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Fractional Greenium Surfaces with Climate Memory

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Abstract

A fractional stochastic evolution is studied for the maturity-dependent spread between green and conventional forward rates. Its Caputo resolvent links the persistence of an initial greenium shock to the local scaling of subsequent fluctuations. The nonlinear equation is globally well posed for every finite probability moment exponent at least two under globally Lipschitz coefficients. In the Gaussian benchmark, covariance asymptotics hold in the trace norm, and so apply directly to the discontinuous maturity weights defining zero-coupon ratios. At positive observation times, the local variance contains a contribution from the reweighting of the entire preceding history. The resulting temporal exponent identifies the memory order, whereas identification of the maturity order depends on information about the noise covariance. These results give a statistical interpretation of climate memory effects without equating physical-measure exponential moments with arbitrage-free prices.

Keywords: fractional stochastic partial differential equations; Caputo memory; stochastic Volterra equations; green bonds; covariance asymptotics; fractal regularity

MSC: Primary 60H15, 35R11, 91G30; Secondary 60G15, 45D05, 47D06, 91G80

1. Introduction

A green bond carries an environmental designation attached to the use of its proceeds. Its contractual value also depends on the issuer's credit quality. Here, "greenium" denotes a yield difference relative to a matched conventional security, with the sign determined by the chosen convention. Studies of matched securities have found substantial variation across market segments and issuer types [1–4]. Corporate issuance effects and the ownership of green securities support an interpretation based on investor demand rather than an exogenously fixed discount [5,6]. External verification changes the information available to investors, as documented by Dorfleitner, Utz, and Zhang [7] and discussed by Ehlers and Packer [8]. The Green Bond Principles provide a voluntary framework for labelled issuance, with practical interpretation developed in the Guidance Handbook [9,10]. The structural analysis of Agliardi and Agliardi [11] permits either sign of the premium, consistent with a signed maturity-dependent state variable.

A scalar spread cannot represent the way in which transition information changes refinancing costs at different horizons; therefore, the natural state variable is the difference between a conventional forward curve and its green counterpart, represented as a random surface indexed by calendar time and time-to-maturity. Integration along the maturity coordinate gives the logarithm of every matched green-0brown zero-coupon ratio, so the infinite-dimensional state retains a direct relation with observable fixed income quantities. The construction follows the principle underlying Heath0-Jarrow-0Morton term structure

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theory [12], while the choice of a Hilbert maturity space is consistent with the functional analytic formulations developed by Filipović [13] and by Carmona and Tehranchi [14]. The equation is a reduced statistical model on a fixed time-to-maturity grid. The deterministic change of this coordinate is explained in Section 2.2; no equivalence with a transported HJM equation is asserted. A risk-neutral construction additionally requires a specification of the traded claims and their discounting convention.

A policy announcement can affect climate-sensitive valuation beyond the date of its release. A fractional response kernel represents this delayed adjustment without imposing a single exponential relaxation scale. The underlying resolvent theory was developed by Podlubny [15] and by Kilbas, Srivastava, and Trujillo [16], with further treatment in Mainardi [17] and in Gorenflo, Kilbas, Mainardi, and Rogosin [18]. The operator formulation uses the fractional evolution theory of Bazhlekova [19] and the Volterra framework of Prüss [20]. Weak integro-differential equations were treated by Zacher [21], while Baeumer, Meerschaert, and Nane [22] connected fractional Cauchy problems with stochastic time changes.

Fractional calculus also supplies models in which an observed field is reconstructed from noisy data. Ben-Loghfyry and Charkaoui [23] combined a Caputo derivative with regularised nonlinear diffusion in magnetic resonance image processing. Ben-Loghfyry [24] treated the fractional time order as an unknown in a PDE-constrained image denoising problem. The connection with the present analysis concerns the choice of a memory law and the recovery of its order from observations. The data generating mechanisms are different, as image denoising is not evidence for a particular greenium parameter. Here, the admissible time order is constrained by stochastic integrability, and its population identification follows from the increment variance.

The stochastic construction combines fractional resolvents with the Hilbert space integration theory of Da Prato and Zabczyk [25]. Cylindrical and jump forcing have been discussed more generally by Peszat and Zabczyk [26]. Time-fractional stochastic equations appear in Mijena and Nane [27], whereas Kirchner and Willems [28] analysed a related fractional parabolic class. These formulations use different placements of the fractional operators, so their kernels cannot be interchanged without checking the equation. The memory systems of Baños, Cordoni, Di Nunno, Di Persio, and Røse [29] provide a complementary infinite-dimensional precedent. The wind energy study of Di Persio, Fraccarolo, and Veronese [30] provides a sustainable energy application based on fractional Brownian motion, which is distinct from the Brownian forcing filtered by a deterministic resolvent considered here.

The proposed fractional structure is also consistent with the scale dependence observed in financial data. Sustainable preferences alter equilibrium returns [31], carbon exposure enters asset prices [32], and climate news innovations generate hedging demands [33], while green issuance also affects shareholder value [34]. The probabilistic study of non-classical scaling began with Mandelbrot [35], while Cont [36] documented empirical regularities that are difficult to reconcile with smooth diffusion models. Fractional Brownian analysis provides one route to persistent dynamics [37], while rough volatility relates singular kernels to fine-scale observations [38] and affine Volterra processes show that non-Markovian kernels can coexist with transform tractability [39]. Brownian forcing is retained here, but is filtered by a Caputo resolvent, so the non-classical exponent is generated by deterministic climate memory and remains visible in the variance of economically meaningful curve observations.

Nonlocality along maturity is represented by a fractional power of a positive elliptic operator. Its order controls the rate at which high-frequency maturity modes are damped, and so describes how a transition shock spreads across separated tenor regions. The

concrete Dirichlet realisation gives explicit spectral formulae, while the proofs only require positivity, self-adjointness, a compact resolvent, and a strictly positive lower spectral bound. The Neumann and Robin realisations remain available when the endpoints of the observed maturity interval require a different economic convention.

The central mathematical link is that one resolvent determines both the stochastic surface and the scaling of its observations. A weighted fixed-point argument constructs the nonlinear evolution in $L^p(\Omega; H)$ for every $2 \leq p < \infty$ allowed by the initial moment. The same kernel estimates control approximation by finitely many maturity modes. The quantitative Galerkin rate is stated under an additional graph-domain hypothesis, for which verifiable coefficient conditions are provided.

In the additive Gaussian model, this link becomes an operator identity for the covariance. A trace norm expansion covers every square-integrable maturity weight, including an exact zero-coupon exposure. Reweighting the stochastic history adds a non-zero term to the local variance at a positive observation date. Its exponent identifies the Caputo order. At long horizons, the covariance converges, but its maturity spectrum also depends on the spatial colour of the noise. The non-diagonal covariance formula specifies this dependence and separates conditional identification from an unrestricted inverse problem.

The financial construction in Section 2 defines the observations to which the analysis applies; Section 3 develops the resolvent estimates used in the nonlinear theory of Section 4, while Section 5 derives the Gaussian consequences for those observations; the economic interpretation is drawn together in Section 6, and Section 7 formulates the additional structure needed for risk-neutral valuation; finally, the appendices supply the technical tools used in the proofs.

2. Greenium Formulation and Financial Interpretation

2.1. Forward Curves and Zero-Coupon Relative Values

A zero-coupon claim makes a single payment at its maturity date. Calendar maturity is denoted by T, whereas $\tau = T - t$ is the remaining time until payment. A positive differentiable price curve determines its continuously compounded forward curve through $f^i(t, \tau) = -\partial_\tau \log P^i(t, \tau)$, where $i = B$ denotes the conventional benchmark and $i = G$ the green claim. The benchmark notation does not assert a particular emissions classification. Matching means that the comparison controls for contractual differences; it does not identify two differently priced default-free claims with the same payoff in one frictionless market.

Fix a maximal maturity horizon $L > 0$. For calendar time $t \geq 0$ and time-to-maturity $\tau \in [0, L]$, let

$$P^B(t, \tau) = \exp\left(- \int_0^\tau f^B(t, u)\, \mathrm{d}u \right), \qquad P^G(t, \tau) = \exp\left(- \int_0^\tau f^G(t, u)\, \mathrm{d}u \right)$$

denote a conventional benchmark zero-coupon price and the price of a matched green claim. Matching concerns the financial characteristics that are external to the environmental label, so the residual difference can be interpreted through sustainable fixed income. Define the greenium forward surface by

$$X(t, u) = f^B(t, u) - f^G(t, u). \tag{1}$$

The sign convention implies that $X(t, u) > 0$ corresponds to a green forward rate below the benchmark forward rate.

The yield greenium at a maturity $\tau > 0$ is the average forward spread, since

$$y^B(t, \tau) - y^G(t, \tau) = \frac{1}{\tau} \int_0^\tau X(t, u)\, \mathrm{d}u, \qquad y^i(t, \tau) = -\frac{\log P^i(t, \tau)}{\tau}. \tag{2}$$

Therefore, the surface X retains information that is lost when one maturity is represented by a scalar yield difference.

Proposition 1 (Pathwise green–brown ratio)**.** *Suppose that $f^B(t,\cdot)$ and $f^G(t,\cdot)$ belong to $L^1(0,L)$ for a fixed $t \geq 0$. Then, for every $\tau \in [0,L]$,*

$$\frac{P^G(t,\tau)}{P^B(t,\tau)} = \exp\left(\int_0^\tau X(t,u)\,\mathrm{d}u\right). \tag{3}$$

The right-hand side is well defined whenever $X(t,\cdot) \in L^2(0,L)$.

Proof. The definitions give

$$\log P^G(t,\tau) - \log P^B(t,\tau) = -\int_0^\tau f^G(t,u)\,\mathrm{d}u + \int_0^\tau f^B(t,u)\,\mathrm{d}u.$$

The two integrals combine into $\int_0^\tau X(t,u)\,\mathrm{d}u$, and exponentiation proves (3). If $X(t,\cdot) \in L^2(0,L)$, then

$$\int_0^\tau |X(t,u)|\,\mathrm{d}u \leq \tau^{1/2}\|X(t,\cdot)\|_{L^2(0,L)},$$

so the integral is finite. □

Let $H = L^2(0,L)$ with the usual real inner product. For $w \in H$, write

$$Y_w(t) = \langle X(t), w\rangle_H. \tag{4}$$

The choice $w = \mathbf{1}_{[0,\tau]}$ gives the exact logarithm in (3). Smooth weights represent maturity buckets for which the loading changes continuously, while signed weights allow for relative trades between tenor regions. No regularity beyond H is required for the zero-coupon identity, which becomes important when short-time covariance asymptotics are established for every $w \in H$.

Remark 1 (Probability measure)**.** *The pathwise identity does not impose an arbitrage condition. Unless stated otherwise, the probability measure $\mathbb{P}$ below represents the statistical law used for inference or stress analysis. Therefore, the quantity $\mathbb{E}\exp(Y_w(t))$ is an exponential relative value functional. A numeraire is a strictly positive traded asset used as the unit in which other prices are expressed. An equivalent martingale measure has the same null events as the statistical measure and makes the appropriately discounted traded prices local martingales. The Heath–Jarrow–Morton restriction specifies the admissible forward rate drift in such a pricing model. None of these properties follows from the pathwise ratio identity alone. The corresponding conditional pricing equation on an enlarged state space is stated in Section 7.*

2.2. Coordinate Convention and Climate Memory

The Musiela coordinate is the remaining maturity rather than a fixed calendar maturity. If $\widehat{f}(t,T)$ is a forward rate indexed by its payment date, then

$$f(t,\xi) = \widehat{f}(t,t+\xi), \qquad \partial_t f(t,\xi) = \partial_t \widehat{f}(t,t+\xi) + \partial_\xi f(t,\xi) \tag{5}$$

for a differentiable deterministic curve. The same additional transport term occurs in the drift of an Itô forward rate model. For example, a calendar curve $\widehat{f}(t,T) = g(T)$ appears as $g(t+\xi)$ in remaining maturity, even though no new market information has arrived. This deterministic ageing is the Musiela translation.

