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Time-invariant portfolio strategies in structured products with guaranteed minimum equity exposure

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Abstract

We introduce a new exotic option to be used within structured products to address a key disadvantage of standard time-invariant portfolio protection: the well-known *cash-lock risk*. Our approach suggests enriching the framework by including a threshold in the allocation mechanism so that a guaranteed minimum equity exposure (GMEE) is ensured at any point in time. To be able to offer such a solution still with hard capital protection, we apply an option-based structure with a dynamic allocation logic as underlying. We provide an in-depth analysis of the prices of such new exotic options, assuming a Heston–Vasicek-type financial market model, and compare our results with other options used within structured products. Our approach represents an interesting alternative for investors aiming at downsizing protection via time-invariant portfolio protection strategies, meanwhile being also afraid to experience a cash-lock event triggered by market turmoils.

KEYWORDS

guaranteed minimum equity exposure, portfolio insurance strategies, stochastic volatility, structured products, time-invariant portfolio protection

1 | INTRODUCTION

Dating from the 2008 worldwide economic crisis, financial arenas have started to see an increase in the equity market along with a related decrease in interest rate levels, until a generalized drop happened in 2018, then accompanied by abrupt volatility changes caused by the *Covid Crisis* in 2020. As a consequence, many investors are increasingly interested in investment products that can provide strong capital protection at maturity. In particular, they have commenced looking at investment products to protect their portfolios from significant losses and, at the same time, to offer participation in equity (or equity indices) with an attractive participation rate. In order to address these needs, structured investment products have been offered in the market. These products provide capital protection and the opportunity to participate in equity markets through embedded financial derivatives.

The idea of considering structured products that include a capital guarantee at a fixed future date is very common among life insurance products, see for example Reference 1, where the authors provide a detailed overview of variable annuities with the so-called Guaranteed Minimum Benefit of type "x" (GMxB). In a financial framework, designing a structured investment product able to combine high capital protection with high participation is challenging because such features are highly correlated. Indeed, high capital protection negatively affects the amount of the available risk budget and, consequently, the amount that can be invested in financial derivatives. Moreover, the capital protection level is affected by market interest rates, and the participation rate is affected by both interest rate levels and market volatility. However, as argued in References 2-4, a combination of a low interest rate and high market volatility leads to a very low participation rate, making the guarantee structure unattractive for investors.

For this reason, a provider of a structured investment product with capital protection is interested in increasing the attractiveness of his/her product to an investor by simultaneously providing a high protection level and a high market participation rate. Assuming that the protection level of a structured investment product is fixed, the main factor affecting the participation rate is the price of the embedded options. In order to obtain the above desirable features from the standpoint of a product provider, the price of the embedded option should not depend significantly on the market volatility, and it should decrease with decreasing market interest rate levels.

In the past, to obtain the above features, investment product providers replaced traditional European call and put options with more complex exotic derivatives written on the same underlying risky asset, such as barrier options, Asian options, etc. Recently, a new family of structured investment products with capital protection emerged on the market. Their distinguishing characteristic is not an embedded option with an exotic payoff but a plain vanilla option linked to a modified underlying risky asset. In particular, traditional risky assets such as stock indices are replaced with volatility target (VolTarget) strategies. Investment mechanisms of this nature have been extensively utilized in both portfolio management, as demonstrated by Reference 5, and pension funds or annuity provisions, as evidenced by Reference 6. In the context of financial applications, Reference 7 offers the first comprehensive analysis of the VolTarget strategies, characterizing them as the underlying asset in an option embedded in a structured investment product.

In the present paper, we continue this line of research by considering an underlying asset for the option embedded in a structured investment with capital protection, different dynamic asset allocation algorithms, the so-called *portfolio insurance strategies*. Among all the possible portfolio insurance techniques described in the literature and applied to the financial industry, *Constant Proportion-portfolio insurance* (CPPI) is undoubtedly the most used and appreciated alternative, see, for example, Reference 8. The main reason behind to use CPPI approach lies in its ability to provide market participation ensuring capital preservation at the same time, see, for example, References 9-13. In layman's terms, the CPPI algorithm depends on two parameters, namely the multiplier and the floor, established as constant input parameters. The former serves as the adaptation for the present value, and the latter measures the degree of protection for the invested resources, according to the current risk-free rate.

Inspired by this setup, Reference 14 argued that the CPPI strategy could be further improved by assuming a time-varying floor in the so-called time-invariant portfolio protection (TIPP). The solution proposed was to modify the floor at each rebalancing time taking on board its previous values, as long as they were not excessively low. Reference 14 carried out an empirical analysis demonstrating the ability of the TIPP strategy to perform the role it was designed for, namely to limit losses in the event of a market collapse at the price of a lower return compared to a passive position on the index. Starting from the pioneering work by Reference 14, several authors have studied the role of TIPP and its applications either in the risk management context, see, for example, References 9, 15-17. The horizon effects suffered by the TIPP mechanism are investigated in Reference 18, while a performance study of TIPP, compared to other portfolio strategies, is presented in Reference 19.

However, as argued in Reference 7, both CPPI and TIPP are characterized by path-dependency and bear the so-called *cash-lock risk*: in extreme market scenarios, when the value of a risky asset decreases significantly, the whole portfolio needs to be invested into a risk-less asset. As a result, the investor loses the possibility to participate in potential upward movements of the risky asset. Hence, the TIPP strategy is not eligible to be viewed as the underlying of European option embedded in a structured investment product with capital protection.

Consequently, we first propose to modify the standard TIPP strategy by adding an investment threshold in the risky asset allocation, the so-called guaranteed minimum equity exposure (GMEE). The GMEE has been introduced in Reference 20 within the CPPI framework to buffer the so-called cash-lock risk. Also for the TIPP allocation mechanisms, a guaranteed minimum equity exposure could overcome the risk of low market participation in a V-shapes market environment.

For our approach, we introduce and evaluate suitable equity-linked instruments, namely European-type OTC call options whose underlying is the TIPP strategy endowed with a GMEE feature. We evaluate these new types of exotic options and compare them with other well-known strategies. More precisely, we look at forms of options where the underlying is a pure risky equity index, the standard TIPP logic, and the CPPI methodology, with and without the guaranteed minimum equity exposure. From a practical perspective, such a comparison makes a roundup of the possible proposals that an advisor may present to his clients, according to the views on future market developments.

We present an accurate and comprehensive numerical analysis for option prices in a Vasicek–Heston framework, involving different time horizons for option maturities and various rebalancing frequencies without considering transaction costs. When selecting a financial model, we are guided by two fundamental criteria, as exemplified by Reference 21. First, we aim to exercise control over the primary distributional properties of returns, which is a hallmark of stochastic volatility models. Second, we require a more efficient method for valuing present values and/or payoffs of financial instruments, particularly for very long maturities, as well as for implementing effective hedging strategies. These are key economic characteristics of stochastic interest rate models. Consequently, we have chosen to utilize a Heston model as the underlying model, which includes a Vasicek-type risk-free rate to account for negative interest rate values. Our numerical findings, corroborated by an in-depth sensitivity analysis, reveal that the TIPP strategy endowed with GMEE is the cheapest procedure among those examined for short and medium-time horizons, further confirming both the effectiveness and the usefulness of our new approach.

We have organized the rest of this paper as follows: Section 2 encompasses an overview of the portfolio insurance strategies analyzed in the following. To better clarify both the advantages and the shortcomings of each procedure, as well as to stress the role of the guaranteed minimum equity exposure, the main definitions and basic formulas are supported by historical simulations; more general results are shown in Section 3, where we describe the theoretical reference framework, perform an in-depth numerical analysis via Monte Carlo simulations, also proposing a detailed sensitivity analysis of our results w.r.t. both endogenous and exogenous parameters; in Section 4 we report our main conclusions and findings.

2 | AN OVERVIEW ON PROPORTION PORTFOLIO INSURANCE STRATEGIES

Portfolio insurance strategies are designed to recover at maturity a given proportion of the capital invested at $t = 0$. Formally, such strategies are defined by exploiting suitable stochastic processes having support on proper probability spaces, see Section 3.2 for the necessary technical details. The most widespread portfolio insurance strategy is the Constant Proportion-Portfolio Insurance strategy (CPPI), see, for example, Reference 10. The CPPI mechanism is based on a specific dynamic allocation, able to guarantee a predetermined value. The latter is known as *floor* $F = \{F_t\}_{t\in[0,T]}$, see, for example, Reference 22, and represents the minimum value of the portfolio which is acceptable for an investor at any time t during the investment period $[0, T]$, with $T < \infty$. The floor dynamic is

$$\mathrm{d}F_t^{\mathrm{CPPI}} = r_t F_t^{\mathrm{CPPI}}\mathrm{d}t,\ \ t \in [0, T], \tag{1}$$

whose initial value is $F_0^{\mathrm{CPPI}} = G \cdot e^{-\int_0^T r_s \mathrm{d}s}$, where G is the *guarantee*, defined as the product between the capital initially invested V_0 and the percentage of V_0 that the investor aims to recover at least at maturity T, the so-called *protection level* $\mathrm{PL} \in (0, 1]$. The CPPI strategy at time $t \in [0, T]$ is obtained by self-financing the investment in a riskless security $B = \{B_t\}_{t\in[0,T]}$, representing a cash reserve, such as a money market account, and a risky security $S = \{S_t\}_{t\in[0,T]}$, such as an equity index (see Section 3.2 for technical details). For the investment to be successful, at each time t we must determine the reserve (the so-called *cushion*) to be set aside so that a risky asset component is always maintained. The cushion is defined as follows

$$C_t^{\mathrm{CPPI}} = \left(V_t^{\mathrm{CPPI}} - F_t^{\mathrm{CPPI}}\right)^+,\ \ t \in [0, T], \tag{2}$$

see, for example, Reference 22. Equation (2) reveals that keeping the portfolio value higher than its floor is equivalent to ensuring that the cushion is always positive. For the CPPI method, the key assumption is that the exposure to the risky asset E_t^{CPPI} is proportional to the cushion. Hence, we have

$$E_t^{\mathrm{CPPI}} = M \cdot C_t^{\mathrm{CPPI}} = M \cdot \left(V_t^{\mathrm{CPPI}} - F_t^{\mathrm{CPPI}}\right)^+,\ \ t \in [0, T], \tag{3}$$

for a given parameter $M \in \mathbb{N}^+$, called the *multiplier*. Equation (3) implies that the investment in the risky asset might be potentially unbounded. Therefore, to limit the latter potential leverage effect in the allocation, the market practice suggests introducing the so-called *maximum leverage effect factor*, $L_{\max}$, such that

$$E_t^{\mathrm{CPPI}} = \min\left\{L_{\max} \cdot V_t^{\mathrm{CPPI}}, M \cdot C_t^{\mathrm{CPPI}}\right\}. \tag{4}$$

