L is not the mechanical energy. However, E_L may correspond to the Hamiltonian that can be obtained by applying Legendre transformation to L. An example can be found in [22]. Moreover, the non-uniqueness of the Lagrangian[6] suggests to look for a Lagrangian $L = L(t, \mathbf{q}, \dot{\mathbf{q}})$ that admits the maximal number of Noether point symmetries [11, 12], i.e.

$$\Gamma = \xi(t, \mathbf{q})\partial_t + \sum_{k=1}^{n} \eta_k(t, \mathbf{q})\partial_{q_k},$$

[4] A regular Lagrangian is such that its Hessian is invertible and consequently the Legendre transformation is well defined.

[5] The free nonholonomic particle with linear constraint has been presented in several works, see e.g. [1, 24].

[6] In [19] many Lagrangians were determined for linear and nonlinear oscillators, and in [18] different Lagrangians were presented for the system of n rotors.

that yield the corresponding Noether first integrals [17, 23]

$$I = \xi L + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}(\eta_k - \dot{q}_k \xi) - f\,,$$

where $f = f(t, \mathbf{q})$ is a function to be determined by means of the condition

$$L\frac{\mathrm{d}\xi}{\mathrm{d}t} + (\Gamma + (\eta_k - \dot{q}_k\xi)\partial_{\dot{q}_k})(L) = \frac{\mathrm{d}f}{\mathrm{d}t}.$$

In this paper we consider Noether point symmetries only.

2.2. Moving energies

Moving energies have been introduced in [10] in terms of time-dependent changes of coordinates that transform a nonholonomic systems with affine constraint in the velocity into a nonholonomic systems with linear constraints. If the transformed nonholonomic system is time independent then it admits a conserved energy, and the moving energy is nothing but the pullback of the conserved energy of the nonholonomic system with linear constraints via the time-dependent change of coordinates. The moving energy is a first integral of the system with affine constraints, and this fact is also related to the presence of symmetry in the system. In [8, 14] moving energies have been thoroughly studied. In particular, a more general definition has been introduced in [8]. If a nonholonomic system described by a natural Lagrangian L and a constraint which is an affine function of the velocities is considered, then a moving energy associated to a vector field Y on the configuration space is the restriction to the constraint manifold of a function given by the difference between the energy and the momentum of the vector field Y. Precisely let $(L, Q, \mathcal{M})$ be a nonholonomic systems with affine constraints, where L is the natural Lagrangian, Q the configuration space and $\mathcal{M}$ the constant rank (affine) distribution defined by the nonholonomic constraint. We regard $\mathcal{M}$ as a submanifold of TQ and name it the constraint manifold. Then for any vector field Y on Q we define

$$E_{L,Y} := E_L - \langle p, Y\rangle\,,$$

where p is the momentum vector $p = \nabla_{\dot{q}} L$. A function $f\colon \mathcal{M} \longrightarrow \mathbb{R}$ is called a *moving energy* for a nonholonomic system $(L, Q, \mathcal{M})$ if there exists a vector field Y on Q, called generator of f, such that f equals the restriction of $E_{L,Y}$ to the constrained manifold

$$f = E_{L,Y}|_{\mathcal{M}}\,.$$

Not all moving energies defined in this manner are conserved, but only those whose associated vector field satisfies certain conditions.

3. A rolling sphere on a turntable

We consider the system of an homogeneous sphere that rolls without sliding on a table that uniformly rotates with uniform angular velocity Ω about an orthogonal axes. The system is well-known (see e.g. [16]) and the equations of motion can be integrated explicitly and one can easily see that the center of mass describes circles of a given frequency, that depends on the velocity of rotation of the table and on the inertia of the sphere on a fixed reference frame. We start by describing the holonomic system of a homogeneous sphere moving on a plane. The configuration space of the system is diffeomorphic to $\mathbb{R}^2 \times \mathrm{SO}(3)$ endowed with coordinates $(x, y, \phi, \psi, \theta)$, where (x, y) are the coordinates of the center of mass of the sphere, and (ϕ, ψ, θ) are Euler's angles that parameterize $\mathrm{SO}(3)$. The phase space, upon left trivialization of the $T\mathrm{SO}(3)$ component, is diffeomorphic to $\mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^2 \times \mathbb{R}^3$ endowed with coordinates $(x, y, \phi, \psi, \theta, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z)$, where $(\dot{x}, \dot{y})$ represent the velocities of the center of mass and $\omega = (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$ represents the angular velocity of the sphere in the space representation. The sphere is free to move and so the Lagrangian is, up to a factor m that is the mass:

$$\mathcal{L}(x, y, \phi, \psi, \theta, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}ca^2\|\omega\|^2, \tag{2}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^2$, a is the radius of the sphere, and ca^2 is the moment of inertia divided by the mass.