The present equation is specified directly for a statistical greenium surface on a fixed tenor interval. It does not derive from an HJM model by deleting a transport term. A Caputo derivative integrates earlier observations over time, so a time-dependent spatial change of variables also changes the spatial arguments inside that integral. The absence of an explicit $\partial_\xi X$ in the model is a reduced-form modelling assumption, not a fractional change-of-coordinates identity. The bond ratio identity remains valid at every observation date because it only concerns integration along the maturity coordinate.

The memory order is a structural parameter to be inferred from the stochastic law, rather than a number fixed by the definition of a green bond; throughout the paper, $1/2 < \beta < 1$. The upper endpoint recovers first-order time evolution, while the lower threshold is imposed by Brownian stochastic integrability, as proved in Remark 3. In the non-degenerate Gaussian benchmark, Corollary 6 identifies β from the local variance exponent. Proposition 12 makes the associated two-scale calibration statement precise. No numerical value is inferred without observations.

Time is measured in a fixed reference unit. Under $t = t_* s$, the rescaled process $\widetilde{X}(s) = X(t_* s)$ has coefficients $t_*^{\beta} A$, $t_*^{\beta} F$, and $t_*^{\beta-1/2} G$, driven by the Brownian motion $t_*^{-1/2} W(t_* s)$. Consequently, the memory order is dimensionless, but a comparison between different orders must retain a common time unit and recalibrate coefficient magnitudes.

The deterministic Caputo memory uses two complementary kernels:

$$g_{1-\beta}(r) = \frac{r^{-\beta}}{\Gamma(1-\beta)}, \qquad k_\beta(r) = \frac{r^{\beta-1}}{\Gamma(\beta)}, \qquad r > 0. \tag{6}$$

For an absolutely continuous trajectory u, one has ${}^{C}D^{\beta} u = g_{1-\beta} * u'$ and $k_\beta * g_{1-\beta} = 1$ on positive times. Thus, the fractional integral $k_\beta * v$ records the accumulated effect of a forcing history v. This convolution is the memory potential in the present model; it is distinct from a spatial potential energy or a financial risk premium.

Proposition 2 (Diffusive representation of the memory potential)**.** *For $0 < \beta < 1$, define the positive measure*

$$\nu_\beta(\mathrm{d}r) = \frac{\sin(\pi\beta)}{\pi} r^{-\beta}\,\mathrm{d}r. \tag{7}$$

For $t > 0$, the kernel satisfies

$$k_\beta(t) = \int_0^\infty e^{-rt} \nu_\beta(\mathrm{d}r). \tag{8}$$

If $v : [0, T] \to H$ is continuous and $z_r(t) = \int_0^t e^{-r(t-s)} v(s)\,\mathrm{d}s$, then

$$(k_\beta * v)(t) = \int_0^\infty z_r(t) \nu_\beta(\mathrm{d}r), \qquad \partial_t z_r(t) = -r z_r(t) + v(t), \quad z_r(0) = 0. \tag{9}$$

The integral in H is absolutely convergent.

Proof. The substitution $q = rt$ gives

$$\int_0^\infty e^{-rt} r^{-\beta}\,\mathrm{d}r = \Gamma(1-\beta) t^{\beta-1}.$$

Euler's reflection formula $\Gamma(\beta)\Gamma(1-\beta) = \pi/\sin(\pi\beta)$ proves (8). If $M = \sup_{s \le T} \|v(s)\|$, then $\|z_r(t)\| \le M \min\{t, r^{-1}\}$. Its integral against ν_β is finite because $r^{-\beta}$ is integrable near zero and $r^{-\beta-1}$ is integrable at infinity. Therefore, Bochner Fubini proves the first identity in (9). Differentiating the ordinary exponential convolution proves the differential equation. Finally, direct beta integration gives

$$(k_\beta * g_{1-\beta})(t) = \frac{1}{\Gamma(\beta)\Gamma(1-\beta)} \int_0^t (t-s)^{\beta-1} s^{-\beta}\, \mathrm{d}s = 1,$$

which verifies the relation between the two kernels. □

The representation describes a continuum of exponentially relaxing memory states. The rate r determines how quickly one such state forgets a disturbance, while ν_β determines their aggregate weighting. This measure has infinite total mass, so it is not a probability distribution of relaxation times. For a temporary deterministic input $v(s) = v_0 \mathbf{1}_{[a,b]}(s)$, the potential is exactly

$$(k_\beta * v)(t) = \frac{(t-a)_+^\beta - (t-b)_+^\beta}{\Gamma(1+\beta)} v_0, \qquad (r)_+ = \max\{r, 0\}. \tag{10}$$

Thus, the input continues to contribute after b. In the evolution equation, the restoring operator modifies this bare memory response into the resolvent P_β. Polynomial relaxation of the mean, proved in Theorem 8, is not by itself a claim that stationary return autocovariances are non-integrable.

2.3. Fractional Maturity Propagation

Let $-\Delta_D$ denote the Dirichlet Laplacian on $(0, L)$. Its normalised eigenfunctions and eigenvalues are

$$e_n(\xi) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi\xi}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n \geq 1.$$

For $\alpha \in (1, 2]$, $\kappa > 0$ and $\lambda_0 > 0$, define

$$A = \kappa(-\Delta_D)^{\alpha/2} + \lambda_0 \,\mathrm{Id}, \qquad Ae_n = \mu_n e_n, \qquad \mu_n = \kappa\left(\frac{n\pi}{L}\right)^\alpha + \lambda_0. \tag{11}$$

The parameter α determines the spectral penalty attached to maturity oscillations. A smaller value reduces the separation between high and low modes and allows for wider transmission of shocks across maturity regions.

The Dirichlet realisation imposes zero endpoint values when the profile has sufficient trace regularity. It is an anchoring assumption of this particular model that every observed greenium vanishes at either endpoint; however, this is not an empirical assertion. The operator estimates remain valid for a positive self-adjoint generator with a compact resolvent and positive spectral gap. A different one-dimensional elliptic realisation changes the endpoint convention, while its continuous-profile conclusions use the corresponding spatial embedding. If a deterministic non-zero reference curve $\bar{X}$ is separated from a residual Z, the total spread is $X = \bar{X} + Z$ and the bond ratio acquires the explicit factor $\exp\{\int_0^\tau \bar{X}(t, u)\, \mathrm{d}u\}$. Such a separation does not remove the reference contribution from the price identity.

For $\theta \geq 0$, define

$$D(A^\theta) = \left\{ x = \sum_{n\geq 1} x_n e_n \in H : \sum_{n\geq 1} \mu_n^{2\theta} |x_n|^2 < \infty \right\}, \qquad A^\theta x = \sum_{n\geq 1} \mu_n^\theta x_n e_n.$$

The graph norm is equivalent to $\|A^\theta x\|$ because $\mu_1 > 0$.

Lemma 1 (Spectral compactness)**.** *If $\theta > 0$, then $D(A^\theta)$ is compactly embedded in H. If $\theta_2 > \theta_1 \geq 0$, then the embedding $D(A^{\theta_2}) \hookrightarrow D(A^{\theta_1})$ is continuous and compact.*

Proof. Let $(x^m)_{m\geq 1}$ be bounded in $D(A^\theta)$ and write $x^m = \sum_{n\geq 1} x_n^m e_n$. For the spectral projection $\Pi_N x = \sum_{n=1}^N \langle x, e_n\rangle e_n$,

$$\|(\mathrm{Id} - \Pi_N)x^m\|^2 = \sum_{n>N} |x_n^m|^2 \leq \mu_{N+1}^{-2\theta} \sum_{n>N} \mu_n^{2\theta} |x_n^m|^2.$$

The right-hand side converges to zero uniformly in m as $N \to \infty$. Each projected sequence $(\Pi_N x^m)_m$ lies in a finite-dimensional bounded set. Successive extraction and a diagonal argument yield a subsequence that is Cauchy in H. The first assertion follows.

For $\theta_2 > \theta_1$,

$$\left\|A^{\theta_1} x\right\|^2 \leq \mu_1^{-2(\theta_2-\theta_1)} \left\|A^{\theta_2} x\right\|^2,$$

which proves continuity. The corresponding tail estimate is

$$\sum_{n>N} \mu_n^{2\theta_1} |x_n|^2 \leq \mu_{N+1}^{-2(\theta_2-\theta_1)} \sum_{n>N} \mu_n^{2\theta_2} |x_n|^2.$$

The same diagonal argument proves compactness. □

Temporal memory and nonlocal maturity propagation are separate operations. The latter uses a fractional power of $B = -\Delta_D$ at one observation time, and can be described without any integration over the financial past.

Proposition 3 (Nonlocal maturity interaction)**.** *Let $0 < q < 1$ and $x \in D(B)$. Then,*

$$B^q x = \frac{q}{\Gamma(1-q)} \int_0^\infty [x - e^{-sB}x] \frac{\mathrm{d}s}{s^{1+q}}, \tag{12}$$

where the integral converges in H. In the maturity model, $q = \alpha/2$ when $1 < \alpha < 2$; the endpoint $\alpha = 2$ corresponds directly to B.

Proof. The identity $x - e^{-sB}x = \int_0^s e^{-rB} Bx \, \mathrm{d}r$ gives the bound $s\|Bx\|$ near zero, whereas contraction of e^{-sB} gives the bound $2\|x\|$ at infinity. The integral in (12) is consequently Bochner integrable. Pairing it with e_n reduces the assertion to

$$\int_0^\infty (1 - e^{-s\lambda_n}) s^{-1-q} \, \mathrm{d}s = \lambda_n^q \int_0^\infty (1 - e^{-r}) r^{-1-q} \, \mathrm{d}r = \lambda_n^q \frac{\Gamma(1-q)}{q}.$$

Integration by parts proves the last equality, with vanishing boundary terms because $0 < q < 1$. Equality of every spectral coefficient proves the operator identity. □

The variable s in (12) is a smoothing parameter, not calendar time. The heat operator averages the current profile over maturity before the differences are combined. Consequently, α determines spatial interaction while β determines temporal memory. Their joint appearance in the limiting covariance spectrum reflects the interaction of these two mechanisms, not an identification of them.

2.4. Nonlinear Transition Feedback

A finite-rank transition feedback can be written as

$$F(x) = b_0 - \rho x + \chi\big(\langle \omega, x\rangle\big) v, \qquad x \in H, \tag{13}$$

where $b_0, v, \omega \in H$, $\rho \geq 0$, and $\chi : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz. The scalar projection $\langle \omega, x\rangle$ extracts a climate-sensitive factor from the curve, while v determines the maturity direction

through which that factor enters the drift. The specification can represent a bounded marginal response of the greenium curve to a market-wide transition signal.