Motivated by regulatory constraints, see, for example, Reference 23, $L_{\max}$ is typically set to 100%, or 200%. Therefore, the CPPI portfolio dynamics are given by

$$dV_t^{\text{CPPI}} = V_t^{\text{CPPI}} \left(\alpha_t^{\text{CPPI}} \frac{dS_t}{S_t} + (1 - \alpha_t^{\text{CPPI}}) \frac{dB_t}{B_t} \right), \; t \in [0, T], \tag{5}$$

where

$$\alpha_t^{\text{CPPI}} = \min \left\{ L_{\max}, \frac{M \cdot C_t^{\text{CPPI}}}{V_t^{\text{CPPI}}} \right\}.$$

The CPPI strategy may be affected by the so-called *cash-lock* event: due to abrupt changes in market levels, the risky asset loses significantly in value. As a result, the exposure E_t^{CPPI} drops to zero, so the portfolio is entirely invested into the riskless asset until the end of the investment time horizon T without the chance to recover. Since a cash-locked position prevents any equity market participation in case of subsequent market rise, it can be considered a critical risk, especially for long-horizon investments. To overcome this problem, a possible solution is to modify the base protection mechanism of the CPPI, namely the floor. More precisely, we look for a mechanism able to protect a predetermined fraction of its maximum portfolio value up to time t, instead of providing a flat threshold. Indeed, the latter represents the key aspect of the CPPI allocation algorithm known as *time invariant portfolio protection* strategy (TIPP), see, for example, Reference 14. The TIPP floor is

$$F_t^{\text{TIPP}} = \max \left\{ F_t^{\text{CPPI}}, \text{PL} \cdot \sup_{s \le t} V_t^{\text{TIPP}} \right\}, \tag{6}$$

where PL is the protection level mentioned above. As for the CPPI allocation mechanism, the TIPP is a self-financing strategy. Hence, V_t^{TIPP} is the solution of

$$dV_t^{\text{TIPP}} = V_t^{\text{TIPP}} \left(\alpha_t^{\text{TIPP}} \frac{dS_t}{S_t} + \left(1 - \alpha_t^{\text{TIPP}}\right) \frac{dB_t}{B_t} \right), \; t \in [0, T], \tag{7}$$

where

$$\alpha_t^{\text{TIPP}} = \frac{E_t^{\text{TIPP}}}{V_t^{\text{TIPP}}} = \min \left\{ L_{\max}, \frac{M \cdot C_t^{\text{TIPP}}}{V_t^{\text{TIPP}}} \right\}. \tag{8}$$

The exposure E_t^{TIPP} and the cushion C_t^{TIPP} in Equation (8) are respectively given by

$$E_t^{\text{TIPP}} = \min \left\{ L_{\max} \cdot V_t^{\text{TIPP}}, M \cdot \left(V_t^{\text{TIPP}} - F_t^{\text{TIPP}}\right)^+ \right\}, \tag{9}$$

$$C_t^{\text{TIPP}} = \left(V_t^{\text{TIPP}} - F_t^{\text{TIPP}}\right)^+. \tag{10}$$

The new definition of the floor F_t^{TIPP} can partially avoid the cash-lock event associated with the CPPI strategy. Indeed, when a cash-lock occurs within the CPPI mechanism, previously obtained gains get lost. Instead, by exploiting the TIPP strategy, we can maintain the pre-specified percentage of past gains. Therefore, even if a cash-lock event occurs at $t^* \in [0, T]$, the TIPP strategy is able to retain the equity market participation obtained during the period $[0, t^*]$. For the TIPP to obtain potential equity market participation for any $t \in [t^*, T]$, we introduce in the risky exposure a further threshold $\alpha_{min} \in [0, 1]$ defined as the GMEE. Thus, we obtain a new portfolio insurance strategy, dubbed TIPP with GMEE (G-TIPP), whose building blocks are given in (6) (the G-TIPP floor), (7) (the G-TIPP dynamics), and (10) (the G-TIPP cushion), while the G-TIPP exposure is

$$E_t^{\text{G-TIPP}} = \max \left\{ \min \left\{ L_{\max} \cdot V_t^{\text{G-TIPP}}, M \cdot \left(V_t^{\text{G-TIPP}} - F_t^{\text{TIPP}}\right)^+ \right\}, \alpha_{min} \cdot V_t^{\text{G-TIPP}} \right\}. \tag{11}$$

Remark 1. It is worth noting that the introduction of the GMEE is not exclusive to time-invariant strategies, as we have shown above. The construction of such an allocation mechanism has recently been introduced in

TABLE 1 Main features of constant proportion-portfolio insurance (CPPI), time-invariant portfolio protection (TIPP), and TIPP with guaranteed minimum equity exposure (GMEE).

	CPPI	TIPP	TIPP with GMEE
Portfolio structure	Dynamic allocation between risky and riskless asset	Dynamic allocation between risky and riskless asset	Static allocation within structured investment product (assets plus call structure), where the underlying of the embedded option is a GMEE-TIPP strategy
Protection	Only at maturity	Paid immediately	Only at maturity
Maturity	Fixed maturity (usually 5-10 years)	No maturity (open-end)	Fixed maturity (usually 5–10 years)
Availability	Typically available within predefined period of, for example 3 months	Ongoing	Typically available within a predefined period of for example, 3 months
Floor	Sensitive to interest rates level and maturity	Insensitive to interest rates level and there is no maturity for the strategy	Sensitive to interest rates level and maturity through the asset plus call setup
Cash-lock event	Allocation of risky assets stay at zero until maturity; no equity market participation after cash-lock event	Investors can sell strategy immediately at the guaranteed level; high-protection level, but it must be closed since no further equity market participation is possible.	Guaranteed minimum equity exposure always enables participation, investor can stay in the strategy until maturity

the literature in the case of CPPI strategies, see, for example, Reference 20. Here we do not provide the details of the CPPI strategy with GMEE (which will be denoted as G-CPPI, for the sake of notation uniformity), but we deem it appropriate to take it into account, as it might be used as an additional benchmark for derivative pricing, as we will see in detail in Section 3.3.

For ease of convenience, we summarize the main features of the portfolio insurance strategies introduced in Table 1.

To better understand how the CPPI, the TIPP, and the G-TIPP strategies work, as well as brought advantages and shortcomings, we perform a simulation study by comparing the aforementioned strategies when the underlying risky asset is the MSCI World Net TR Eur Index. Since portfolio insurance strategies' risk-returns are path-dependent, being also strongly influenced by market conditions, we perform historical simulation studies over two different time windows. The first one ranges from December 31, 2007, to December 28, 2017, while the second one covers the period from June 29, 2012, to June 29, 2022. We select such time windows since a market crash occurs either at the beginning (time window from December 31, 2007 to December 28, 2017) or at the end (time window from June 29, 2012, to June 29, 2022) of the period under investigation. As for the parameters' strategies, we set a multiplier of $M = 5$, a protection level at 100% (resp., 90%) of the capital initially invested for CPPI (resp., for TIPP), and a maximum leverage factor, $L_{\max}$, equal to 100%. We also consider both daily and weekly rebalancing frequencies.

2.1 | Historical simulations: 2007–2017

In this Subsection, we report a comparison between CPPI and TIPP when the market shows an early-falling-and rising trend. Results concerning both portfolio evolution and corresponding risky asset exposures, for each investment strategy in Figure 1, w.r.t. different rebalancing disciplines.

It is worth noticing that the beginning of the period under investigation is characterized by an initial turbulent phase due to the subprime crisis: at the beginning of 2009, the risky asset losses are approximately equal to the 50% of the initial value. Subsequently, we can see considerable growth for the risky asset. At the end of 2014, the index gained more than 70% over its opening value, also showing a positive upward trend that was confirmed until the end of the time window. Moreover, such a trend generates a positive performance measured through an annualized total return and volatility equal