Now we introduce the nonholonomic constraint of rolling without sliding, which is given by the following two conditions:

$$\dot{x} = a\omega_y - \Omega y, \qquad \dot{y} = -a\omega_x + \Omega x, \tag{3}$$

where, we recall, Ω is the angular velocity of uniform rotation of the table about an axis orthogonal to the plane. The phase space of the nonholonomic system is then an eight-dimensional manifold M_8 diffeomorphic to $\mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^3 \ni (x, y, \phi, \psi, \theta, \omega_x, \omega_y, \omega_z)$. The Lagrangian $\mathcal{L}$ and the constraint (3) are invariant with respect to the attitude of the sphere, thus the equations of motion define a vector field on the reduced space $M_5 \cong \mathbb{R}^2 \times \mathbb{R}^3$, obtained from M_8 simply by cutting the $\mathrm{SO}(3)$ factor. The reduced equations of motion of the system are then

$$\begin{aligned} &\dot{x} = a\omega_y - \Omega y, \qquad \dot{y} = -a\omega_x + \Omega x, \\ &a\dot{\omega}_x = \frac{\alpha}{c}(a\omega_y - \Omega y), \quad a\dot{\omega}_y = \frac{\alpha}{c}(-a\omega_x + \Omega x), \quad \dot{\omega}_z = 0, \end{aligned} \tag{4}$$

after substituting $\alpha = \frac{c\Omega}{1+c}$.

We first observe that the last equation of (4) yields the trivial first integral $I_0 = \omega_z$ and consequently we can consider the restriction of the equations (4) to the level sets of I_0, i.e.:

$$\dot{x} = a\omega_y - \Omega y, \quad \dot{y} = \Omega x - a\omega_x, \quad a\dot{\omega}_x = \frac{\alpha}{c}(a\omega_y - \Omega y), \quad a\dot{\omega}_y = \frac{\alpha}{c}(\Omega x - a\omega_x), \tag{5}$$

in the four unknowns x, y, ω_x, ω_y. If we solve ω_x and ω_y from the first two equations of (5), then following system of two second-order equations in x and y is obtained:

$$\ddot{x} = -\alpha\dot{y}, \qquad \ddot{y} = \alpha\dot{x}. \tag{6}$$

Since system (6) is linear in $\dot{x}, \dot{y}$, then we can use the relationship between the Jacobi last multiplier M and the Lagrangian L as derived in [25], i.e.:

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{u}_i \partial \dot{u}_j}. \tag{7}$$

We recall that the equation of the Jacobi last multiplier [13, 20, 25] for an arbitrary system of two second-order equations $\ddot{x} = f_1(t, x, y, \dot{x}, \dot{y})$, $\ddot{y} = f_2(t, x, y, \dot{x}, \dot{y})$, is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -M\left(\frac{\partial f_1}{\partial \dot{x}} + \frac{\partial f_2}{\partial \dot{y}}\right).$$

Since system (6) has null divergence, then a Jacobi last multiplier is a constant, say 1, and consequently we may consider $M_{11} = M_{22} = 1, M_{12} = 0$ in (7) that yields the following Lagrangian of system (6):

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \alpha\dot{x}y, \tag{8}$$

which is a particular case of the Lagrangian of n rotors as determined in [18]. If we apply Noether's first theorem to this Lagrangian, then we obtain eight Noether symmetries (the maximum possible number [11, 12]), i.e.:

$$\begin{aligned}
&\Gamma_1 = \partial_t,\\
&\Gamma_2 = \cos(\alpha t)\,\partial_t - \tfrac{\alpha}{2}\left[y\cos(\alpha t) + x\sin(\alpha t)\right]\partial_x + \tfrac{\alpha}{2}\left[x\cos(\alpha t) - y\sin(\alpha t)\right]\partial_y,\\
&\Gamma_3 = \sin(\alpha t)\,\partial_t + \tfrac{\alpha}{2}\left[x\cos(\alpha t) - y\sin(\alpha t)\right]\partial_x + \tfrac{\alpha}{2}\left[y\cos(\alpha t) + x\sin(\alpha t)\right]\partial_y,\\
&\Gamma_4 = \partial_y, \quad \Gamma_5 = \partial_x, \quad \Gamma_6 = \cos(\alpha t)\,\partial_x + \sin(\alpha t)\,\partial_y,\\
&\Gamma_7 = \sin(\alpha t)\,\partial_x - \cos(\alpha t)\,\partial_y, \quad \Gamma_8 = y\partial_x - x\partial_y,
\end{aligned} \tag{9}$$