Global Lipschitz continuity is a modelling simplification that excludes superlinear feedback; it is not a structural property of sustainable fixed income. A locally Lipschitz alternative is $F(x) = b_0 - \rho x - \langle \omega, x \rangle^3 \omega$, for which

$$\langle F(x) - F(y), x - y \rangle = -\rho \|x - y\|^2 - (a^3 - b^3)(a - b) \le -\rho \|x - y\|^2, \quad a = \langle \omega, x \rangle, \quad b = \langle \omega, y \rangle. \tag{14}$$

The inequality follows from $(a^3 - b^3)(a - b) = (a - b)^2(a^2 + ab + b^2) \ge 0$. This does not make this cubic map globally Lipschitz. Proposition 5 gives a local construction without that assumption, whereas global continuation requires a separate non-explosion bound. In particular, one-sided drift dissipation cannot be substituted directly for the Lipschitz estimate in the stochastic Volterra contraction.

Proposition 4 (Lipschitz feedback)**.** *If χ has Lipschitz constant L_χ, then*

$$\|F(x) - F(y)\| \le \big(\rho + L_\chi \|\omega\| \|v\|\big) \|x - y\|, \qquad x, y \in H.$$

The map F has linear growth.

Proof. For $x, y \in H$,

$$F(x) - F(y) = -\rho(x - y) + \big[\chi(\langle \omega, x \rangle) - \chi(\langle \omega, y \rangle)\big] v.$$

Cauchy–Schwarz gives

$$|\chi(\langle \omega, x \rangle) - \chi(\langle \omega, y \rangle)| \le L_\chi |\langle \omega, x - y \rangle| \le L_\chi \|\omega\| \|x - y\|.$$

The asserted Lipschitz estimate follows from the triangle inequality. Applying it with $y = 0$ yields

$$\|F(x)\| \le \|F(0)\| + \big(\rho + L_\chi \|\omega\| \|v\|\big) \|x\|.$$

□

3. Fractional Analytic Framework

Let U be a separable Hilbert space and let $W = (W(t))_{t \ge 0}$ be a cylindrical Wiener process on U over a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$ satisfying the usual conditions. Fix $\beta \in (1/2, 1)$. The space $\mathcal{L}(H)$ consists of bounded linear maps on H, with $\|T\|_{\mathcal{L}(H)} = \sup_{\|x\|_H \le 1} \|Tx\|_H$. An un-subscripted norm of an operator acting on H denotes this operator norm; an un-subscripted norm of a vector denotes its H norm. The two-parameter Mittag-Leffler function is

$$E_{\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}, \qquad z \in \mathbb{C},$$

and $E_\beta = E_{\beta,1}$. Spectral calculus defines

$$S_\beta(t)x = E_\beta(-t^\beta A)x = \sum_{n \ge 1} E_\beta(-\mu_n t^\beta) x_n e_n, \qquad t \ge 0, \tag{15}$$

and

$$P_\beta(t)x = t^{\beta - 1} E_{\beta,\beta}(-t^\beta A)x = t^{\beta - 1} \sum_{n \ge 1} E_{\beta,\beta}(-\mu_n t^\beta) x_n e_n, \qquad t > 0. \tag{16}$$

The family S_β replaces the semigroup associated with a first-order evolution equation. The kernel P_β enters both deterministic and stochastic convolutions.

3.1. Resolvent Bounds and Difference Estimates

The Mittag-Leffler estimates recorded in Appendix A imply the following operator bounds. Their proofs are included because the exponents determine every subsequent regularity threshold.

Lemma 2 (Fractional resolvent estimates)**.** *For every $\theta \in [0,1]$, a constant C_θ exists such that*

$$\left\|A^\theta S_\beta(t)\right\|_{\mathcal{L}(H)} \le C_\theta t^{-\beta\theta}, \qquad t > 0, \tag{17}$$

$$\left\|A^\theta P_\beta(t)\right\|_{\mathcal{L}(H)} \le C_\theta t^{\beta(1-\theta)-1}, \qquad t > 0. \tag{18}$$

Moreover,

$$\left\|P_\beta(t)\right\|_{\mathcal{L}(H)} \le \frac{Ct^{\beta-1}}{(1+\mu_1 t^\beta)^2}, \qquad t > 0. \tag{19}$$

The family S_β is strongly continuous on $[0,\infty)$, and P_β is strongly continuous on $(0,\infty)$.

Proof. For $x = \sum_n x_n e_n$, Parseval's identity gives

$$\left\|A^\theta S_\beta(t)x\right\|^2 = \sum_{n\ge 1} \mu_n^{2\theta}\left|E_\beta(-\mu_n t^\beta)\right|^2 |x_n|^2.$$

Writing $r = \mu_n t^\beta$ yields

$$\mu_n^\theta\left|E_\beta(-\mu_n t^\beta)\right| = t^{-\beta\theta} r^\theta \left|E_\beta(-r)\right|.$$

The bound $\sup_{r\ge 0} r^\theta\left|E_\beta(-r)\right| < \infty$ proves (17). The same calculation gives

$$\mu_n^\theta t^{\beta-1}\left|E_{\beta,\beta}(-\mu_n t^\beta)\right| = t^{\beta(1-\theta)-1} r^\theta \left|E_{\beta,\beta}(-r)\right|,$$

which proves (18).

The sharper scalar estimate,

$$\left|E_{\beta,\beta}(-r)\right| \le \frac{C}{(1+r)^2}, \qquad r \ge 0,$$

implies

$$\left\|P_\beta(t)\right\|_{\mathcal{L}(H)} \le Ct^{\beta-1}\sup_{n\ge 1}(1+\mu_n t^\beta)^{-2} \le \frac{Ct^{\beta-1}}{(1+\mu_1 t^\beta)^2}.$$

For strong continuity of S_β, let $t_j \to t \ge 0$. Each spectral coefficient converges, and

$$\left|E_\beta(-\mu_n t_j^\beta) - E_\beta(-\mu_n t^\beta)\right|^2 |x_n|^2 \le C|x_n|^2.$$

Dominated convergence over n proves $S_\beta(t_j)x \to S_\beta(t)x$. The proof for P_β is identical when $t > 0$, because t_j then remains in a compact subset of $(0,\infty)$ and the spectral multipliers are uniformly bounded. □

Lemma 3 (Differentiability away from the origin)**.** *The map $t \mapsto P_\beta(t)x$ is continuously differentiable on $(0,\infty)$ for every $x \in H$, and*

$$P_\beta'(t)x = t^{\beta-2}E_{\beta,\beta-1}(-t^\beta A)x. \tag{20}$$

There is a constant C such that

$$\left\|P'_\beta(t)\right\|_{\mathcal{L}(H)} \le Ct^{\beta-2}, \qquad t > 0. \tag{21}$$

If $x \in D(A)$, then $S_\beta(\cdot)x$ is continuously differentiable on $(0,\infty)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}S_\beta(t)x = -AP_\beta(t)x. \tag{22}$$

More generally, if $x \in D(A^\vartheta)$ with $\vartheta \in (0,1]$, then

$$\left\|(S_\beta(t) - S_\beta(s))x\right\| \le C|t-s|^{\beta\vartheta}\left\|A^\vartheta x\right\|, \qquad s,t \ge 0. \tag{23}$$

Proof. The scalar identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{\gamma-1}E_{\beta,\gamma}(-\mu t^\beta)\right] = t^{\gamma-2}E_{\beta,\gamma-1}(-\mu t^\beta)$$

follows by termwise differentiation of the entire series on compact time intervals. Taking $\gamma = \beta$ proves (20) on each eigenvector. Since $E_{\beta,\beta-1}(-r)$ is bounded for $r \ge 0$, the spectral multiplier in (20) has norm at most $Ct^{\beta-2}$. Difference quotients converge by dominated convergence in the spectral expansion, which proves differentiability in H and (21).

The derivative formula for S_β follows from

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\beta(-\mu t^\beta) = -\mu t^{\beta-1}E_{\beta,\beta}(-\mu t^\beta).$$

For the final estimate, spectral calculus gives

$$(S_\beta(t) - S_\beta(s))x = \big(S_\beta(t) - S_\beta(s)\big)A^{-\vartheta}A^\vartheta x.$$

Assume $t > s \ge 0$. The derivative identity on every spectral mode and the bound

$$\left\|A^{1-\vartheta}P_\beta(r)\right\|_{\mathcal{L}(H)} \le Cr^{\beta\vartheta-1}$$

show that the improper operator-valued integral below converges in operator norm and satisfies

$$\begin{aligned}(S_\beta(t) - S_\beta(s))A^{-\vartheta} &= -\int_s^t A^{1-\vartheta}P_\beta(r)\,\mathrm{d}r,\\ \left\|(S_\beta(t) - S_\beta(s))A^{-\vartheta}\right\|_{\mathcal{L}(H)} &\le C\int_s^t r^{\beta\vartheta-1}\,\mathrm{d}r\\ &= \frac{C}{\beta\vartheta}\big(t^{\beta\vartheta} - s^{\beta\vartheta}\big).\end{aligned}$$

Since $0 < \beta\vartheta \le 1$, concavity of $r \mapsto r^{\beta\vartheta}$ implies $t^{\beta\vartheta} - s^{\beta\vartheta} \le (t-s)^{\beta\vartheta}$. Multiplication by $A^\vartheta x$ proves (23), including the case $s = 0$. □

Lemma 4 (Integrated resolvent increments)**.** *For every $T > 0$, a constant C_T exists such that, whenever $0 < h \le 1$ and $0 \le s \le T$,*

$$\int_0^s \left\|P_\beta(r+h) - P_\beta(r)\right\|_{\mathcal{L}(H)}\,\mathrm{d}r \le C_T h^\beta, \tag{24}$$

$$\int_0^s \left\|P_\beta(r+h) - P_\beta(r)\right\|^2_{\mathcal{L}(H)}\,\mathrm{d}r \le C_T h^{2\beta-1}. \tag{25}$$

Here, $P_\beta(r)$ at $r = 0$ is understood only inside an improper integral.

Proof. Split each integral at $r = h$. On $(0, h)$, Lemma 2 yields

$$\|P_\beta(r+h) - P_\beta(r)\| \le C\big((r+h)^{\beta-1} + r^{\beta-1}\big).$$

Its first power integrates to Ch^β, while its square integrates to $Ch^{2\beta-1}$. On (h, s), the fundamental theorem of calculus and (21) give

$$\|P_\beta(r+h) - P_\beta(r)\| \le \int_r^{r+h} Cq^{\beta-2}\,\mathrm{d}q \le Chr^{\beta-2}.$$

Consequently,

$$h\int_h^s r^{\beta-2}\,\mathrm{d}r \le Ch^\beta$$

and

$$h^2\int_h^s r^{2\beta-4}\,\mathrm{d}r \le Ch^{2\beta-1}.$$

The estimates remain valid when $s < h$, because only the first part is then present. □

3.2. The Semilinear Fractional Equation

Let $F{:}H \to H$ and $G{:}H \to \mathcal{L}_2(U, H)$ be globally Lipschitz. Thus, constants $L_F, L_G \ge 0$ satisfy

$$\|F(x) - F(y)\| \le L_F\|x - y\|, \qquad \|G(x) - G(y)\|_{\mathcal{L}_2(U,H)} \le L_G\|x - y\|. \tag{26}$$

The corresponding linear growth estimates follow by comparison with the value at the origin.