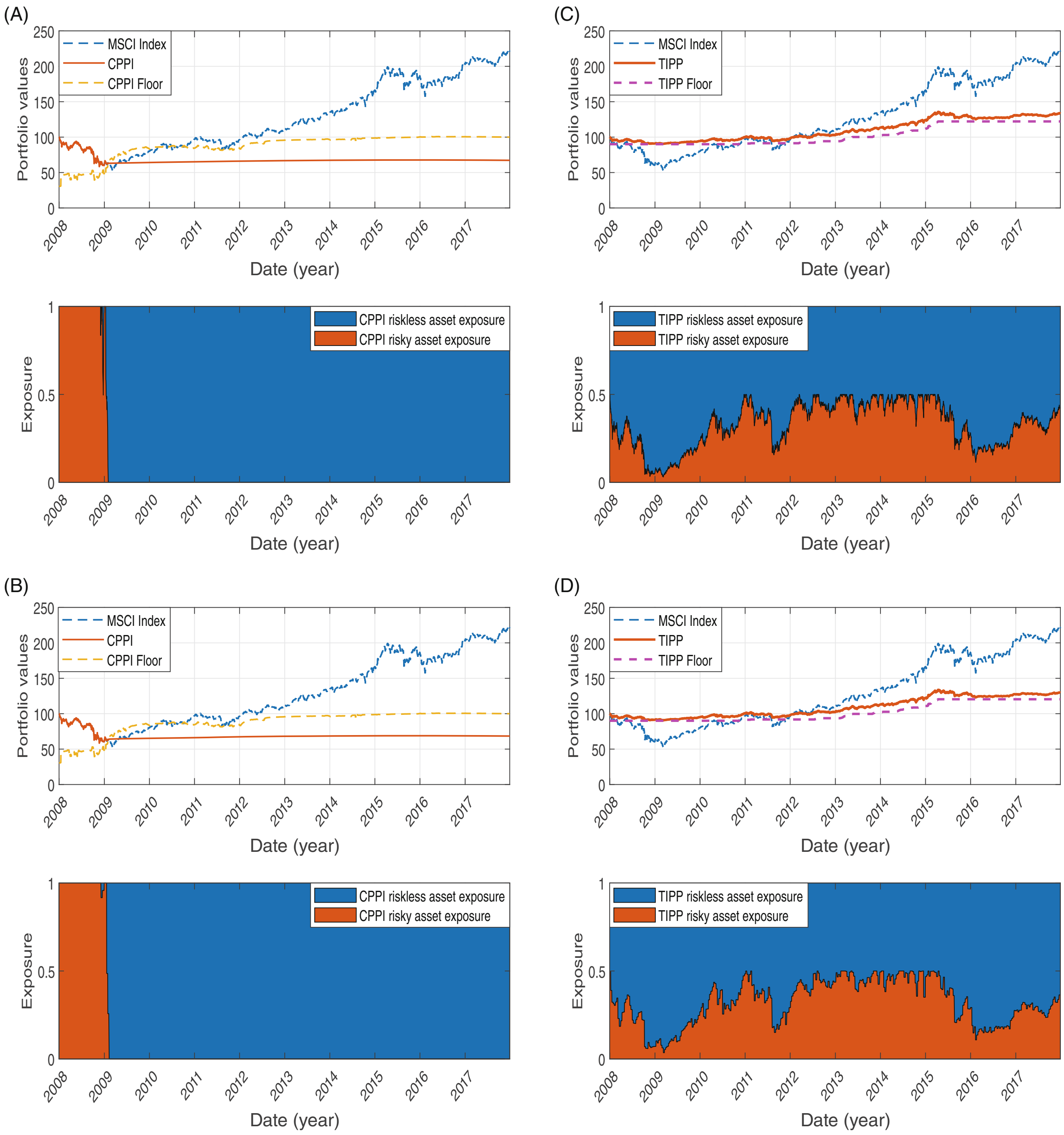

FIGURE 1 Historical simulation and investment exposures for the constant proportion-portfolio insurance (CPPI) strategy (left charts) and the time-invariant portfolio protection (TIPP) strategy (right charts) linked to the MSCI equity index. The time window ranges between December 31, 2007, and December 28, 2017. At the top of each subfigure, we show the investment strategies' trend, while at the bottom we depict the assets' exposures. (A) CPPI with 1-day rebalancing frequency; (B) CPPI with 1-week rebalancing frequency; (C) TIPP with 1-day rebalancing frequency; (D) TIPP with 1-week rebalancing frequency.

to 8.24% and 17.67%, respectively, over the entire 10-year time horizon, and a 1-day maximum loss of −8.56%. Hence, from the perspective of an investor whose goal is to minimize the risk of suffering losses, this type of equity investment might be too volatile.

For this reason, we focus on the CPPI allocation strategy and its modified version, the TIPP. We start our analysis by considering the CPPI procedure with a daily rebalancing frequency. From Figure 1, we notice that, at the very beginning, the CPPI strategy exactly follows the trend of risky security, so that it exhibits a 100%-exposure to the asset. However, when the market deteriorates (due to the US subprime crisis), the CPPI portfolio flattens out on the floor. Such a circumstance sheds light on two main issues related to the CPPI mechanism:

1. We experience a cash-lock phenomenon, as the strategy fails to follow the market trend also in the presence of significant market improvements. This is because the floor depends on the investment time horizon and the riskless rate level. If we consider a 10-year frame with near-zero interest rate levels, we observe that the fluctuation of the floor at the beginning of the reference period will be strongly affected by the long time-to-maturity, whereby the floor will be significantly lower than the initial investment value. This determines that the entire portfolio will be invested in the risky asset, failing to guarantee protection, even in case of risky asset price decreases;
2. Since the CPPI floor is not indexed to the asset but depends exclusively on the risk-free rate, at the onset of the subprime crisis the risk-free interest rate also plummets, causing the floor to rise rapidly. The sharp rise in the floor, combined with the simultaneous market crash, translates into a cash-lock event, indicating an irreversible situation: the exposure to the risky security falls to 0% so that at maturity the initial investment will not be guaranteed.

As a consequence of the aforementioned drawbacks, the CPPI strategy reaches an annualized return equal to −3.89%. The situation remains completely unchanged even if the frequency of rebalancing is reduced (weekly instead of daily observations), as it is shown in Figure 1B. To overcome such drawbacks, a first attempt is to consider an alternative floor definition, namely, we move to the TIPP strategy. Looking at Figure 1C, we note that the initial TIPP exposure is 50%. The TIPP exposure is far lower than the CPPI one. However, after the market collapse, mainly because of the financial crisis, the exposure stands at 3% instead of canceling. Thus, when the market gets back on track, that is, between early 2009 and the end of 2018, the risky exposure gradually widens, without exceeding 50% of the value of the entire TIPP portfolio. This behavior of the TIPP mechanism, which is completely different from that of the CPPI strategy, is because the TIPP floor does not depend on the time horizon of the investment, nor it is sensitive to the level of interest rates. More precisely, the TIPP floor depends on the level of protection, exogenously set to 90% of the initial investment, as well as on the risk index performance. Thus, at the beginning of the investment horizon, when the risky asset loses value under the blows of the financial crisis, the TIPP mechanism remains constant and never falls below 90%. This is why the initial exposure is much lower than the levels obtained with the CPPI strategy. Therefore, should a further sudden market crash occur, the TIPP strategy will be less exposed than the CPPI mechanism. Moreover, in the recovery phase, the TIPP floor starts to gradually increase at a rate comparable to the market, ensuring a slight but steady growth in exposure and, thus, in equity market participation.

Again, the behavior described above remains unchanged even if we vary the rebalancing frequency, as can be seen from Figure 1D.

2.2 | Historical simulations: 2012–2022

The analysis carried out in Section 2.1 suggests that the introduction of the TIPP strategy contributes to preventing the occurrence of cash-lock events. Unfortunately, this is not always sufficient: as an example, we proceed by running a study on a different time frame. More precisely, in this subsection, we report a comparison between CPPI and TIPP strategies when the market shows a falling–and–rising trend at the end of the period under investigation. The results are depicted in Figure 2, where both the evolution of the portfolios and the corresponding exposure to the risky asset for each investment strategy are shown, based on different rebalancing disciplines.

This time horizon features two main characteristics. On the one hand, it depicts a long upward trend (from 2012 to early 2020), ending in a turbulent phase due to the Covid-19 outbreak, while, on the other hand, this decade is typified by constant risk-free interest rates close to zero. Because of these issues, the CPPI strategy behaves in a very particular way: with the floor remaining almost constant throughout the investment horizon and at very low levels, the exposure to the risky asset is kept constant and equal to 100%, perfectly matching the pure risky asset investment. Moreover,

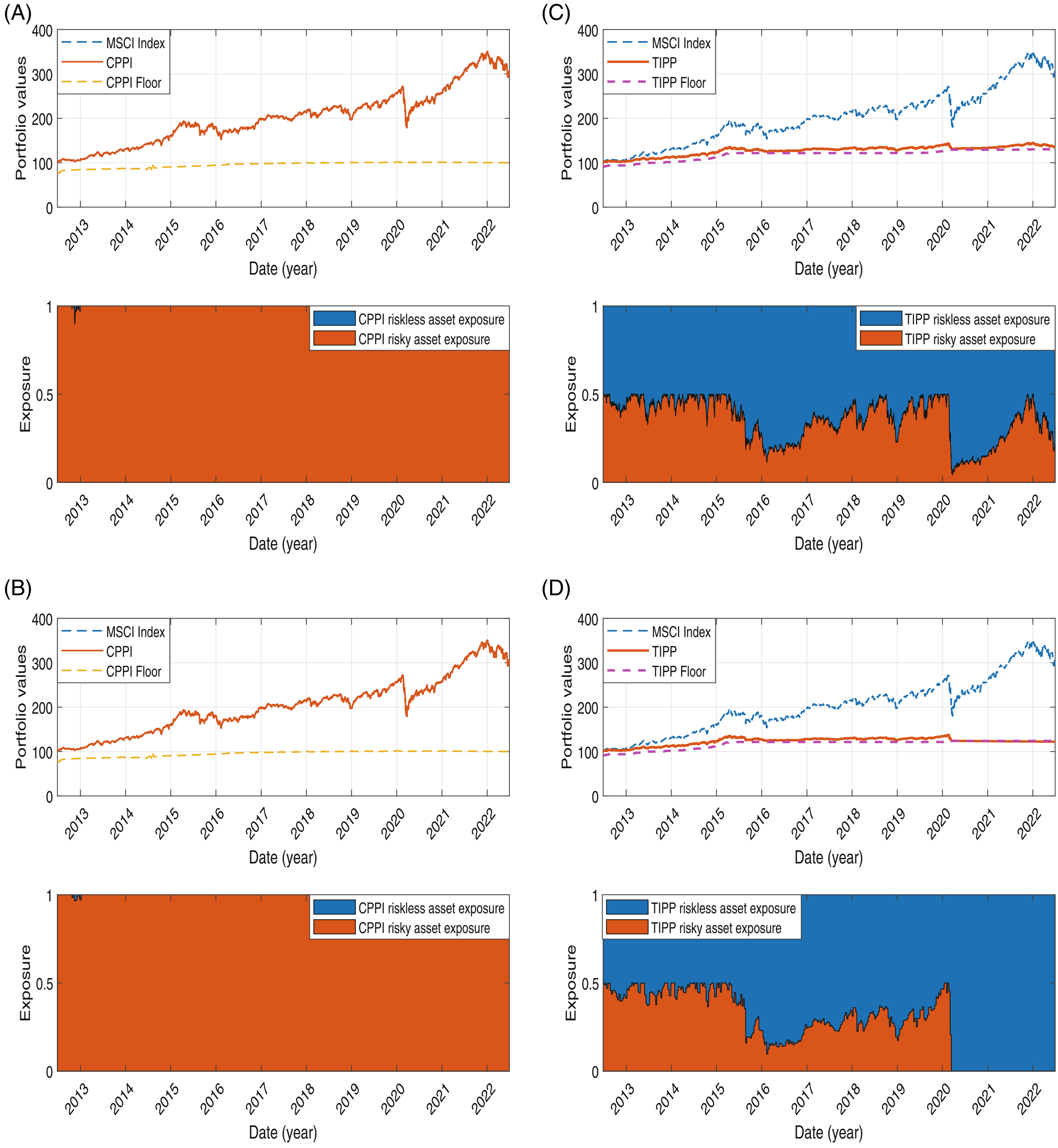

FIGURE 2 Historical simulation and investment exposures for the constant proportion-portfolio insurance (CPPI) strategy (left charts) and the time-invariant portfolio protection (TIPP) strategy (right charts) linked to the MSCI equity index. The time window ranges between June 29, 2012, and June 29, 2022. At the top of each subfigure, we show the investment strategies' trend, while at the bottom we depict the assets' exposures. (A) CPPI with 1-day rebalancing frequency; (B) CPPI with 1-week rebalancing frequency; (C) TIPP with 1-day rebalancing frequency; (D) TIPP with 1-week rebalancing frequency.

due to the very low level of the floor, the negative shock due to the pandemic outbreak does not affect the CPPI portfolio, guaranteeing a full investment in the risky security (and a consequent maximum exposure), as it is shown in Figure 2A. Thus, although the performance of the CPPI strategy is very high within such a market scenario, the level of protection the strategy should inherently provide is zero. Consequently, the CPPI mechanism fails to meet the demands of potential investors who are seeking hedging against market risk. The same occurs with weekly rebalancing frequencies, as it is shown in Figure 2B. This is another reason for deciding to replace the CPPI strategy with the TIPP mechanism.