and consequently the following eight first integrals of system (6) are derived, i.e.:

$$\begin{aligned}
&I_1 = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2), \quad I_2 = \left[\tfrac{\alpha}{2}(y\dot{x} - x\dot{y}) - \tfrac{\dot{x}^2+\dot{y}^2}{2}\right]\cos(\alpha t) + \tfrac{\alpha}{2}(x\dot{x} + y\dot{y})\sin(\alpha t),\\
&I_3 = -\tfrac{\alpha}{2}(x\dot{x} + y\dot{y})\cos(\alpha t) + \left[\tfrac{\alpha}{2}(y\dot{x} - x\dot{y}) - \tfrac{\dot{x}^2+\dot{y}^2}{2}\right]\sin(\alpha t),\\
&I_4 = -\dot{y} + \alpha x, \quad I_5 = -\dot{x} - \alpha y, \quad I_6 = -\dot{x}\cos(\alpha t) - \dot{y}\sin(\alpha t),\\
&I_7 = -\dot{x}\sin(\alpha t) + \dot{y}\cos(\alpha t), \quad I_8 = \tfrac{-\alpha}{2}(x^2 + y^2) + (x\dot{y} - y\dot{x}).
\end{aligned} \tag{10}$$

Four first integrals depend on time explicitly. However, we can suitably combine them in order to obtain two further first integrals independent of time, i.e.:

$$I_2^2 + I_3^2 = I_1^2 - \alpha I_1 I_8, \qquad I_6^2 + I_7^2 = 2I_1. \tag{11}$$

Obviously, only three autonomous first integrals are functionally independent from each other, e.g. I_1, I_4, I_5. In fact,

$$I_8 = \frac{1}{2\alpha}(2I_1 - I_4^2 - I_5^2).$$

If we make the substitutions $\dot{x} = a\omega_y - \Omega y$, $\dot{y} = \Omega x - a\omega_x$ into the three first integrals I_1, I_4, I_5, then we obtain three autonomous first integrals of system (4), the last two already known [10]:

$$I_1 = \frac{1}{2}\left[(a\omega_y - \Omega y)^2 + (\Omega x - a\omega_x)^2\right], \quad I_4 = a\omega_x - \frac{\alpha}{c}x, \quad I_5 = -a\omega_y + \frac{\alpha}{c}y. \tag{12}$$

In [10] the following first integral of system (4) was derived by introducing the so-called moving energy:

$$\begin{aligned}
E_{L,M_5} = {}& \frac{1}{2}\left[(a\omega_y - y\Omega)^2 + (x\Omega - a\omega_x)^2\right] + \frac{1}{2}ca^2(\omega_x^2 + \omega_y^2 + \omega_z^2)\\
&- \Omega^2(x^2 + y^2) + \Omega a(x\omega_x + y\omega_y) - ca^2\Omega\omega_z,
\end{aligned} \tag{13}$$

that is:

$$E_{L,M_5} = -\frac{1+c}{c}\left(I_1 - \frac{1+c}{2}I_4^2 - \frac{1+c}{2}I_5^2\right) + \frac{1}{2}ca^2 I_0^2 - ca^2\Omega I_0. \tag{14}$$

Thus, the moving energy is just a combination of the autonomous first integrals I_1, I_4, I_5 in (12), that one can find by means of Noether's first Theorem applied to the Lagrangian (8), and the trivial integral I_0.

4. A nonholonomic particle with affine constraint

In this section we generalize the classical *nonholonomic particle* [24]. In the first example we consider a nonholonomic constraint affine in the velocities, in the second one we add a potential force acting on the particle.