The formal equation is

$${}^{C}D_t^\beta X(t) = -AX(t) + F(X(t)) + G(X(t))\dot{W}(t), \qquad X(0) = X_0. \tag{27}$$

The notation is symbolic because $\dot{W}$ is distribution valued in time. The mild formulation is

$$X(t) = S_\beta(t)X_0 + \int_0^t P_\beta(t-s)F(X(s))\,\mathrm{d}s + \int_0^t P_\beta(t-s)G(X(s))\,\mathrm{d}W(s). \tag{28}$$

Definition 1 (Mild solution)**.** *Let $2 \le p < \infty$ and $T > 0$. An adapted process $X : [0, T] \times \Omega \to H$ is an L^p mild solution if $t \mapsto X(t)$ is continuous from $[0, T]$ into $L^p(\Omega; H)$, if*

$$\sup_{0\le t\le T} \mathbb{E}\|X(t)\|^p < \infty,$$

and if (28) holds in $L^p(\Omega; H)$ for every $t \in [0, T]$.

Lemma 5 (Continuity of Volterra convolutions)**.** *Let $2 \le p < \infty$, and let $u{:}[0, T] \to L^p(\Omega; H)$ be adapted and continuous with bounded pth moments. Then,*

$$t \longmapsto \int_0^t P_\beta(t-s)F(u(s))\,\mathrm{d}s$$

and

$$t \longmapsto \int_0^t P_\beta(t-s)G(u(s))\,\mathrm{d}W(s)$$

are continuous from $[0, T]$ into $L^p(\Omega; H)$.

Proof. The linear growth bounds imply

$$\sup_{0\le s\le T}\left(\mathbb{E}\|F(u(s))\|^p\right)^{1/p}<\infty,\qquad \sup_{0\le s\le T}\left(\mathbb{E}\|G(u(s))\|_{\mathcal{L}_2}^p\right)^{1/p}<\infty.$$

Let $t=s+h$ with $h>0$. For the deterministic convolution, separate the integral over (s,t) from the difference of kernels over $(0,s)$. Minkowski's inequality, (18) with $\theta=0$, and (24) give

$$\left(\mathbb{E}\left\|\int_s^t P_\beta(t-r)F(u(r))\,\mathrm{d}r\right\|^p\right)^{1/p}\le Ch^\beta$$

and

$$\left(\mathbb{E}\left\|\int_0^s [P_\beta(t-r)-P_\beta(s-r)]F(u(r))\,\mathrm{d}r\right\|^p\right)^{1/p}\le Ch^\beta.$$

For the stochastic convolution, the Burkholder–Davis–Gundy inequality in Hilbert spaces yields

$$\left(\mathbb{E}\left\|\int_s^t P_\beta(t-r)G(u(r))\,\mathrm{d}W(r)\right\|^p\right)^{1/p}\le C_p\left(\int_s^t (t-r)^{2\beta-2}\,\mathrm{d}r\right)^{1/2}\le Ch^{\beta-1/2}.$$

The old history term is bounded by

$$C_p\left(\int_0^s \left\|P_\beta(t-r)-P_\beta(s-r)\right\|^2\mathrm{d}r\right)^{1/2}\le Ch^{\beta-1/2}$$

through (25). The estimates prove right continuity. Left continuity follows by the same decomposition with the roles of s and t reversed. Continuity at zero follows directly from the corresponding new history estimates. □

4. Nonlinear Well-Posedness and Regularity

4.1. Global Moment Well-Posedness

For $2\le p<\infty$, $T>0$ and $\eta>0$, let $\mathbb{M}^p_{\eta,T}$ be the space of adapted processes u such that $t\mapsto u(t)$ is continuous from $[0,T]$ into $L^p(\Omega;H)$ and

$$\|u\|_{p,\eta,T}=\sup_{0\le t\le T} e^{-\eta t/p}\left(\mathbb{E}\|u(t)\|^p\right)^{1/p}<\infty.$$

An adapted process that is continuous in L^p is stochastically continuous, and so admits a progressively measurable version under the usual conditions on the filtration. Each element of $\mathbb{M}^p_{\eta,T}$ is identified with such a version when it appears in a stochastic integral. The space is complete: a Cauchy sequence converges uniformly in the weighted norm to a continuous $L^p(\Omega;H)$-valued map, adaptedness is preserved because $L^p(\Omega,\mathcal{F}_t;H)$ is closed for every t, and the limiting process again has a progressively measurable version.

Theorem 1 (Global mild well-posedness). *Let $2\le p<\infty$, let $X_0\in L^p(\Omega,\mathcal{F}_0;H)$, and suppose that (26) holds. For every $T>0$, Equation (28) has a unique L^p mild solution on $[0,T]$. A constant $C_{p,T}$ exists such that*

$$\sup_{0\le t\le T}\mathbb{E}\|X(t)\|^p\le C_{p,T}\left(1+\mathbb{E}\|X_0\|^p\right). \tag{29}$$

The solutions obtained on finite intervals agree on overlaps, and so define a global process.

Proof. For $u \in \mathbb{M}^p_{\eta,T}$ define

$$(\Phi u)(t) = S_\beta(t)X_0 + \int_0^t P_\beta(t-s)F(u(s))\,\mathrm{d}s + \int_0^t P_\beta(t-s)G(u(s))\,\mathrm{d}W(s).$$

Lemma 5 shows that Φu is continuous in $L^p(\Omega;H)$ and adapted. The initial term satisfies

$$e^{-\eta t/p}\big(\mathbb{E}\|S_\beta(t)X_0\|^p\big)^{1/p} \le C\big(\mathbb{E}\|X_0\|^p\big)^{1/p}.$$

The deterministic convolution is estimated by Minkowski's inequality. Since

$$\big(\mathbb{E}\|F(u(s))\|^p\big)^{1/p} \le \|F(0)\| + L_F e^{\eta s/p}\|u\|_{p,\eta,T},$$

one obtains

$$\begin{aligned} &e^{-\eta t/p}\left(\mathbb{E}\left\|\int_0^t P_\beta(t-s)F(u(s))\,\mathrm{d}s\right\|^p\right)^{1/p} \\ &\quad\le C\big(\|F(0)\| + L_F\|u\|_{p,\eta,T}\big)\int_0^t r^{\beta-1}e^{-\eta r/p}\,\mathrm{d}r \\ &\quad\le C\Gamma(\beta)\left(\frac{p}{\eta}\right)^\beta\big(\|F(0)\| + L_F\|u\|_{p,\eta,T}\big). \end{aligned}$$

For the stochastic convolution, the Hilbert-space Burkholder–Davis–Gundy inequality gives

$$\begin{aligned} &e^{-\eta t/p}\left(\mathbb{E}\left\|\int_0^t P_\beta(t-s)G(u(s))\,\mathrm{d}W(s)\right\|^p\right)^{1/p} \\ &\quad\le C_p\left(\int_0^t e^{-2\eta(t-s)/p}\|P_\beta(t-s)\|^2 e^{-2\eta s/p}\big(\mathbb{E}\|G(u(s))\|^p_{\mathcal{L}_2}\big)^{2/p}\,\mathrm{d}s\right)^{1/2} \\ &\quad\le C_p\big(\|G(0)\|_{\mathcal{L}_2} + L_G\|u\|_{p,\eta,T}\big)\left(\int_0^\infty r^{2\beta-2}e^{-2\eta r/p}\,\mathrm{d}r\right)^{1/2} \\ &\quad\le C_{p,\beta}\eta^{-(\beta-1/2)}\big(\|G(0)\|_{\mathcal{L}_2} + L_G\|u\|_{p,\eta,T}\big). \end{aligned}$$

The integral is finite because $2\beta - 2 > -1$. These bounds prove that Φ maps $\mathbb{M}^p_{\eta,T}$ into itself.

For $u, v \in \mathbb{M}^p_{\eta,T}$, the same estimates and the Lipschitz inequalities give

$$\|\Phi u - \Phi v\|_{p,\eta,T} \le \Big(CL_F\eta^{-\beta} + C_pL_G\eta^{-(\beta-1/2)}\Big)\|u - v\|_{p,\eta,T}. \tag{30}$$

The coefficient tends to zero as $\eta \to \infty$. Choosing η so that it is less than one, the Banach fixed-point theorem produces a unique fixed point in $\mathbb{M}^p_{\eta,T}$.

The estimates for ΦX also yield

$$\|X\|_{p,\eta,T} \le C\big(\mathbb{E}\|X_0\|^p\big)^{1/p} + C\eta^{-\beta}\|F(0)\| + C_p\eta^{-(\beta-1/2)}\|G(0)\|_{\mathcal{L}_2} + q_\eta\|X\|_{p,\eta,T},$$

where $q_\eta < 1$. Absorption of the final term, followed by multiplication by $e^{\eta T/p}$ and raising to the pth power, proves (29). Uniqueness makes the fixed points on nested time intervals consistent. □

Remark 2 (Scope of the probability-moment exponent)**.** *The exponent p refers to the probability norm $L^p(\Omega;H)$, while $H = L^2(0,L)$ remains the maturity space. Theorem 1 applies to each fixed $2 \le p < \infty$ for which $X_0 \in L^p(\Omega;H)$. Its constants depend on p and need not remain bounded as*

p increases. If the initial condition has all finite moments, uniqueness in L^2 makes the solutions constructed at different exponents the same process.

For $1 \le p < 2$, an L^2 initial condition gives an L^p solution by the inequality $\|Z\|_{L^p(\Omega;H)} \le \|Z\|_{L^2(\Omega;H)}$. This observation supplies no theorem for an initial condition known only to belong to L^p and no uniqueness assertion among all such weaker solutions. The endpoint $p = \infty$ is excluded: a non-degenerate Gaussian observation at any positive time is not essentially bounded.

Remark 3 (Sharp Brownian integrability threshold)**.** *The threshold $\beta > 1/2$ is a stochastic integrability restriction. If $\Sigma \neq 0$, an eigenvector e_n exists with $\Sigma^* e_n \neq 0$, and*

$$\left\|\Sigma^* P_\beta(r) e_n\right\|_U^2 \sim \frac{\|\Sigma^* e_n\|_U^2}{\Gamma(\beta)^2} r^{2\beta-2} \qquad r \downarrow 0. \tag{31}$$

Thus, the variance integral in this single forced mode diverges when $0 < \beta \le 1/2$. Spatial smoothness of a non-zero additive coefficient does not remove the divergence. A model in the lower range requires a different temporal forcing or a different solution concept; the zero-noise deterministic problem only requires $\beta > 0$.