Indeed, inspecting Figure 2C, we note that the TIPP portfolio is no longer overlapping the risky security as was the case with the CPPI logic. On the contrary, it always stays close to the floor. Even, though the floor is reached during the peak of the crisis, the TIPP mechanism allows the portfolio to bounce off the floor (thanks also to a high rebalancing frequency). This allows the investor to *breathe a sigh of relief*: as proof of this, we note that the risky exposure is between 11% and 50% over the entire time horizon under consideration, with one exception: a 4%-downward spike coinciding with the lockdown in most European countries. Thereafter, it nicely recovers to pre-Covid levels.

Comparing the returns of the two proposed strategies, we also notice that the annualized return of TIPP is equal to 3.12%, that is, slightly more than a quarter of the annualized return of CPPI (equal to 11.72%). In this case, the overall return is less than the CPPI one, but it provides a consistent level of protection. For this reason, even when the interest rate level is zero, TIPP meets the needs of an investor whose only goal is to be always hedged from adverse market trends.

On the other hand, when we decide to rebalance the portfolio once a week, we observe an even more extreme behavior: the outbreak of the pandemic is once again and even more effectively a turning point in investments, the latter being subject to the cash-lock risk: as shown in Figure 2D, once the floor is reached, the TIPP portfolio is no longer able to bounce off this lower limit. Consequently, the allocation shifts entirely to the riskless stock, eliminating any equity market participation in the event of favorable market conditions such as the market rise in early 2022.

2.3 | Historical simulations with guaranteed minimum equity exposure

As shown in Sections 2.1 and 2.2, the TIPP exposure to the risky asset is generally lower than the CPPI one, and changes more regularly over time. This is due to a higher level of protection that the TIPP allocation strategy can guarantee. Furthermore, previous historical simulations have revealed that the introduction of the TIPP mechanism in place of the CPPI logic is not enough to avoid cash-lock events.

This further implies the need to introduce the GMEE. We simulate the TIPP with GMEE over the same historical data as before, to highlight the effect of introducing such a new threshold, choosing a value for the guaranteed minimum equity exposure equal to $\alpha_{\min} = 30\%$. The results are shown in Figure 3, where both the evolution of the portfolios and the exposure to the risky asset are shown for each investment strategy, based on different rebalancing disciplines.

We start by studying the 2007–2017 decade, assuming a daily portfolio rebalancing in Figure 3A. We note an extreme variability of the TIPP portfolio during the most turbulent phase of the market. Such instability culminates in breaking through the floor. Therefore, we argue that the presence of the guaranteed minimum equity exposure induces a transitory cash-lock effect. However, the intrinsic mechanism of the TIPP logic with the GMEE ensures a prompt achievement of an adequate safety level, such that the risky security remains mostly fixed (at the level of the value assumed for the GMEE, or 30%). The same pattern occurs also with weekly rebalancing, as shown in Figure 3C. The role played by GMEE in defending the investment from preventing equity market participation is even more highlighted during the 2012–2022 decade. From Figure 3D we observe that the presence of a $\alpha_{\min}$ enables to stem the losses in case of strong market turbulence (see, e.g., the beginning of the Covid-19 pandemic). Therefore, the GMEE acts as a buffer and guarantees participation in bull markets. For ease of reading, we summarize the strategies' performances w.r.t. the time windows used for the historical simulations. More precisely, we report the 2007–2017 analysis in Table 2 and the 2012–2022 analysis in Table 3.

3 | OPTION PRICING WITH PORTFOLIO INSURANCE STRATEGIES

Portfolio insurance strategies are designed to secure, at maturity, at least part of the capital invested at the beginning of the financial transaction. On the other hand, due to the cash-lock event, the same strategies do not guarantee to benefit from higher returns when the market experiences significant upturns. As discussed in Section 2, the introduction

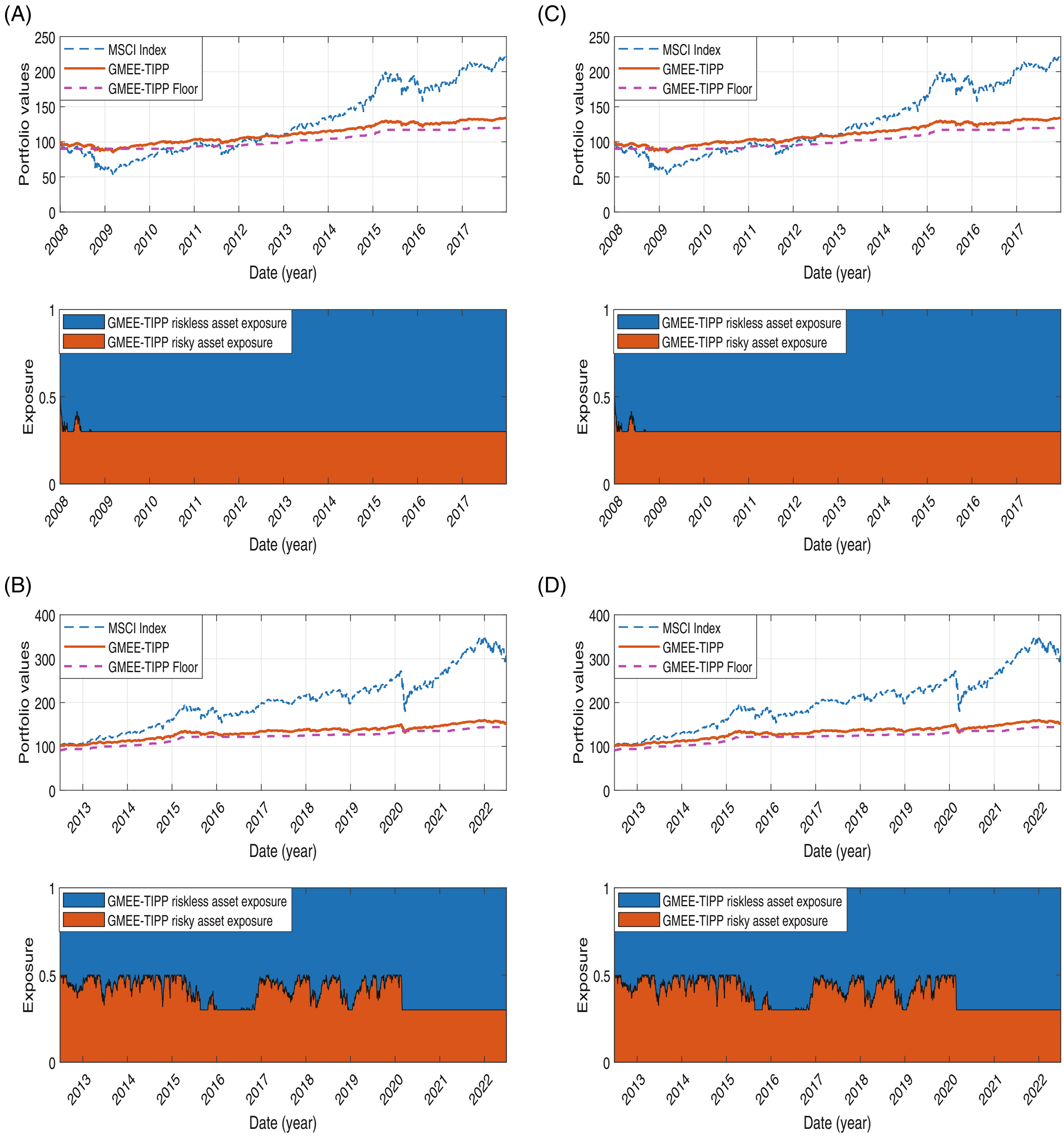

FIGURE 3 Historical simulation and risky exposures for time-invariant portfolio protection with guaranteed minimum equity exposure (GMEE-TIPP), linked to MSCI equity index. The time windows are December 31, 2007–December 28, 2012 (top charts), January 1, 2003–December 31, 2012 (middle charts), January 1, 2012–June 29, 2022 (bottom charts). (A) GMEE-TIPP with 1-day rebalancing frequency (2007–2017); (B) GMEE-TIPP with 1-day rebalancing frequency (2012–2022); (C) GMEE-TIPP with 1-week rebalancing frequency (2007–2017); (D) GMEE-TIPP with 1-week rebalancing frequency (2012–2022).

TABLE 2 Risk-return parameters (in %) of MSCI World Net (Eur), constant Proportion-portfolio insurance (CPPI) strategy, TIPP strategy, G-TIPP strategy with daily (d) and weekly (w) rebalance frequencies.

Investment strategy	MSCI World	TIPP (d)	CPPI (d)	G-TIPP (d)	TIPP (w)	CPPI (w)	G-TIPP (w)
Annualized return (%)	8.24	2.92	−3.89	2.96	2.63	−3.72	2.96
Max loss 1-day (%)	−8.56	−1.53	−8.56	−2.56	−1.87	−8.56	−2.56
Annualized volatility (%)	17.67	4.53	11.09	5.33	4.67	11.12	5.34
Average exposure (%)	100.00	32.97	10.47	30.18	32.35	10.69	30.18
Maximum average exposure (%)	100.00	50.00	100.00	50.00	50.00	100.00	50.00
Minimum average exposure (%)	100.00	3.07	0.00	30.00	3.55	0.00	30.00

Note: The time window ranges from December 31, 2007 to December 28, 2017.