4.1. Nonholonomic free particle with affine constraint

Consider a free particle in $\mathbb{R}^3$, with kinetic Lagrangian $\mathcal{L} = \frac{1}{2}(v_1^2 + v_2^2 + v_3^2)$, where (x_1, x_2, x_3) are Cartesian coordinates on $\mathbb{R}^3$ and (v_1, v_2, v_3) the corresponding velocities. The particle is subjected to the affine nonholonomic constraint

$$v_3 = v_1 x_2 + \mu\,, \tag{15}$$

where $\mu \in \mathbb{R}$ is a fixed parameter. The phase space is then the subset $\mathcal{M} \cong \mathbb{R}^5$ of $\mathbb{R}^6$ defined by (15) and can be parameterized by $(x_1, x_2, x_3, v_1, v_2)$. The equations of motion are the restriction to $\mathcal{M}$ of Lagrange equations for L with Lagrange multiplier and read

$$\dot{x}_1 = v_1\,, \quad \dot{x}_2 = v_2\,, \quad \dot{x}_3 = v_1 x_2 + \mu\,, \quad \dot{v}_1 = -\frac{x_2 v_1 v_2}{1 + x_2^2}\,, \quad \dot{v}_2 = 0\,. \tag{16}$$

One can easily check that the energy is not conserved, however the moving energy

$$E_{\mathcal{L},Y}|_{\mathcal{M}} = \frac{1}{2}\left[(v_1^2 + v_2^2 + v_3^2) - \mu v_3\right]\Big|_{\mathcal{M}} = \frac{1}{2}\left(v_1^2(1 + x_2^2) + v_2^2 - \mu^2\right), \tag{17}$$

where $Y = \mu\partial_{x_3}$ is the infinitesimal generator of the translation along the x_3 direction, is conserved.

We now show that the moving energy also in this case can be obtained as a Noether symmetry for a suitable Lagrangian that yields part of the equation (16). We first observe that the existence of the first integral defined by the last equation of (16), allows to integrate the second equation: $x_2(t) = v_{2,0}\,t + x_{2,0}$, where $x_2(0) := x_{2,0}$ and $v_2(0) := v_{2,0}$ are constants of integration. Then by substituting in the remaining three equation one obtains

$$\dot{x}_1 = v_1\,, \quad \dot{x}_3 = v_1(v_{2,0}t + x_{2,0}) + \mu\,, \quad \dot{v}_1 = -\frac{v_1 v_{2,0}(v_{2,0}\,t + x_{2,0})}{1 + (v_{2,0}\,t + x_{2,0})^2}\,. \tag{18}$$

Now deriving the first equation of (18) with respect to time and substituting into the last equation yields the following second-order ordinary differential equation on x_1:

$$\ddot{x}_1 = -\frac{\dot{x}_1 v_{2,0}(v_{2,0}\,t + x_{2,0})}{1 + (v_{2,0}\,t + x_{2,0})^2}\,. \tag{19}$$

We recall that the Jacobi last multiplier [13] of a second-order differential equation $\ddot{x} = f(t, x, \dot{x})$ is any solution of the following equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -M\,\frac{\partial f}{\partial \dot{x}}\,, \tag{20}$$

and that a Lagrangian can be obtained by means of the following formula

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}. \tag{21}$$

Consequently, a Jacobi last multiplier of equation (19) is

$$M_0 = \sqrt{1 + (v_{2,0}\,t + x_{2,0})^2}\,, \tag{22}$$

that yields (by double integration) the Lagrangian

$$L = \frac{1}{2}\dot{x}_1^2\sqrt{1+(v_{2,0}\,t+x_{2,0})^2}\,. \tag{23}$$

By applying Noether's first theorem to this Lagrangian we obtain five Noether symmetries and then five first integrals. One can easily realize that the symmetry

$$\sqrt{1+(v_{2,0}\,t+x_{2,0})^2}\,\partial_t$$

generates the first integral

$$I_1 = \frac{1}{2}\dot{x}_1^2(1+(v_{2,0}\,t+x_{2,0})^2)\,,$$

that in the original coordinates reads

$$I_1 = \frac{1}{2}v_1^2(1+v_2^2)\,,$$

and then coincides with the moving energy (17) up to an addictive constant.