4.2. Stability and Temporal Regularity

Theorem 2 (Stability with respect to the initial curve)**.** *Let $X^{(1)}$ and $X^{(2)}$ solve (28) with initial conditions $X_0^{(1)}, X_0^{(2)} \in L^p(\Omega, \mathcal{F}_0; H)$ and with the same coefficients and Wiener process. For every $T > 0$,*

$$\sup_{0 \le t \le T} \mathbb{E}\left\|X^{(1)}(t) - X^{(2)}(t)\right\|^p \le C_{p,T} \mathbb{E}\left\|X_0^{(1)} - X_0^{(2)}\right\|^p. \tag{32}$$

Proof. Set $Z = X^{(1)} - X^{(2)}$. Subtraction of the two mild equations gives

$$\begin{aligned} Z(t) = {} & S_\beta(t)(X_0^{(1)} - X_0^{(2)}) \\ & + \int_0^t P_\beta(t-s)\left[F(X^{(1)}(s)) - F(X^{(2)}(s))\right] \mathrm{d}s \\ & + \int_0^t P_\beta(t-s)\left[G(X^{(1)}(s)) - G(X^{(2)}(s))\right] \mathrm{d}W(s). \end{aligned}$$

The Bielecki estimates used in (30) imply

$$\|Z\|_{p,\eta,T} \le C\left(\mathbb{E}\left\|X_0^{(1)} - X_0^{(2)}\right\|^p\right)^{1/p} + q_\eta \|Z\|_{p,\eta,T}.$$

Choose η with $q_\eta < 1$ and absorb the final term. Removing the exponential weight proves (32). □

Theorem 3 (Robustness under coefficient perturbations)**.** *For $i \in \{1, 2\}$, let $F_i{:}H \to H$ and $G_i{:}H \to \mathcal{L}_2(U, H)$ satisfy the global Lipschitz conditions with common constants L_F and L_G. Assume that*

$$\Delta_F = \sup_{x \in H} \|F_1(x) - F_2(x)\| < \infty, \qquad \Delta_G = \sup_{x \in H} \|G_1(x) - G_2(x)\|_{\mathcal{L}_2(U,H)} < \infty.$$

Let $X^{(i)}$ be the solution associated with $(X_0^{(i)}, F_i, G_i)$ and driven by the same cylindrical Wiener process. For every $T > 0$,

$$\sup_{0\le t\le T}\big(\mathbb{E}\big\|X^{(1)}(t)-X^{(2)}(t)\big\|^{p}\big)^{1/p} \le C_{p,T}\Big[\big(\mathbb{E}\big\|X_0^{(1)}-X_0^{(2)}\big\|^{p}\big)^{1/p}+\Delta_F+\Delta_G\Big]. \tag{33}$$

Proof. Set $Z = X^{(1)} - X^{(2)}$. The drift difference is decomposed as

$$F_1(X^{(1)}) - F_2(X^{(2)}) = F_1(X^{(1)}) - F_1(X^{(2)}) + F_1(X^{(2)}) - F_2(X^{(2)}),$$

and the diffusion difference is treated in the same way. Applying the deterministic and stochastic Bielecki estimates from Theorem 1 gives

$$\begin{aligned}\|Z\|_{p,\eta,T} \le C\big(\mathbb{E}\big\|X_0^{(1)}-X_0^{(2)}\big\|^{p}\big)^{1/p} + q_\eta\|Z\|_{p,\eta,T}\\ + C\eta^{-\beta}\Delta_F + C_p\eta^{-(\beta-1/2)}\Delta_G,\end{aligned}$$

where $q_\eta = CL_F\eta^{-\beta} + C_pL_G\eta^{-(\beta-1/2)}$. Choose η so that $q_\eta < 1$, absorb the second term, and remove the exponential weight. The constants depending on the chosen η are absorbed into $C_{p,T}$. □

The coefficients of a greenium equation are inferred from a finite panel of labelled and conventional bonds, so the stochastic model is necessarily used with estimated drift and diffusion maps. Estimate (33) shows that bounded specification errors in those maps cannot produce an uncontrolled displacement of the entire maturity surface on a fixed horizon.

Theorem 4 (Quantitative temporal increments)**.** *Let $2 \le p < \infty$, and let X be the solution from Theorem 1. Suppose that $X_0 \in L^p(\Omega; D(A^\vartheta))$ for some $\vartheta \in (0,1]$. Then, for every $T > 0$ and every $0 \le s < t \le T$,*

$$\big(\mathbb{E}\|X(t)-X(s)\|^{p}\big)^{1/p} \le C_{p,T}\big(1+\big(\mathbb{E}\big\|A^{\vartheta}X_0\big\|^{p}\big)^{1/p}\big)|t-s|^{\gamma_\vartheta}, \tag{34}$$

where

$$\gamma_\vartheta = \min\Big\{\beta\vartheta,\ \beta-\frac{1}{2}\Big\}. \tag{35}$$

Proof. Write $h = t - s$. By (23), the initial contribution satisfies

$$\big(\mathbb{E}\big\|(S_\beta(t)-S_\beta(s))X_0\big\|^{p}\big)^{1/p} \le Ch^{\beta\vartheta}\big(\mathbb{E}\big\|A^{\vartheta}X_0\big\|^{p}\big)^{1/p}.$$

The difference of deterministic convolutions is

$$\begin{aligned}D(t)-D(s) = \int_s^t P_\beta(t-r)F(X(r))\,\mathrm{d}r\\ + \int_0^s [P_\beta(t-r)-P_\beta(s-r)]F(X(r))\,\mathrm{d}r.\end{aligned}$$

The moment estimate (29) and linear growth of F provide a uniform bound for $(\mathbb{E}\|F(X(r))\|^p)^{1/p}$. Minkowski's inequality gives an estimate of $O(h^\beta)$ for the first term, while (24) gives the same order for the second.

The stochastic convolution difference has the analogous decomposition

$$\begin{aligned}M(t)-M(s) = \int_s^t P_\beta(t-r)G(X(r))\,\mathrm{d}W(r)\\ + \int_0^s [P_\beta(t-r)-P_\beta(s-r)]G(X(r))\,\mathrm{d}W(r).\end{aligned}$$

The Burkholder–Davis–Gundy inequality followed by (18) yields

$$\left(\mathbb{E}\left\|\int_s^t P_\beta(t-r)G(X(r))\,\mathrm{d}W(r)\right\|^p\right)^{1/p} \le C\left(\int_0^h r^{2\beta-2}\,\mathrm{d}r\right)^{1/2} \le Ch^{\beta-1/2}.$$

The old history term is bounded by $Ch^{\beta-1/2}$ through (25). Since $h^\beta \le C_T h^{\beta-1/2}$ on $[0,T]$, combination of the estimates proves (34). □

Corollary 1 (Pathwise temporal Hölder continuity)**.** *Under the assumptions of Theorem 4, suppose that $p\gamma_\vartheta > 1$. The solution has a modification wfor which the trajectories are Hölder continuous in H with every exponent*

$$0 < \delta < \gamma_\vartheta - \frac{1}{p}.$$

Proof. Equation (34) implies

$$\mathbb{E}\|X(t) - X(s)\|^p \le C|t-s|^{p\gamma_\vartheta}.$$

The Banach-valued Kolmogorov continuity theorem applies because $p\gamma_\vartheta > 1$ and yields every exponent below $(p\gamma_\vartheta - 1)/p$. □

4.3. Local Feedback and Continuation

The globally Lipschitz model has a controlled local extension when the coefficients are Lipschitz on bounded balls. Localisation must retain the complete Volterra history rather than restart the equation at an exit time.

Proposition 5 (Localisation with retained memory)**.** *Suppose that $F{:}H \to H$ and $G{:}H \to \mathcal{L}_2(U,H)$ are Lipschitz on every bounded ball, with constants allowed to depend on the radius. Let $X_0 = x_0 \in D(A)$ be deterministic. There is a unique adapted continuous local mild solution up to an increasing limit τ_∞ of exit times. On the event $\tau_\infty < \infty$,*

$$\sup_{0\le t<\tau_\infty} \|X(t)\| = \infty. \tag{36}$$

If the truncated solutions below satisfy

$$\sup_{R>\|x_0\|} \mathbb{E} \sup_{0\le t\le T\wedge\tau_R} \|X_R(t)\|^m \le C_T < \infty \tag{37}$$

for some $m > 0$ and every $T > 0$, then $\tau_\infty = \infty$ almost surely.

Proof. For an integer $R > \|x_0\|$, let π_R be the metric projection of H onto its closed radius-R ball. This map is non-expansive. Hence, $F_R = F \circ \pi_R$ and $G_R = G \circ \pi_R$ are globally Lipschitz and bounded. Theorem 1 gives a global solution X_R. Choose a finite p with $p(\beta - 1/2) > 1$. Since $x_0 \in D(A)$, Theorem 4 and Corollary 1 give a continuous modification of X_R. Define

$$\tau_R = \inf\{t \ge 0 : \|X_R(t)\| \ge R\}, \qquad \inf \varnothing = \infty.$$

These are stopping times and are strictly positive.

For integers $S > R$, set $\sigma = \tau_R \wedge \tau_S$ and $D = X_R - X_S$. At every deterministic t, the difference equation on $\{t \le \sigma\}$ agrees with the equation whose integrands are multiplied by $\mathbf{1}_{\{s\le\sigma\}}$. For the stochastic integral, this follows by stopping the martingale with fixed

terminal kernel $P_\beta(t-s)$; its part after σ vanishes on $\{t \le \sigma\}$. On $\{s \le \sigma\}$, both truncated coefficients equal the original coefficients evaluated at their respective states. With

$$d(t) = \left(\mathbb{E}\left[\mathbf{1}_{\{t\le\sigma\}}\|D(t)\|^p\right]\right)^{1/p}, \qquad d_{\eta,T} = \sup_{t\le T} e^{-\eta t/p} d(t),$$

Minkowski's inequality and the stochastic convolution estimate give

$$d_{\eta,T} \le \left(CL_S\eta^{-\beta} + C_p L_S \eta^{-(\beta-1/2)}\right) d_{\eta,T},$$

where L_S bounds both local Lipschitz constants on the radius-S ball. The quantity $d_{\eta,T}$ is finite, since X_R and X_S have bounded pth moments on $[0,T]$. Choosing η so that the coefficient is smaller than one yields $d(t) = 0$ for every $t \le T$. Countably many rational times and continuity imply $X_R = X_S$ throughout $[0,\sigma]$ almost surely.

If $\tau_S < \tau_R$, continuity and equality at $\sigma = \tau_S$ would give $\|X_R(\tau_S)\| = S > R$ before X_R reached its radius-R boundary, which is impossible. Thus, $\tau_R \le \tau_S$ and the two processes agree up to τ_R. Define $\tau_\infty = \lim_R \tau_R$ and $X(t) = X_R(t)$ whenever $t < \tau_R$. This definition is consistent and preserves adaptedness and continuity. Its mild equation uses the integral over the original interval $[0,t]$. The same stopped difference estimate proves local uniqueness.

If $\tau_\infty < \infty$, every sufficiently large R is reached by the continuous local path before τ_∞, or at a time increasing to it; the path cannot remain bounded, which proves (36). Finally, (37) implies

$$\mathbb{P}\{\tau_R \le T\} \le R^{-m}\mathbb{E}\sup_{t\le T\wedge\tau_R}\|X_R(t)\|^m \le C_T R^{-m}.$$

The events decrease as R increases, so $\mathbb{P}\{\tau_\infty \le T\} = 0$. Taking an integer T proves global continuation. □

The deterministic regular initial condition in this proposition supplies the continuous paths needed to define the norm exit times. The global moment theorem has the weaker initial assumption stated in Theorem 1. The uniform stopped supremum bound (37) is an additional condition; it does not follow from moment estimates at each deterministic time. For the cubic feedback in (14), the energy identity

$$\langle x, F(x)\rangle = \langle x, b_0\rangle - \rho\|x\|^2 - \langle \omega, x\rangle^4$$

provides a dissipative drift contribution, but the required stochastic memory estimate remains separate from this algebraic fact.