TABLE 3 Risk-return parameters (in %) of MSCI World Net (Eur), constant Proportion-portfolio insurance (CPPI) strategy, time-invariant portfolio protection (TIPP) strategy, G-TIPP strategy with daily (d) and weekly (w) rebalance frequencies.

Investment strategy	MSCI World	TIPP (d)	CPPI (d)	G-TIPP (d)	TIPP (w)	CPPI (w)	G-TIPP (w)
Annualized return (%)	11.72	3.34	11.72	5.43	3.09	11.72	5.43
Max loss 1-day (%)	−8.09	−1.60	−8.09	−2.43	−2.18	−8.09	−2.43
Annualized volatility (%)	14.73	4.58	14.73	5.84	4.49	14.73	5.84
Average exposure (%)	100.00	29.41	100.00	42.09	21.85	100.00	42.09
Maximum average exposure (%)	100.00	80.00	100.00	80.00	80.00	100.00	80.00
Minimum average exposure (%)	100.00	0.00	100.00	30.00	0.00	100.00	30.00

Note: The time window ranges from June 29, 2012, to June 29, 2022.

of the guaranteed minimum equity exposure avoids the cash-lock problem in bullish markets, but de facto significantly reduces the capital protection effect. This is why both the literature and the industry propose ad hoc structured products.

3.1 | Investments with capital protection and market participation

A structured investment product with capital protection, say $A = \{A\}_{t\in[0,T]}$, is designed so that the initial investment V_0 is split into two parts: the first one is invested in a riskless asset to ensure a prespecified capital protection level (PL), while what remains is invested in financial instruments allowing direct participation in a risky asset S. Our assumption is that the risky asset is represented by an equity index, whereas the riskless asset is represented by a cash account or a liquidity index, such as EONIA. This choice aligns with the structure of the investment strategies employed, where the riskless asset requires a duration of zero by definition, to prevent any inconsistency between the guarantee period and the duration of the riskless asset with embedded risk.*

A typical payoff at maturity T from such a structured product with capital protection is given by

$$A_T = \max\left\{V_0 \cdot \mathrm{PL}, V_0 \cdot \left(\mathrm{PL} + \xi\frac{S_T - S_0}{S_0}\right)\right\} = \mathrm{PL} \cdot V_0 + \xi \cdot \left(\frac{V_0}{S_0}S_T - V_0\right)^+. \tag{12}$$

The first term in (12), namely $\mathrm{PL} \cdot V_0$, is the riskless asset value at maturity T, while the second term is the payoff of a European call option linked to the underlying $\frac{V_0}{S_0}S$, with strike price V_0. Hence, to get the payoff at maturity the issuer should invest at time $t = 0$ the amount $\mathrm{PL} \cdot V_0 \cdot \exp\left\{-\int_0^T r_s ds\right\}$ in the riskless asset. The remaining part of the initial investment V_0 represents the so-called risk budget (RB), namely

$$\mathrm{RB} = V_0 \cdot \left(1 - \mathrm{PL} \cdot \exp\left\{-\int_0^T r_s \mathrm{d}s\right\}\right). \tag{13}$$

*We thank the referees for emphasizing the need to choose risk-free assets other than fixed-income or term deposits.

The parameter ξ appearing in (12) is the so-called market *participation rate*, see, for example, Reference 7 for further details. The financial instrument denoted by Equation (12) bears a resemblance to a variable annuity that incorporates a GMAB-type rider. It pertains to a unit-linked product that disburses to the annuitant the value of the contract at the time of retirement, which could potentially be augmented by an extra premium, see, for example, Reference 24 for further details.

From (12) we find out that the higher the protection level, the higher the protection to the capital initially invested V_0 in case of negative returns from the investment in an equity index. On the other hand, a high participation rate ξ increases the overall payoff of the structured investment product in case of positive returns from the risky asset investment. Thus, the issuer of a structured investment product with capital protection considers his/her product to be attractive if it either admits (i) a high capital protection level PL or *(ii)* a high participation rate ξ. However, the aforementioned features are strongly interrelated: a high capital protection level negatively affects the amount of the available risk budget, leading to a reduction in the participation rate ξ, especially within low interest rate levels.

Therefore, it is crucial to obtain an explicit expression for the participation rate. Concerning the latter, since the underlying risky asset payoff reads

$$g_{dp}(x) = \max\left\{\frac{V_0}{S_0}x - V_0, 0\right\} = V_0 \cdot f_{dp}(x), \tag{14}$$

where

$$f_{dp}(x) = \max\left\{\frac{x}{S_0} - 1, 0\right\} \tag{15}$$

is the payoff function of an ATM-European call option with the underlying risky asset S, then denoting by $\mathcal{O}(g_{dp})$ (resp., $\mathcal{O}(f_{dp})$) the no-arbitrage price at time $t = 0$ of the ATM-European call option with the payoff function g_{dp} (resp., f_{dp}), we have

$$\mathcal{O}(g_{dp}) = V_0 \cdot \mathcal{O}(f_{dp}). \tag{16}$$

Hence, the participation rate ξ equals the number of units of the option with payoff g_{dp} that can be purchased, with the available risk budget, at time $t = 0$, which is given by

$$\xi = \frac{RB}{\mathcal{O}(g_{dp})} = \frac{V_0 \cdot \left(1 - \text{PL} \cdot \exp\left\{-\int_0^T r_s \text{d}s\right\}\right)}{V_0 \cdot \mathcal{O}(f_{dp})} = \frac{1 - \text{PL} \cdot \exp\left\{-\int_0^T r_s \text{d}s\right\}}{\mathcal{O}(f_{dp})}. \tag{17}$$

From (17) we find that the participation rate can be increased by either increasing the risk budget or decreasing the price of the embedded ATM-European call option. More precisely, we observe that the former is mainly affected by the interest rate, while the latter is primarily driven by stock volatility.

Hence, in a market scenario characterized by small interest rates and high volatility, issuers may maintain a sizeable participation rate by reducing the protection level. Unfortunately, such a choice would make these kinds of structured products less attractive to the investor. To overcome the previous drawback, the issuer could invest in derivative products, such that:

(**C1**) the price of the embedded option should not significantly depend on the volatility of the underlying asset, and
(**C2**) the price of the embedded option should decrease with decreasing market interest rate and it should increase with increasing market interest rate.

To meet (**C1**) and (**C2**), providing capital protection also guaranteeing market participation, we introduce a new structured product in which the underlying of embedded option is the TIPP strategy endowed with the guaranteed minimum equity exposure. Investors interested in these structured products are getting a fund or mandate, consisting of an asset and an option written on G-TIPP. It is worth noting that the options on G-TIPP (as well as options on G-CPPI) are not traded on the market. However, the latter can be bought by an investment bank or a re-insurance partner, who is guaranteeing the TIPP-GMEE strategy payoff. The latter must be achieved by the issuer by implementing a dynamic hedging strategy (according to either CPPI or TIPP mechanism), also considering the probability that the OTC option ends in the money or not. Hence, although the TIPP-GMEE is not traded at a public exchange, it is a full-fledged derivative product, which can be purchased by different market participants and financial engineering groups.

3.2 | The model

We start by describing the financial market model we use for the rest of this paper.

Let $0 < T < \infty$ be the investment time horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ the filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ the filtration generated by the whole set of processes involved, hence automatically guaranteeing their adaptness to the latter.

Within such a framework, we assume that the risky asset is described by a real-valued stochastic process $S = \{S_t\}_{t\in[0,T]}$, solution to the following stochastic differential equation

$$dS_t = r_t S_t dt + \sqrt{v_t} S_t dW_t^S, \ t \in [0,T], \tag{18}$$

where the drift, resp. the volatility of the risky asset with dynamic as in (18), is driven by the solution $r = \{r\}_{t\in[0,T]}$ of the Vasicek model, see Reference 25:

$$dr_t = \nu(\beta - r_t)dt + \sigma_r dW_t^r, \ t \in [0,T], \tag{19}$$

resp. by the solution $v = \{v_t\}_{t\in[0,T]}$ of the CIR model, see Reference 26:

$$dv_t = k(\theta - v_t)dt + \sigma_v \sqrt{v_t} dW_t^v, \ t \in [0,T], \tag{20}$$

where $\beta \in \mathbb{R}$ is the long-run mean of the interest rate, $\nu \in \mathbb{R}^+$ is the speed of reversion, $\sigma_r \in \mathbb{R}^+$ is the instantaneous volatility of the interest rate, $k \in \mathbb{R}^+$ is the speed of reversion of the variance process, $\theta \in \mathbb{R}$ is the long-term mean reversion level of the variance process, $\sigma_v \in \mathbb{R}^+$, also known as vol-of-vol (volatility of volatility), indicates the SD of the process v.

The $\mathbb{F}$-adapted Wiener processes $W^S = \{W_t^S\}_{t\in[0,T]}$, $W^v = \{W_t^v\}_{t\in[0,T]}$, $W^r = \{W_t^r\}_{t\in[0,T]}$ appearing in (18), (19), and (20), respectively, are assumed to be correlated, w.r.t. the reference measure $\mathbb{Q}$, as follows

$$\langle dW^S, dW^v \rangle_t = \rho_{S,v} dt, \langle dW^S, dW^r \rangle_t = \rho_{S,r} dt, \langle dW^v, dW^r \rangle_t = \rho_{v,r} dt = \rho_{S,v}\rho_{S,r} dt, \tag{21}$$

being $\rho_{S,v}$ $\rho_{S,r}$, $\rho_{v,r} \in [-1,1]$ the corresponding correlation coefficients.

The riskless asset $B = \{B_t\}_{t\in[0,T]}$ and the corresponding stochastic discount factor $D = \{D_t\}_{t\in[0,T]}$ are, respectively, given by

$$dB_t = r_t B_t dt, \tag{22}$$

$$D_t = \exp\left\{-\int_t^T r_s ds\right\}, \tag{23}$$

where r evolves according to Equation (19).