4.2. Nonholonomic particle with affine constraint and potential force

Consider now the example above of a particle subjected to the affine nonholonomic constrained (15), that moves in $\mathbb{R}^3$, but now under the action of a force of potential energy $V = V(x_2)$. The Lagrangian is then $L = \frac{1}{2}(v_1^2+v_2^2+v_3^2) - V(x_2)$, Let $\mathcal{M} \cong \mathbb{R}^5 \subset \mathbb{R}^6$ be the constraint manifold defined by (15). The equations of motions are then

$$\dot{x}_1 = v_1\,, \quad \dot{x}_2 = v_2\,, \quad \dot{x}_3 = v_1x_2+\mu\,, \quad \dot{v}_1 = -\frac{x_2v_1v_2}{1+x_2^2}\,, \quad \dot{v}_2 = -V'(x_2)\,. \tag{24}$$

As above one can easily verify that the energy is not conserved, however the moving energy

$$E_{\mathcal{L},Y}|_{\mathcal{M}} = \frac{1}{2}\left[(v_1^2+v_2^2+v_3^2)+V(x_2)-\mu v_3\right]\big|_{\mathcal{M}} = \frac{1}{2}\left(v_1^2(1+x_2^2)+v_2^2-\mu^2\right)+V(x_2)\,, \tag{25}$$

where Y has been defined in the previous example, is conserved.

We now show that also in this case the moving energy is obtained as a Noether symmetry of a suitable Lagrangian. We take x_2 as newindependent variable, thus obtaining the non-autonomous system of equations

$$x_1' = \frac{v_1}{v_2}\,, \quad x_3' = \frac{x_2v_1+\mu}{v_2}\,, \quad v_1' = -\frac{x_2v_1}{1+x_2^2}\,, \quad v_2' = -\frac{V'(x_2)}{v_2}\,, \tag{26}$$

where $'$ denote the derivation with respect to x_2. The last equation of (26) yields

$$v_2^2(x_2) = 2(V_0 - V(x_2))\,,$$

where V_0 is a constant of integration. Then system (26) reduces to

$$x_1' = \frac{v_1}{\sqrt{2(V_0-V(x_2))}}\,, \quad x_3' = \frac{x_2v_1+\mu}{\sqrt{2(V_0-V(x_2))}}\,, \quad v_1' = -\frac{x_2v_1}{1+x_2^2}\,. \tag{27}$$

From the first equation of (27) we obtain $v_1 = x_1'\sqrt{2(V_0-V(x_2))}$, that substituted in the third equation yields the following non-autonomous ordinary differential equation:

$$x_1'' = -\frac{x_1'x_2}{(1+x_2^2)} + \frac{x_1'V'(x_2)}{2(V_0-V(x_2))}\,. \tag{28}$$

We then compute the Jacobi last multiplier of (28):

$$M_0 = \sqrt{(V_0 - V(x_2))(1 + x_2^2)},$$

which upon double integration with respect to x_1' yields the Lagrangian

$$L = \frac{(x_1')^2}{2}\sqrt{(V_0 - V(x_2))(1 + x_2^2)}. \tag{29}$$

As in the previous case, the Lagrangian (29) admits five Noether symmetries and consequently five first integrals. In particular, the symmetry

$$\sqrt{(V_0 - V(x_2))(1 + x_2^2)}\,\partial_{x_2}$$

generates the first integral

$$I_1 = (x_1')^2\left((V_0 - V(x_2))(1 + x_2^2)\right),$$

which is the moving energy (25) in the original coordinates up to an addictive constant.

5. Conclusions and future perspectives

In 1931, Bateman [2] determined Lagrangians of both ordinary and partial differential equations by adding a suitable set of complementary equations. In his classical work [7], Douglas provided several examples of two-dimensional systems of second-order differential equations that do not admit a Lagrangian. However, it was shown in [21] that one could determine a different sets of equations compatible with those of Douglas and yet derivable from a variational principle.

Recently, in [4] the classical system of the nonholonomic particle was studied using quasi-velocities and a Jacobi last multiplier was determined. Consequently, one could have easily derived the corresponding Lagrangian (21), which admits the maximum number (five) of Noether symmetries.

In this paper, we have determined a Lagrangian for some classical nonholonomic systems by following Bateman's dictum [2], namely to look for different sets of equations compatible with the original problem and derivable from a variational principle, without recourse to any additional set of equations. The theory of the Jacobi last multiplier allowed us to determine the Lagrangian with the maximal number of Noether symmetries, and therefore we were able to show that the moving energy of each system is obtained from the corresponding Noetherian first integrals.

We conjecture that other nonholonomic systems could be equally framed into a suitable variational problem. Work in this direction is currently in progress.

Data availability statement

No new data were created or analyzed in this study.

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