4.4. Fractional-Domain Regularity

Theorem 5 (Global fractional-domain estimate)**.** *Let $p = 2$, and let*

$$0 \le \theta < 1 - \frac{1}{2\beta}. \tag{38}$$

If $X_0 \in L^2(\Omega; D(A^\theta))$, then $X(t) \in L^2(\Omega; D(A^\theta))$ for every $t \in [0,T]$ and

$$\sup_{0\le t\le T}\mathbb{E}\left\|A^\theta X(t)\right\|^2 \le C_{T,\theta}\left(1 + \mathbb{E}\left\|A^\theta X_0\right\|^2 + \sup_{0\le t\le T}\mathbb{E}\|X(t)\|^2\right). \tag{39}$$

Proof. Define

$$Z_\theta(t) = S_\beta(t)A^\theta X_0 + \int_0^t A^\theta P_\beta(t-s)F(X(s))\,\mathrm{d}s + \int_0^t A^\theta P_\beta(t-s)G(X(s))\,\mathrm{d}W(s).$$

The deterministic integral is well defined because $\beta(1-\theta)-1 > -1$. For the stochastic integral, Itô's isometry and (18) give the kernel exponent $2\beta(1-\theta)-2$, which is greater than -1 exactly under (38).

Let Π_N be the spectral projection onto $\mathrm{span}\{e_1,\dots,e_N\}$. Commutation of Π_N, A^θ, S_β, and P_β gives

$$A^\theta \Pi_N X(t) = \Pi_N Z_\theta(t).$$

As $N \to \infty$, the left input $\Pi_N X(t)$ converges to $X(t)$ in $L^2(\Omega;H)$ and the right-hand side converges to $Z_\theta(t)$ in the same space. The operator $I_{L^2(\Omega)} \otimes A^\theta$ is closed on $L^2(\Omega;H)$. Hence, $X(t)$ belongs to its domain and $A^\theta X(t) = Z_\theta(t)$.

The initial term is bounded by $C\mathbb{E}\|A^\theta X_0\|^2$. Minkowski's inequality gives

$$\left(\mathbb{E}\left\|\int_0^t A^\theta P_\beta(t-s)F(X(s))\,\mathrm{d}s\right\|^2\right)^{1/2} \le C\int_0^t (t-s)^{\beta(1-\theta)-1}\Big(\|F(0)\| + L_F(\mathbb{E}\|X(s)\|^2)^{1/2}\Big)\,\mathrm{d}s.$$

The integral is uniformly finite on $[0,T]$. Itô's isometry gives

$$\mathbb{E}\left\|\int_0^t A^\theta P_\beta(t-s)G(X(s))\,\mathrm{d}W(s)\right\|^2 \le C\int_0^t (t-s)^{2\beta(1-\theta)-2}\Big(\|G(0)\|_{\mathcal{L}_2}^2 + L_G^2\mathbb{E}\|X(s)\|^2\Big)\,\mathrm{d}s.$$

The exponent is greater than -1, so the integral is uniformly bounded. Combining the three estimates proves (39). □

Theorem 6 (Instantaneous maturity smoothing). *Let $X_0 \in L^2(\Omega;H)$ and let θ satisfy (38). For every $0 < \varepsilon < T$,*

$$\sup_{\varepsilon \le t \le T} \mathbb{E}\left\|A^\theta X(t)\right\|^2 \le C_{\varepsilon,T,\theta}\big(1 + \mathbb{E}\|X_0\|^2\big). \tag{40}$$

Proof. The proof of Theorem 5 remains valid for $t > 0$ with the initial term written as $A^\theta S_\beta(t)X_0$. By (17),

$$\mathbb{E}\left\|A^\theta S_\beta(t)X_0\right\|^2 \le Ct^{-2\beta\theta}\mathbb{E}\|X_0\|^2 \le C\varepsilon^{-2\beta\theta}\mathbb{E}\|X_0\|^2$$

for $t \ge \varepsilon$. The convolution estimates are unchanged. □

Corollary 2 (Continuous maturity profiles). *Assume the concrete operator (11). Let θ satisfy*

$$\frac{1}{2\alpha} < \theta < 1 - \frac{1}{2\beta}. \tag{41}$$

If $X_0 \in L^2(\Omega; D(A^\theta))$, then each $X(t)$ admits a version in $C([0,L])$ and

$$\sup_{0 \le t \le T} \mathbb{E}\|X(t)\|_{C([0,L])}^2 < \infty.$$

If X_0 only belongs to $L^2(\Omega; H)$, the same conclusion holds uniformly on $[\varepsilon, T]$ for every $\varepsilon > 0$. The interval in (41) is non-empty precisely when

$$\alpha\left(1 - \frac{1}{2\beta}\right) > \frac{1}{2}. \tag{42}$$

Proof. The spectral Sobolev embedding in Appendix C gives

$$D(A^\theta) \hookrightarrow H^{\alpha\theta}(0, L) \hookrightarrow C([0, L])$$

when $\alpha\theta > 1/2$. Thus, $\|x\|_{C([0,L])} \le C\|A^\theta x\|$ for $x \in D(A^\theta)$. The moment bounds follow from Theorems 5 and 6. Elementary rearrangement shows that a θ satisfying (41) exists exactly under (42). □

Remark 4 (Endpoint convention)**.** *Under the Dirichlet realisation, a continuous profile belonging to $D(A^\theta)$ with sufficient trace regularity vanishes at the endpoints. The condition is the boundary convention of the fixed-tenor model defined in Section 2.2. Neumann or Robin spectral powers preserve the preceding proofs while changing the endpoint behaviour.*

4.5. Variable-Exponent Maturity Observations

A maturity-dependent exponent changes the state-space norm rather than the probability moment exponent in Theorem 1. To separate the two roles, let $q{:}(0, L) \to (1, \infty)$ be measurable, with

$$1 < q_- := \operatorname*{ess\,inf}_\xi q(\xi) \le q_+ := \operatorname*{ess\,sup}_\xi q(\xi) < \infty.$$

Write $E_q = L^{q(\cdot)}(0, L)$, and define its modular and Luxemburg norm by

$$\rho_q(f) = \int_0^L |f(\xi)|^{q(\xi)}\, \mathrm{d}\xi, \qquad \|f\|_{E_q} = \inf\{a > 0 : \rho_q(f/a) \le 1\}. \tag{43}$$

The function-space background is given in [40].

Proposition 6 (Inherited variable-exponent bounds)**.** *If $q_+ \le 2$, then*

$$\|f\|_{E_q} \le (L + 1)^{1/q_-}\|f\|_H, \qquad f \in H. \tag{44}$$

Consequently, the H-valued mild solution belongs to $C([0, T]; L^p(\Omega; E_q))$ for every finite $2 \le p < \infty$ allowed by the initial condition. If q_+ is arbitrary and the hypotheses of Corollary 2 hold, then

$$\sup_{t \le T} \mathbb{E}\|X(t)\|_{E_q}^2 < \infty. \tag{45}$$

For an initial condition in $L^2(\Omega; H)$, the latter assertion holds on each $[\varepsilon, T]$ under the same spatial parameter condition.

Proof. For $f \neq 0$, put $a = \|f\|_H$. On the set where $|f/a| \le 1$, the integrand of $\rho_q(f/a)$ is at most one. On the complementary set, $q(\xi) \le 2$ gives $|f/a|^{q(\xi)} \le |f/a|^2$. Hence, $\rho_q(f/a) \le L + 1$. Division by $(L + 1)^{1/q_-}$ makes the modular at most one and proves (44). The case $f = 0$ is immediate. Applying this bounded linear embedding to $X(t)$ and to $X(t) - X(s)$ proves the first stochastic assertion.

For a bounded f, the lattice property of the Luxemburg norm gives

$$\|f\|_{E_q} \le \|\mathbf{1}\|_{E_q}\|f\|_{C([0,L])}.$$

The constant $\|\mathbf{1}\|_{E_q}$ is finite because the interval has finite measure and $q_+ < \infty$. Corollary 2 now proves (45), including the assertion away from time zero. □

The proposition retains uniqueness in the original H-valued mild-solution class. It does not assert well-posedness for arbitrary initial data in E_q. This distinction already matters for the constant exponent $q = 4$: the function $f(\xi) = \xi^{-1/3}\mathbf{1}_{(0,\min\{L,1\})}(\xi)$ belongs to H but not to $L^4(0, L)$, so an E_q moment bound at the initial time cannot follow from an H moment alone.

An equation posed intrinsically in E_q requires a compatible realisation of A and estimates for its fractional resolvents in that Banach space. Its stochastic coefficient must be treated through Gaussian radonifying operators rather than the Hilbert–Schmidt norm used here. The theory of van Neerven, Veraar, and Weis [41] supplies such stochastic integration in Banach spaces with unconditional martingale differences. The admissibility of a particular variable-exponent space, and of the corresponding kernel estimate, must be established before the present contraction can be transferred. The condition $q_- > 1$ alone is not a replacement for those operator and stochastic integrability hypotheses.

A variable probability exponent has a different interpretation. If $r{:}\Omega \to [1, \infty)$ is measurable with $r_+ < \infty$, then set $P = \max\{2, r_+\}$ and assume $X_0 \in L^P(\Omega; H)$. For the Luxemburg norm built from $\mathbb{E}\|Z\|^{r(\omega)}$, normalisation by $a = \|Z\|_{L^P}$ gives

$$\mathbb{E}\left(\frac{\|Z\|}{a}\right)^{r(\omega)} \le 1 + \mathbb{E}\left(\frac{\|Z\|}{a}\right)^{P} = 2.$$

Thus, $\|Z\|_{L^{r(\cdot)}(\Omega;H)} \le 2^{1/r_-}\|Z\|_{L^P(\Omega;H)}$. The existing solution consequently inherits this variable-moment bound without changing the maturity generator.

4.6. Spectral Galerkin Approximation

The preceding solution theory admits a finite-dimensional approximation that respects the maturity eigenstructure. Let

$$H_N = \operatorname{span}\{e_1, \dots, e_N\}, \qquad \Pi_N x = \sum_{n=1}^{N} \langle x, e_n\rangle e_n.$$

Define X_N by

$$\begin{aligned} X_N(t) = S_\beta(t)\Pi_N X_0 &+ \int_0^t P_\beta(t-s)\Pi_N F(X_N(s))\,\mathrm{d}s \\ &+ \int_0^t P_\beta(t-s)\Pi_N G(X_N(s))\,\mathrm{d}W(s), \end{aligned} \tag{46}$$

where $\Pi_N G(x)$ denotes the composition of $G(x){:}U \to H$ with Π_N. The projected coefficients retain the Lipschitz constants in (26), and Theorem 1 gives a unique solution that remains in H_N. Spectral approximations of stochastic Volterra equations are studied in [42]; a recent treatment of a related fractional evolution class appears in [43].

Theorem 7 (Strong Galerkin consistency). *Let $2 \le p < \infty$ and $X_0 \in L^p(\Omega, \mathcal{F}_0; H)$. For every $T > 0$,*

$$\lim_{N\to\infty} \sup_{0\le t\le T} \left(\mathbb{E}\|X_N(t) - X(t)\|^p\right)^{1/p} = 0. \tag{47}$$

Proof. The lifted projections, still denoted by Π_N, act as contractions on $L^p(\Omega; H)$ and on $L^p(\Omega; \mathcal{L}_2(U, H))$. They converge strongly to the identity in both spaces. The maps

$$t \longmapsto F(X(t)) \quad \text{in } L^p(\Omega; H), \qquad t \longmapsto G(X(t)) \quad \text{in } L^p(\Omega; \mathcal{L}_2(U, H))$$

are continuous because F and G are globally Lipschitz. Their images of $[0, T]$ are compact. Strong convergence of a uniformly bounded sequence of operators is uniform on compact subsets: for a compact set K, a finite ε-net and the contraction property give

$$\sup_{z\in K}\|(\mathrm{Id}-\Pi_N)z\| \le 2\varepsilon + \max_{1\le j\le m}\big\|(\mathrm{Id}-\Pi_N)z_j\big\|,$$

which tends to at most 2ε. Consequently,

$$r_N^F = \sup_{0\le t\le T}\big(\mathbb{E}\|(\mathrm{Id}-\Pi_N)F(X(t))\|^p\big)^{1/p} \longrightarrow 0, \tag{48}$$

$$r_N^G = \sup_{0\le t\le T}\big(\mathbb{E}\|(\mathrm{Id}-\Pi_N)G(X(t))\|_{\mathcal{L}_2}^p\big)^{1/p} \longrightarrow 0. \tag{49}$$

Moreover, $\big(\mathbb{E}\|(\mathrm{Id}-\Pi_N)X_0\|^p\big)^{1/p} \to 0$ by dominated convergence.