Consider a European-type contingent claim with maturity T and the risky asset S as underlying with payoff function f_{dp} given in (15). Due to the market incompleteness, there are many possible choices for a risk-neutral measure. We single out the measure $\mathbb{Q}$ such that the discounted asset price process $\tilde{S} = \left\{\tilde{S}_t\right\}_{t\in[0,T]}$, with $\tilde{S}_t = D(t,T)S_t$, for $t \in [0,T]$, is a $\mathbb{Q}$-martingale. Then, the no-arbitrage price at $t = 0$ of the contingent claim is

$$\mathcal{O}_0 = \mathbb{E}^{\mathbb{Q}}\left[f_{dp}(\tilde{S}_T)\right], \tag{24}$$

where we indicate with $\mathbb{E}^{\mathbb{Q}}$ the expectation w.r.t. the risk-neutral measure $\mathbb{Q}$. For our purposes, in addition to the derivative whose price was defined in (24), we consider additional contingent claims with maturity T, whose underlyings are the portfolio insurance strategies introduced in Section 2, namely the CPPI, TIPP, G–CPPI, G–TIPP strategies. To make the notation as concise as possible, in the following we will indicate with $V^h = \left\{V_t^h\right\}_{t\in[0,T]}$ the real-valued stochastic processes describing the evolution of such portfolios, with $h \in \{\text{CPPI, G–CPPI, TIPP, G–TIPP}\}$.

As argued in Reference 27, the most realistic way to implement portfolio insurance strategies with continuous-time dynamics is to consider discrete-time rebalancing. Namely, we assume that the strategies can be modified at dates $0 = t_0 < t_1 < \cdots < t_n = T$. As a consequence, the portfolio insurance strategies at time t are of buy-and-hold-type, for any $t \in [t_j, t_{j+1})$, $j = 0, \ldots, n$. Hence, the portfolio process V_t^h, can be written as

$$V_t^h = \frac{E_{t_j}^h}{S_{t_j}} S_t + \frac{V_{t_j}^h - E_{t_j}^h}{B_{t_j}} B_t = \beta_{t_j}^h S_t + \gamma_{t_j}^h B_t,\ \ t \in [t_j, t_{j+1}),\ j = 0, \ldots, n, \tag{25}$$

where $\beta_{t_j}^h$ (resp., $\gamma_{t_j}^h$) represents the shares of risky asset S (resp., riskless asset B) held in the portfolios for all $t \in [t_j, t_{j+1})$, such that

$$\lim_{t\to t_j^-} V_t^h = \lim_{t\to t_j^+} V_t^h, j = 1, \ldots, n, \tag{26}$$

for $h \in \{\text{CPPI, G–CPPI, TIPP, G–TIPP}\}$.

Finally, consider the real-valued stochastic process $\tilde{V}^h = \left\{\tilde{V}_t^h\right\}_{t\in[0,T]}$, with $\tilde{V}_t = D(t,T)V_t$, describing the evolution of the discounted version of the previous strategies.

Due to the continuity property (26), it is straightforward to show that the discounted value process $\tilde{V}$ is a $\mathbb{Q}$-martingale. Then, the no-arbitrage price of our contingent claims at time $t = 0$ reads as follows

$$\tilde{\mathcal{O}}_0^h = \mathbb{E}^{\mathbb{Q}}\left[f_{dp}\left(\tilde{V}_T^h\right)\right], \tag{27}$$

for all $h \in \{\text{CPPI, G–CPPI, TIPP, G–TIPP}\}$.

Remark 2. Within this framework, the value of the portfolio insurance strategies endowed with GMEE can be obtained in an explicit form. More precisely, the G-TIPP process, for any $t \in [t_j, t_{j+1})$, $j = 0, \ldots, n$, can be written as

$$V_t^{\text{G–TIPP}} = E_{t_j}^{\text{G–TIPP}} \exp\left\{\int_{t_j}^t \left(r_s - \frac{\nu_s}{2}\right) ds + \int_{t_j}^t \sqrt{\nu_s} dW_s^S\right\} + \left(V_{t_j}^{\text{G–TIPP}} - E_{t_j}^{\text{G–TIPP}}\right) \exp\left\{\int_{t_j}^t r_s ds\right\}, \tag{28}$$

with

$$\begin{cases} E_{t_j}^{\text{G–TIPP}} = \max\left\{\min\left\{L_{\max} \cdot V_{t_j}^{\text{G–TIPP}}, M \cdot \left(V_{t_j}^{\text{G–TIPP}} - F_{t_j}^{\text{G–TIPP}}\right)^+\right\}, \alpha_{\min} \cdot V_{t_j}^{\text{G–TIPP}}\right\} \\ F_{t_j}^{\text{G–TIPP}} = \max\left\{F_{t_j}^{\text{G–CPPI}}, \text{PL} \cdot \sup_{s\le t_j} V_{t_j}^{\text{G–TIPP}}\right\} \end{cases}. \tag{29}$$

Analogously, for any $t \in [t_j, t_{j+1})$, $j = 0, \ldots, n$, the G-CPPI process is

$$V_t^{\text{G–CPPI}} = E_{t_j}^{\text{G–CPPI}} \exp\left\{\int_{t_j}^t \left(r_s - \frac{\nu_s}{2}\right) ds + \int_{t_j}^t \sqrt{\nu_s} dW_s^S\right\} + \left(V_{t_j}^{\text{G–CPPI}} - E_{t_j}^{\text{G–TIPP}}\right) \exp\left\{\int_{t_j}^t r_s ds\right\}, \tag{30}$$

with

$$\begin{cases} E_{t_j}^{\text{G–CPPI}} = \max\left\{\min\left\{L_{\max} \cdot V_{t_j}^{\text{G–CPPI}}, M \cdot \left(V_{t_j}^{\text{G–CPPI}} - F_{t_j}^{\text{G–CPPI}}\right)^+\right\}, \alpha_{\min} \cdot V_{t_j}^{\text{G–CPPI}}\right\} \\ F_{t_j}^{\text{G–CPPI}} = F_0 \exp\left\{\int_{t_0}^{t_j} r_s ds\right\} \end{cases}, \tag{31}$$

hence obtaining the standard CPPI and TIPP strategies, by taking $\alpha_{\min} = 0$ in (31) and (29).

TABLE 4 Parameters used in the numerical experiments for the stochastic interest rate model (Vasicek) and the stochastic volatility model (Heston model).

	Vasicek model	Heston model
Long-run mean	$\beta = 0.05$	$\theta = 0.04$
Rate of mean reversion	$k = 1.25$	$\nu = 1.25$
Volatility	$\sigma_r = 0.025$	$\sigma_v = 0.2$
Correlation	$\rho_{S,r} = -0.2$	$\rho_{S,v} = -0.5$

3.3 | Numerical results

In what follows, because of the nonexistence of a closed-form pricing formula of the hybrid Vasicek–Heston model we are considering, we rely on numerical methods to evaluate expressions (24) and (27), exploiting parameter values provided in Reference 2, see Table 4, as to illustrate the benefits of using financial derivatives that are linked to portfolio insurance strategies. In particular, we compare the values of the ATM European call options on risky asset S with the ATM European call options when the underlying is either the CPPI strategy, the TIPP strategy, and their modified version, that is the G-CPPI and the G-TIPP strategies.

More precisely, we consider Monte Carlo simulations, according to the following scheme:

Step 1. given a daily partition $\pi = \{t_0, t_1, \dots, t_n\}$ of $[0, T]$, with $\Delta t := |t_j - t_{j-1}| = \frac{T}{j}$, for $j = 0, \dots, n$, we discretize the SDEs (18), (19), and (20) according to the Milstein scheme, to obtain the simulated paths for the underlying risky asset;

Step 2. we simulate the trajectories of the portfolio insurance strategies, according to the formulae provided in Section 2, for the floor, the cushion, the exposure, and the portfolio value, for each strategy. It is worth noting that, when we consider nondaily rebalancing frequencies, we must update the risky exposure as follows

$$E_{t_{j+1}} = \begin{cases} \max\left\{\min\left\{L_{\max}, M\frac{C_{j+1}}{V_j}\right\}, \alpha_{min}\right\} & \text{if } j+1 \equiv 0 \bmod \tau^* \\ E_{t_j} & \text{otherwise} \end{cases}; \tag{32}$$

where τ^* represents the fixed rebalancing frequency;

Step 3. we compute the price of the ATM European call option by using the trapezoidal rule to evaluate the stochastic discount factor.

We compare the option prices by considering different levels of the guaranteed minimum equity exposure $\alpha_{\min}$ and several time horizons T, obtained results have been reported in Figure 4.

Starting with an initial volatility level of 20% and an initial interest rate equal to 0%, we observe that the price obtained by considering the risky asset as the derivative underlying is higher than the ones obtained by taking any portfolio insurance strategy as underlying, whatever the maturity T and the GMEE level $\alpha_{\min}$. Our results show that using European call options linked to portfolio insurance strategies within structured investment products would lead to a significant increase in the participation rate, for a given capital protection level (and therefore for the same risk budget). Furthermore, by looking at the European call option prices linked to portfolio insurance strategies we obtained, we notice that the TIPP options are always cheaper than the CPPI ones. This result is confirmed with any investment time horizon and strategy parameters, that is, multiplier, protection level, and rebalancing frequency). Therefore, according to the reasoning described in Section 3.1, we argue that it is more convenient to invest in a structured product where the risky component is a European call option linked to the TIPP logic, since this improves the attractiveness for investors, by increasing the related participation rate (thanks to Equation 17). Such a result still holds even when we consider European call options associated with portfolio insurance strategies endowed with GMEE, that is, the G-TIPP and the G-CPPI. The option on G-TIPP is cheaper than the one written on G-CPPI.