Set $Z_N = X_N - X$. Commutation of Π_N with S_β and P_β gives

$$\begin{aligned} Z_N(t) = {} & S_\beta(t)(\Pi_N-\mathrm{Id})X_0 \\ & + \int_0^t P_\beta(t-s)\Pi_N\big[F(X_N(s))-F(X(s))\big]\,\mathrm{d}s \\ & + \int_0^t P_\beta(t-s)(\Pi_N-\mathrm{Id})F(X(s))\,\mathrm{d}s \\ & + \int_0^t P_\beta(t-s)\Pi_N\big[G(X_N(s))-G(X(s))\big]\,\mathrm{d}W(s) \\ & + \int_0^t P_\beta(t-s)(\Pi_N-\mathrm{Id})G(X(s))\,\mathrm{d}W(s). \end{aligned}$$

The Bielecki estimates used in the fixed-point proof imply

$$\begin{aligned} \|Z_N\|_{p,\eta,T} \le {} & C\big(\mathbb{E}\|(\mathrm{Id}-\Pi_N)X_0\|^p\big)^{1/p} + q_\eta\|Z_N\|_{p,\eta,T} \\ & + C\eta^{-\beta}r_N^F + C_p\eta^{-(\beta-1/2)}r_N^G. \end{aligned}$$

Choose η independently of N so that $q_\eta < 1$. After absorption, the right-hand side converges to zero by (48) and (49). Removing the exponential weight proves (47). □

Corollary 3 (Fractional-domain Galerkin rate)**.** *Assume the hypotheses of Theorem 7. Impose the following additional graph-domain hypothesis, which is not a consequence of global Lipschitz continuity on H. For some $\delta > 0$, let $X_0 \in L^p(\Omega; D(A^\delta))$. Require $F(X(t)) \in D(A^\delta)$ almost surely and $G(X(t)) \in \mathcal{L}_2(U, D(A^\delta))$ almost surely for each t, with*

$$\sup_{0\le t\le T}\left[\big(\mathbb{E}\Big\|A^\delta F(X(t))\Big\|^p\big)^{1/p} + \big(\mathbb{E}\Big\|A^\delta G(X(t))\Big\|_{\mathcal{L}_2}^p\big)^{1/p}\right] < \infty. \tag{50}$$

Then,

$$\sup_{0\le t\le T}\big(\mathbb{E}\|X_N(t)-X(t)\|^p\big)^{1/p} \le C_{p,T,\delta}\mu_{N+1}^{-\delta}\left[\big(\mathbb{E}\Big\|A^\delta X_0\Big\|^p\big)^{1/p}+1\right], \tag{51}$$

where the constant also depends on the bound in (50).

Proof. For $x \in D(A^\delta)$, spectral calculus gives

$$\|(\mathrm{Id}-\Pi_N)x\| \le \mu_{N+1}^{-\delta}\Big\|A^\delta x\Big\|.$$

For $B \in \mathcal{L}_2(U,H)$ with $A^\delta B \in \mathcal{L}_2(U,H)$, application of this estimate to every vector of an orthonormal basis of U gives

$$\|(\mathrm{Id} - \Pi_N)B\|_{\mathcal{L}_2(U,H)} \le \mu_{N+1}^{-\delta} \left\|A^\delta B\right\|_{\mathcal{L}_2(U,H)}.$$

Insert these bounds into the last Bielecki estimate in the proof of Theorem 7 and absorb $q_\eta \|Z_N\|_{p,\eta,T}$. The result follows after the weight is removed. □

Proposition 7 (Verifiable graph-domain hypotheses for the Galerkin rate)**.** *Fix $\delta > 0$ and a finite $2 \le p < \infty$. Equip $H_\delta = D(A^\delta)$ with the Hilbert norm $\|x\|_{H_\delta} = \|A^\delta x\|$. In addition to the global H-Lipschitz assumptions, suppose that $F{:}H_\delta \to H_\delta$ and $G{:}H_\delta \to \mathcal{L}_2(U,H_\delta)$ are globally Lipschitz in the indicated graph norms. If $X_0 \in L^p(\Omega, \mathcal{F}_0; H_\delta)$, then*

$$\sup_{t \le T} \mathbb{E}\left\|A^\delta X(t)\right\|^p \le C_{p,T,\delta}\left(1 + \mathbb{E}\left\|A^\delta X_0\right\|^p\right), \tag{52}$$

and the extra hypothesis (50) holds. Therefore, the Galerkin rate (51) follows under conditions stated directly on the coefficients and the initial curve.

Proof. The map $J_\delta{:}H_\delta \to H$ defined by $J_\delta x = A^\delta x$ is an isometric isomorphism, with inverse $A^{-\delta}$ because $\mu_1 > 0$. Define

$$\widetilde{F}(y) = A^\delta F(A^{-\delta}y), \qquad \widetilde{G}(y) = A^\delta G(A^{-\delta}y).$$

The graph-norm hypotheses give global Lipschitz continuity of these maps on H, with values in H and $\mathcal{L}_2(U,H)$ respectively. Their values at zero are finite in the same spaces. Apply Theorem 1 to the initial condition $A^\delta X_0$ and coefficients $\widetilde{F}, \widetilde{G}$, obtaining a mild solution Y. The operator $A^{-\delta}$ is bounded and commutes with S_β and P_β. Therefore, it can be passed through both integrals, showing that $A^{-\delta}Y$ solves the original equation. Uniqueness in H identifies it with X, and $Y = A^\delta X$. The moment estimate for Y is (52).

Let $L_{F,\delta}$ and $L_{G,\delta}$ denote the graph Lipschitz constants. Linear growth gives

$$\begin{aligned}
\left\|A^\delta F(X(t))\right\| &\le \left\|A^\delta F(0)\right\| + L_{F,\delta}\left\|A^\delta X(t)\right\|, \\
\left\|A^\delta G(X(t))\right\|_{\mathcal{L}_2(U,H)} &\le \left\|A^\delta G(0)\right\|_{\mathcal{L}_2(U,H)} + L_{G,\delta}\left\|A^\delta X(t)\right\|.
\end{aligned}$$

Taking L^p norms and the supremum over $[0,T]$ proves (50). □

For the feedback (13), the conditions $b_0, v \in D(A^\delta)$ and $\omega \in H$ are sufficient for the drift hypothesis because

$$\left\|A^\delta[F(x) - F(y)]\right\| \le \left[\rho + L_\chi \left\|A^\delta v\right\| \|\omega\| \mu_1^{-\delta}\right] \left\|A^\delta(x-y)\right\|.$$

An additive diffusion satisfies the noise hypothesis whenever $A^\delta \Sigma \in \mathcal{L}_2(U,H)$. No restriction $\delta < 1 - 1/(2\beta)$ is needed in this graph-preserving case, since the coefficient already takes values in the higher-regularity space. By contrast, the domain gain in Theorem 5 assumes only H-valued coefficients and is limited by the singular stochastic kernel.

The additional hypothesis is substantive. For a fixed $b \in H \setminus D(A^\delta)$, the map $F(x) = b$ is globally Lipschitz with constant zero, but $A^\delta F(X(t))$ is undefined; therefore, global Lipschitz continuity on H cannot imply (50).

Corollary 4 (Convergence of observable greenium ratios)**.** *For every $w \in H$,*

$$\sup_{0\le t\le T} \left(\mathbb{E}|\langle X_N(t) - X(t), w\rangle|^p\right)^{1/p} \longrightarrow 0.$$

Taking $w = \mathbf{1}_{[0,\tau]}$ proves uniform L^p convergence of the projected logarithmic green–brown zero-coupon ratios for every $\tau \in [0, L]$.

Proof. Cauchy–Schwarz gives $|\langle X_N(t) - X(t), w\rangle| \le \|w\|\|X_N(t) - X(t)\|$, and Theorem 7 completes the proof. □

The Galerkin result provides a direct approximation route for empirical term structure panels. Truncation acts on maturity modes rather than on the temporal memory kernel, so the anomalous time scaling is retained at every fixed spectral dimension, while the complete greenium surface is recovered as N increases.

5. Gaussian Benchmark and Observable Asymptotics

The remainder of the analysis considers

$$F \equiv 0, \qquad G(x) \equiv \Sigma \in \mathcal{L}_2(U, H), \tag{53}$$

with deterministic $X_0 \in H$. The solution is

$$X(t) = S_\beta(t)X_0 + Z(t), \qquad Z(t) = \int_0^t P_\beta(t-s)\Sigma\, \mathrm{d}W(s). \tag{54}$$

The stochastic convolution is Gaussian. Its covariance operator is

$$Q_t = \int_0^t P_\beta(r)\Sigma\Sigma^* P_\beta(r)\, \mathrm{d}r, \tag{55}$$

where the integral is taken in the trace-class norm.

5.1. Gaussian Law and Covariance Operators

Proposition 8 (Gaussian structure)**.** *For every $t \ge 0$, $X(t)$ is Gaussian in H with mean $S_\beta(t)X_0$ and covariance Q_t. For $w, z \in H$,*

$$\mathbb{E}Y_w(t) = \langle S_\beta(t)X_0, w\rangle, \tag{56}$$

$$\mathrm{Cov}(Y_w(t), Y_z(t)) = \int_0^t \langle \Sigma^* P_\beta(r)w, \Sigma^* P_\beta(r)z\rangle_U\, \mathrm{d}r. \tag{57}$$

The map $t \mapsto Q_t$ is continuous in the trace norm.

Proof. The stochastic integral in (54) is an H-valued centred Gaussian random variable because its integrand is deterministic. Itô's isometry gives

$$\mathbb{E}\langle Z(t), w\rangle\langle Z(t), z\rangle = \int_0^t \langle \Sigma^* P_\beta(r)^* w, \Sigma^* P_\beta(r)^* z\rangle_U\, \mathrm{d}r.$$

The operator $P_\beta(r)$ is self-adjoint, which proves (57). The integrand in (55) is positive and trace-class, with

$$\left\|P_\beta(r)\Sigma\Sigma^* P_\beta(r)\right\|_{\mathcal{L}_1(H)} = \left\|P_\beta(r)\Sigma\right\|^2_{\mathcal{L}_2(U,H)} \le Cr^{2\beta-2}\|\Sigma\|^2_{\mathcal{L}_2}.$$

The right-hand side is integrable at zero. Continuity of Q_t follows from absolute continuity of the trace-class Bochner integral. □

5.2. Polynomial Persistence and the Limiting One-Time Law

Theorem 8 (Sharp polynomial persistence)**.** *For every $x \in H$,*

$$t^{\beta} S_{\beta}(t)x \longrightarrow \frac{1}{\Gamma(1-\beta)} A^{-1}x \qquad \text{in } H \tag{58}$$

as $t \to \infty$. Consequently,

$$\mathbb{E}Y_w(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)} \left\langle A^{-1}X_0, w \right\rangle + o(t^{-\beta})$$

for every $w \in H$.