We further observe that the G-TIPP (resp. G-CPPI) call option price is slightly higher than the option associated with the standard TIPP (resp. standard CPPI) logic. This is due to the introduction of the GMEE threshold, which triggers a

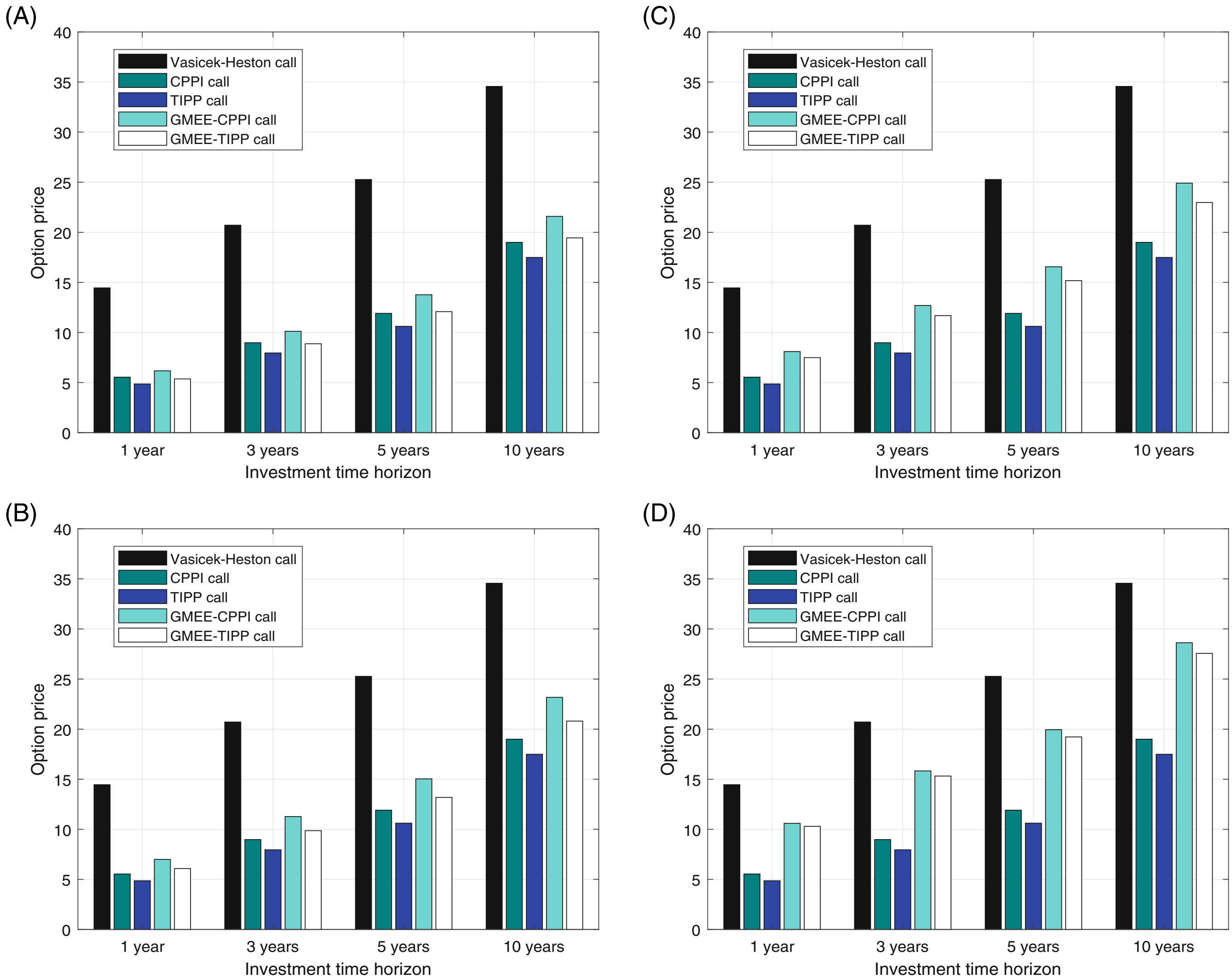

FIGURE 4 Comparison among ATM call option pricing under different underlyings, maturities, and level of guaranteed minimum equity exposure. The model parameters are: $v_0 = 20\%$, $k = 1.25$, $\theta = v_0^2$, $\sigma_v = 0.2$, $\rho_{S,v} = -0.5$ (Heston), $r_0 = 0$, $\beta = 0.02$, $\nu = 1.25$, $\sigma_r = 0.025$, $\rho_{S,r} = -0.2$ (Vasicek). The remaining parameters associated to the portfolio insurance strategies are: $L_{max} = 100\%$, $M = 4$, PL = 90%, $\alpha_{min} = \{0.3,\ 0.4,\ 0.5,\ 0.7\}$. We performed 10^5 Monte Carlo simulations with $S_0 = 100$. (A) Call option pricing with different underlyings, $\alpha_{min} = 30\%$; (B) Call option pricing with different underlyings, $\alpha_{min} = 40\%$; (C) Call option pricing with different underlyings, $\alpha_{min} = 50\%$; (D) Call option pricing with different underlyings, $\alpha_{min} = 70\%$.

larger number of paths ending above the strike price (given by the capital V_0 to be protected) at maturity, leading to an increase in the corresponding option price at $t = 0$. Therefore, at the TIPP optionsderate decrease in the participation rate, the introduction of the threshold α_{min} can solve one of the main problems related to portfolio insurance strategies, namely the achievement of equity market participation.

To conclude, since the option written on G-TIPP solves the problem of cash-lock and simultaneously leads to the smallest decrease in the participation rate (compared to G-CPPI), it is more suitable to be included in a structured investment product with capital protection.

3.4 | Sensitivity analysis: option pricing versus model parameters

In Section 3.3, we showed that the highest participation rate is attained by considering structured products with the risky component given by European ATM call options linked to the G-TIPP. To confirm that such options are a winning

TABLE 5 ATM call option prices on risky asset, G-CPPI and G-TIPP, for different values of initial interest rate (r_0) and initial annual volatility (ν_0).

Panel A: Option on pure risky asset						
	Initial annual volatility (ν_0)					
Initial interest rate (r_0)	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**RPR$^\nu$**
0.01	10.08	14.51	18.26	21.58	24.68	1.45
0.03	11.02	15.44	19.14	22.48	25.62	1.32
0.05	12.12	16.42	20.09	23.29	26.25	1.17
0.07	13.24	17.40	20.80	24.03	26.96	1.04
0.10	14.94	18.86	22.27	25.41	28.08	0.88
RPR^r	0.48	0.30	0.22	0.18	0.14	
Panel B: Option on G-CPPI						
	Initial annual volatility (ν_0)					
Initial interest rate (r_0)	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	***RPR$^\nu$***
0.01	4.47	6.34	8.03	9.39	10.60	1.37
0.03	5.80	7.73	9.33	10.84	12.21	1.10
0.05	7.30	9.16	10.83	12.22	13.39	0.83
0.07	8.81	10.59	12.08	13.46	14.66	0.66
0.10	11.07	12.75	14.26	15.64	16.64	0.50
RPR^r	1.48	1.01	0.78	0.67	0.57	
Panel C: Option on G-TIPP						
	Initial annual volatility (ν_0)					
Initial interest rate (r_0)	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**RPR$^\nu$**
0.01	3.99	5.47	6.71	7.79	8.81	1.21
0.03	5.19	6.63	7.84	8.91	9.90	0.91
0.05	6.51	7.86	9.02	10.03	10.99	0.69
0.07	7.90	9.10	10.15	11.15	12.07	0.53
0.10	10.03	11.03	11.98	12.91	13.73	0.37
RPR^r	1.52	1.02	0.79	0.66	0.56	

Notes: The model parameters are: $k = 1.25$, $\theta = \nu_0^2$, $\sigma_\nu = 0.2$, $\rho_{S,\nu} = -0.5$ (Heston), $\beta = r_0$, $\nu = 1.25$, $\sigma_r = 0.025$, $\rho_{S,r} = -0.2$ (Vasicek). The remaining parameters associated to the portfolio insurance strategies are: $L_{\max} = 100\%$, $M = 4$, $\alpha_{min} = 30$, % PL = 90%. We performed 10^5 MC simulations with $S_0 = 100$.

choice, we need to verify that properties (***C1***) and (***C2***) are valid. The latter implies that the corresponding prices are less influenced by the variance of the risky asset and more influenced by the interest rate levels. This is why we evaluate the prices of ATM call options linked to the pure risky asset, the G-CPPI strategy, and the G-TIPP mechanism, for different combinations of initial interest rate and variance, related results being shown in Table 5.

Moreover, we quantify the dependence of option prices on both the initial variance level of the underlying risky asset and the initial interest rate. To do this, we resort to the so called *relative range of option prices* (RPR), see, for example, Reference 2 for further details. We first define the relative range of option prices for a fixed initial interest rate level as

$$\text{RPR}^\nu := \frac{\mathcal{O}_0(\nu_0^{\max}) - \mathcal{O}_0(\nu_0^{\min})}{\mathcal{O}_0(\nu_0^{\min})}, \tag{33}$$

where $v_0^{\max}$ (resp., $v_0^{\min}$) is the highest (resp., the smallest) variance level we consider in the numerical analysis, see last column in Table 5 for results.

We observe the following:

(i) the highest RPR^v value is obtained for call options linked to the pure risky asset for given initial interest rates. Therefore, we conclude that the options linked to portfolio insurance strategies are less affected by the initial variance of the underlying risky asset;
(ii) the RPR^v associated with G-TIPP options is always lower than the RPR^v for G-CPPI options. Such a result stems from the floor rebalancing mechanism within the G-TIPP logic, which implies a smaller exposure to the risky asset. Therefore, we get an overall lower variance for the G-TIPP logic than the G-CPPI one, making the G-TIPP options less affected by the level of the initial variance.

We now study the connection between the option price and the interest rate initial level. Looking at Table 5, we note that option prices increase as the initial level of the interest rate rises, regardless of the type of underlying and the initial level of variance. To quantify the dependence of the option price on the interest rate levels, we compute the relative range of option prices for a fixed initial variance level as

$$\text{RPR}^r := \frac{\mathcal{O}(r_0^{\max}) - \mathcal{O}(r_0^{\min})}{\mathcal{O}(r_0^{\min})}, \tag{34}$$

where $r_0^{\max}$ (resp., $r_0^{\min}$) is the highest (resp., the smallest) interest rate level we consider in the numerical analysis. The results are depicted in the last row of each panel in Table 5. In particular, for a fixed level of initial variance, the lowest RPR^r is attained for options on purely risky stocks. Therefore, the price of the latter is much less sensitive to interest rate increases. We also observe that RPR^r associated with the G-CPPI strategy is comparable to that of G-TIPP. Therefore, concerning property **(C2)**, we can argue that both strategies are suitable to be included in a structured product with capital protection. However, the only option that also fulfills condition **(C1)** and provides the highest participation rate is the one written on the G-TIPP. This makes the derivative the most attractive structured product for investors.

3.5 | Sensitivity analysis: option pricing versus portfolio insurance strategies parameters

The results in Section 3.4 show that the G-TIPP option is better suited than any other to be included in a structured product. However, the price of this type of option is highly influenced by the exogenous parameters characterizing the strategy. To strengthen the results obtained in Section 3.4, we show that the lowest prices are achieved when we consider the G-TIPP logic as underlying, regardless of the protection level, the multiplier, or the rebalancing frequency, further proving that the properties (**C1**) and (**C2**) are still preserved.

3.5.1 | The role of the multiplier

We consider the European call option prices as a function of the multiplier, for different values of the initial volatility. The results are depicted in Table 6.

As a preliminary comment, we observe that for any v_0 value, the greater the multiplier, the more expensive the European call option price, whatever the portfolio strategy involved. This is not surprising, since M is an amplifying factor for the risk budget, and directly affects the risky exposure. However, we observe that the options on G-TIPP (Panel C) is always less expensive than the G-CPPI one (Panel B), for each level of the multiplier considered. Therefore, this reinforces the previous results: the inclusion of the G-TIPP mechanism within a structured investment product with capital protection is able to increase the corresponding participation rate, regardless of the multiplier set at $t = 0$ by the fund manager. Furthermore, by analyzing the RPR^v, we note that such an indicator is the smallest when we consider the G-TIPP as underlying for the ATM European call option. This implies that the option on G-TIPP is less affected by the variance levels of the underlying.

TABLE 6 ATM call option prices on pure risky asset, on the G-CPPI and the G-TIPP strategies, for different values of multiplier (M) and initial annual volatility (ν_0).

Panel A: Option on pure risky asset						
	Initial annual volatility (ν_0)					
Multiplier (M)	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	RPR^{ν}
-	9.55	14.07	17.83	21.11	24.33	1.55
Panel B: Option on G-CPPI						
	Initial annual volatility (ν_0)					
Multiplier (M)	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	RPR^{ν}
4	3.78	5.70	7.32	8.65	9.97	1.64
5	4.51	6.55	8.16	9.42	10.66	1.37
6	5.13	7.18	8.71	9.84	11.05	1.15
10	6.46	8.25	9.50	10.54	11.60	0.80
Panel C: Option on G-TIPP						
	Initial annual volatility (ν_0)					
Multiplier (M)	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	RPR^{ν}
4	3.40	4.92	6.16	7.23	8.27	1.44
5	4.00	5.65	6.94	8.04	9.10	1.28
6	4.57	6.29	7.60	8.71	9.78	1.14
10	6.35	8.07	9.26	10.16	10.99	0.73

Notes: The model parameters are: $k = 1.25$, $\theta = \nu_0^2$, $\sigma_\nu = 0.2$, $\rho_{S,\nu} = -0.5$ (Heston), $\beta = r_0$, $\nu = 1.25$, $\sigma_r = 0.025$, $\rho_{S,r} = -0.2$ (Vasicek). The remaining parameters associated to the portfolio insurance strategies are: PL = 90%, $L_{\max} = 100\%$, $\alpha_{min} = 30\%$. We performed 10^5 MC simulations with $S_0 = 100$.

3.5.2 | The role of the protection level

We consider the European call option prices as a function of the initial volatility and the protection level. The results are depicted in Table 7.

Again, the protection level impacts the risk budget. In particular, the lower the protection level, the greater the risk budget and the risky exposure. Therefore, as the protection level decreases, the European call option increases, whatever the portfolio insurance strategy involved. Also in this case we observe that the European call options written on G-TIPP are:

- less affected by the initial variance of the underlying risky asset, and
- the cheapest for any combination of protection level and initial volatility.

For this reason, also in this case they represent the best choice among the contingent claims to be included in a structured investment product with guaranteed capital.

3.5.3 | The role of the rebalancing frequency

We consider the European call option prices as a function of the initial volatility and the rebalancing frequencies. The results are depicted in Table 8.

Up to now, we have only considered daily rebalancing for the portfolio strategies involved. However, these assumptions might be violated in practice: due to transaction costs and liquidity constraints, it is not always possible to rebalance the portfolio daily. For this reason, portfolio managers might also consider lower rebalancing frequencies. This

TABLE 7 ATM call option prices on pure risky asset for different values of initial annual volatility (ν_0).

Panel A: Option on pure risky asset						
	Initial annual volatility (ν_0)					
Protection level (PL)	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**RPR$^\nu$**
-	9.55	14.07	17.83	21.11	24.33	1.55
Panel B: Option on G-CPPI						
	Initial annual volatility (ν_0)					
Protection level (PL)	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**RPR$^\nu$**
0.70	9.04	12.69	15.39	17.54	19.42	1.15
0.75	8.29	11.49	13.83	15.71	17.37	1.10
0.80	6.99	9.77	11.82	13.45	14.94	1.14
0.85	5.38	7.72	9.53	10.98	12.36	1.30
0.90	3.78	5.70	7.32	8.65	9.97	1.64
Panel C: Option on G-TIPP						
	Initial annual volatility (ν_0)					
Protection level (PL)	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**RPR$^\nu$**
0.70	9.04	12.69	15.38	17.53	19.41	1.15
0.75	8.23	11.37	13.67	15.50	17.12	1.08
0.80	6.54	9.06	10.94	12.45	13.81	1.11
0.85	4.90	6.86	8.37	9.62	10.79	1.20
0.90	3.40	4.92	6.16	7.23	8.27	1.44

Notes: ATM call option prices on G-CPPI and G-TIPP for different values of protection level (PL) and initial annual volatility (ν_0). $k = 1.25$, $\theta = \nu_0^2$, $\sigma_\nu = 0.2$, $\rho_{S,\nu} = -0.5$ (Heston), $\beta = r_0$, $\nu = 1.25$, $\sigma_r = 0.025$, $\rho_{S,r} = -0.2$ (Vasicek). The remaining parameters associated to the portfolio insurance strategies are: $M = 4$, $L_{\max} = 100\%$, $\alpha_{\min} = 30\%$. We performed 10^5 MC simulations with $S_0 = 100$.

accentuates the risk of a floor violation between two consecutive trading dates. Hence, we also consider different frequencies for portfolio rebalancing: weekly (w), biweekly (2w), monthly (m), quarterly (3m), and three-times-a-year (4m). We observe that the portfolio rebalancing impacts the European option pricing related to the strategies involved in our analysis: options get cheaper as the rebalancing frequency intensifies, due to the smaller premium required by the issuer. As a consequence, we focus on the comparison between G-TIPP and G-CPPI option prices (Panel C and B, respectively). Our findings show that even using different rebalancing disciplines, the G-TIPP call option is always lower-priced than the G-CPPI one. Such a result might be further clarified by considering the following reasoning: by construction, the European option prices on portfolio insurance strategies heavily depend on the strategy's risky exposure, seen as a metric to evaluate the cash-lock probability. The latter increases as the rebalancing frequency decreases. As seen in Section 2, the average exposure to the risky asset for the G-TIPP strategy is lower than for the G-CPPI one, hence a cash-lock event is less likely to occur. This turns into a lower price for G-TIPP. Even in this case, it will be more convenient to include the G-TIPP within a structured investment product capital as there will be an increase in the participation rate. This is true for any combination of the rebalancing frequency and the initial level of variance. Furthermore, by looking at the corresponding RPR$^\nu$ we observe that the European options linked to the G-TIPP logic are less affected by the initial variance level of the underlying risky asset.

4 | CONCLUSIONS

In this paper, we introduce a new type of portfolio insurance strategy, namely G-TIPP as an underlying asset for options embedded in structured investment products with capital protection. Through detailed numerical simulations within

TABLE 8 ATM call option prices on G-TIPP and G-CPPI for different rebalancing disciplines (f) and and initial annual volatility (v_0).

Panel A: Option on pure risky asset						
	Initial annual volatility (v_0)					
Rebalancing frequencies	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**RPR**
—	9.55	14.07	17.83	21.11	24.33	1.55
Panel B: Option on G-CPPI						
	Initial annual volatility (v_0)					
Rebalancing frequencies	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**RPR**
d	3.78	5.70	7.32	8.65	9.97	1.64
w	3.82	5.79	7.45	8.82	10.21	1.68
2w	3.86	5.88	7.60	9.05	10.44	1.71
m	3.89	5.95	7.70	9.21	10.60	1.73
3m	3.99	6.18	8.05	9.64	11.11	1.78
4m	4.01	6.19	8.08	9.72	11.16	1.78
Panel C: Option on G-TIPP						
	Initial annual volatility (v_0)					
Rebalancing frequencies	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**RPR**
d	3.40	4.92	6.16	7.23	8.27	1.44
w	3.41	4.95	6.19	7.27	8.32	1.44
2w	3.43	4.99	6.24	7.33	8.38	1.44
m	3.46	5.02	6.28	7.38	8.44	1.44
3m	3.57	5.19	6.50	7.65	8.75	1.45
4m	3.61	5.26	6.60	7.77	8.89	1.46

Notes: The model parameters are: $v_0 = 20\%$, $k = 1.25$, $\theta = v_0^2$, $\sigma_v = 0.2$, $\rho_{S,v} = -0.5$ (Heston), $r_0 = 0$, $\beta = r_0$, $\nu = 1.25$, $\sigma_r = 0.025$, $\rho_{S,r} = -0.2$ (Vasicek). The remaining parameters associated to the portfolio insurance strategies are: PL = 90% $L_{\max} = 100\%$, $M = 4$, $\alpha_{\min} = 30\%$. The rebalancing frequency is daily (d), weekly (w), biweekly (2w), monthly (m). We performed 10^5 MC simulations with $S_0 = 100$.

the hybrid Heston–Vasicek model, we obtain that European call options linked to G-TIPP strategies exhibit better characteristics than options with pure risky securities and other types of portfolio insurance strategies (such as G-CPPI) as underlying assets. In particular, we find that European call options linked to G-TIPP are cheaper than those linked to pure risky securities and G-CPPI, which leads to a significant increase in the participation rate. Furthermore, we show that options on G-TIPP are the ones less sensitive to market volatility.

Although G-TIPP options are not traded on the market, the corresponding payoff can be obtained by the issuer by implementing appropriate dynamic hedging strategies. The construction of such hedging strategies is the subject of our ongoing research.

For all these reasons, the inclusion of G-TIPP-linked options in a structured investment product with capital protection makes it possible to maintain a relatively high participation rate (capital protection being equal), regardless of the level of market volatility and interest rate levels. Therefore, we believe this new G-TIPP option represents an attractive and appropriate approach to structured products. We anticipate that, especially in a market environment characterized by low-interest rate levels and high volatility, portfolio solutions of the latter types are destinated to widely spread over markets.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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