Proof. Write $x = \sum_{n\geq 1} x_n e_n$. The scalar asymptotic

$$E_{\beta}(-r) = \frac{1}{r\Gamma(1-\beta)} + O(r^{-2}), \qquad r \to \infty$$

gives the following for each n:

$$t^{\beta} E_{\beta}(-\mu_n t^{\beta}) \longrightarrow \frac{1}{\mu_n \Gamma(1-\beta)}.$$

The estimate $\left|E_{\beta}(-r)\right| \leq C(1+r)^{-1}$ implies

$$t^{\beta} \left| E_{\beta}(-\mu_n t^{\beta}) \right| \leq \frac{C}{\mu_n} \leq \frac{C}{\mu_1}.$$

Therefore, the difference between the two spectral coefficients is bounded by a constant independent of n and t. Dominated convergence with respect to the summable sequence $(|x_n|^2)_n$ proves (58). Taking the inner product with w gives the scalar assertion. □

Theorem 9 (Long-horizon Gaussian limit)**.** *The operator*

$$Q_{\infty} = \int_0^{\infty} P_{\beta}(r) \Sigma \Sigma^{*} P_{\beta}(r) \, \mathrm{d}r \tag{59}$$

is positive, self-adjoint, and trace-class. For $t \geq 1$,

$$\|Q_{\infty} - Q_t\|_{\mathcal{L}_1(H)} \leq C t^{-2\beta-1} \|\Sigma\|_{\mathcal{L}_2}^2. \tag{60}$$

The law of $X(t)$ converges weakly on H to the centred Gaussian measure $\mathcal{N}(0, Q_{\infty})$.

Proof. Near the origin, the trace norm of the integrand in (59) is bounded by $Cr^{2\beta-2}\|\Sigma\|_{\mathcal{L}_2}^2$, which is integrable. For $r \geq 1$, (19) gives

$$\left\| P_{\beta}(r) \right\| \leq C r^{-\beta-1},$$

hence,

$$\int_1^{\infty} \left\| P_{\beta}(r) \Sigma \Sigma^{*} P_{\beta}(r) \right\|_{\mathcal{L}_1} \mathrm{d}r \leq C \|\Sigma\|_{\mathcal{L}_2}^2 \int_1^{\infty} r^{-2\beta-2} \, \mathrm{d}r < \infty.$$

This proves the existence and trace-class properties of Q_{∞}. The same estimate gives

$$\|Q_{\infty} - Q_t\|_{\mathcal{L}_1} \leq C \|\Sigma\|_{\mathcal{L}_2}^2 \int_t^{\infty} r^{-2\beta-2} \, \mathrm{d}r = C t^{-2\beta-1} \|\Sigma\|_{\mathcal{L}_2}^2.$$

Theorem 8 shows that the mean converges to zero, while (60) gives convergence of the covariance in the trace norm, so the Gaussian convergence criterion recorded in Appendix D applies directly. For completeness, $Q_t \le Q_\infty$ in the order of positive operators, and the spectral projection R_N onto the first N eigenvectors of Q_∞ satisfies

$$\sup_{t\ge 0} \operatorname{Tr}\big((\mathrm{Id} - R_N)Q_t\big) \le \operatorname{Tr}\big((\mathrm{Id} - R_N)Q_\infty\big) \longrightarrow 0.$$

The set $\{S_\beta(t)X_0 : t \ge 0\}$ is relatively compact, as the map is continuous and converges to zero. Finite-dimensional Gaussian estimates and the displayed tail bound imply tightness; the characteristic functionals converge according to

$$\exp\left(i\langle S_\beta(t)X_0, h\rangle - \frac{1}{2}\langle Q_t h, h\rangle\right) \longrightarrow \exp\left(-\frac{1}{2}\langle Q_\infty h, h\rangle\right), \qquad h \in H,$$

and the uniqueness of Gaussian characteristic functionals proves the asserted weak convergence. □

Theorem 10 (Quantitative Wasserstein convergence)**.** *Let $\mathcal{W}_2$ denote the quadratic Wasserstein distance on probability measures over H with finite second moment. For $t \ge 1$,*

$$\mathcal{W}_2\big(\mathcal{L}aw(X(t)), \mathcal{N}(0, Q_\infty)\big) \le Ct^{-\beta}\|X_0\| + Ct^{-\beta-1/2}\|\Sigma\|_{\mathcal{L}_2(U,H)}. \tag{61}$$

Proof. On an auxiliary probability space carrying a cylindrical Wiener process $\widetilde{W}$ on U, define

$$Z_t^\sharp = \int_0^t P_\beta(r)\Sigma\,\mathrm{d}\widetilde{W}(r), \qquad Z_\infty^\sharp = \int_0^\infty P_\beta(r)\Sigma\,\mathrm{d}\widetilde{W}(r).$$

The second integral is well defined by Theorem 9. The random variables $Z_t^\sharp$ and $Z_\infty^\sharp$ have laws $\mathcal{N}(0, Q_t)$ and $\mathcal{N}(0, Q_\infty)$, respectively. Consequently,

$$\big(S_\beta(t)X_0 + Z_t^\sharp, Z_\infty^\sharp\big)$$

is an admissible coupling of the two measures in (61). Orthogonality of the deterministic mean and the centred stochastic tail gives

$$\begin{aligned}\mathcal{W}_2^2\big(\mathcal{L}aw(X(t)), \mathcal{N}(0, Q_\infty)\big) &\le \big\|S_\beta(t)X_0\big\|^2 + \mathbb{E}\Big\|Z_\infty^\sharp - Z_t^\sharp\Big\|^2 \\ &= \big\|S_\beta(t)X_0\big\|^2 + \int_t^\infty \big\|P_\beta(r)\Sigma\big\|_{\mathcal{L}_2}^2\,\mathrm{d}r.\end{aligned}$$

The scalar bound for E_β yields $\big\|S_\beta(t)\big\| \le Ct^{-\beta}$ for $t \ge 1$, and (19) yields

$$\int_t^\infty \big\|P_\beta(r)\Sigma\big\|_{\mathcal{L}_2}^2\,\mathrm{d}r \le C\|\Sigma\|_{\mathcal{L}_2}^2 \int_t^\infty r^{-2\beta-2}\,\mathrm{d}r \le Ct^{-2\beta-1}\|\Sigma\|_{\mathcal{L}_2}^2.$$

Taking the square roots proves (61). □

The rate separates the fading initial greenium surface from the residual long-horizon dispersion. The deterministic contribution decays as $t^{-\beta}$, while the missing part of the limiting stochastic convolution decays faster by one-half power of time.

Remark 5 (Memory and stationarity)**.** *The limiting measure describes one-time marginals. The resolvent does not satisfy the semigroup identity, so convergence of one-time marginals does not supply a Markov transition law for the observed surface. A memory-state enlargement is discussed*

in Section 7. A stationary Gaussian moving average can be constructed with a two-sided Wiener process and the kernel P_β, since the kernel belongs to $L^2(0,\infty;\mathcal{L}_2(U,H))$ after composition with Σ.

5.3. Short-Time Covariance in the Trace Norm

Set

$$c_\beta = \frac{1}{\Gamma(\beta)^2(2\beta-1)}. \tag{62}$$

The next theorem gives an operator statement that applies simultaneously to every maturity weight.

Theorem 11 (Trace norm covariance scaling)**.** *As $t \downarrow 0$,*

$$t^{1-2\beta}Q_t \longrightarrow c_\beta \Sigma\Sigma^* \qquad \text{in } \mathcal{L}_1(H). \tag{63}$$

Consequently, for every $w, z \in H$,

$$\mathrm{Cov}(Y_w(t), Y_z(t)) = c_\beta \langle \Sigma^* w, \Sigma^* z\rangle_U t^{2\beta-1} + o(t^{2\beta-1}). \tag{64}$$

In particular, (64) applies to the exact zero-coupon weights $w = \mathbf{1}_{[0,\tau]}$.

Proof. Define

$$B(r) = r^{1-\beta}P_\beta(r) = E_{\beta,\beta}(-r^\beta A), \qquad r > 0.$$

The family is uniformly bounded and converges strongly to $\Gamma(\beta)^{-1}\,\mathrm{Id}$ as $r \downarrow 0$. Strong convergence together with uniform boundedness implies

$$\left\| B(r)\Sigma - \Gamma(\beta)^{-1}\Sigma \right\|_{\mathcal{L}_2(U,H)} \longrightarrow 0. \tag{65}$$

Indeed, for an orthonormal basis (u_k) of U, each term $\left\|(B(r) - \Gamma(\beta)^{-1}\,\mathrm{Id})\Sigma u_k\right\|^2$ converges to zero and is dominated by $C\|\Sigma u_k\|^2$.

Let $C_\Sigma = \Sigma\Sigma^*$. The ideal inequalities for Hilbert–Schmidt operators give

$$\begin{aligned}
&\left\| B(r)C_\Sigma B(r) - \Gamma(\beta)^{-2}C_\Sigma \right\|_{\mathcal{L}_1} \\
&\quad \le \left\| B(r)\Sigma - \Gamma(\beta)^{-1}\Sigma \right\|_{\mathcal{L}_2} \|\Sigma^* B(r)\|_{\mathcal{L}_2} \\
&\qquad + \Gamma(\beta)^{-1}\|\Sigma\|_{\mathcal{L}_2} \left\| \Sigma^* B(r) - \Gamma(\beta)^{-1}\Sigma^* \right\|_{\mathcal{L}_2},
\end{aligned}$$

which converges to zero by (65). Substituting $r = tu$ in (55) yields

$$t^{1-2\beta}Q_t = \int_0^1 u^{2\beta-2} B(tu) C_\Sigma B(tu)\,\mathrm{d}u.$$

The integrand converges in the trace norm to $u^{2\beta-2}\Gamma(\beta)^{-2}C_\Sigma$ and is dominated by $Cu^{2\beta-2}\|C_\Sigma\|_{\mathcal{L}_1}$. Since $2\beta - 2 > -1$, dominated convergence in the trace-class Banach space gives

$$\lim_{t\downarrow 0} t^{1-2\beta}Q_t = \Gamma(\beta)^{-2}C_\Sigma \int_0^1 u^{2\beta-2}\,\mathrm{d}u = c_\beta C_\Sigma.$$

Pairing the operator limit with w and z proves (64). □

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Corollary 5 (Exact zero-coupon variance). *Let $X_0 \in H$ be deterministic and let $\tau \in (0, L]$. Then,*

$$\mathrm{Var}\left(\log \frac{P^G(t,\tau)}{P^B(t,\tau)}\right) = c_\beta \left\|\Sigma^* \mathbf{1}_{[0,\tau]}\right\|_U^2 t^{2\beta-1} + o(t^{2\beta-1}).$$

If $\Sigma^* \mathbf{1}_{[0,\tau]} \neq 0$, the leading coefficient is strictly positive.

Proof. Proposition 1 identifies the logarithm with $Y_{\mathbf{1}_{[0,\tau]}}(t)$. The assertion follows from Theorem 11. □

5.4. Second-Order Short-Time Expansion

Lemma 6 (Two-term kernel expansion). *For every $w \in D(A^2)$,*

$$P_\beta(t)w = \frac{t^{\beta-1}}{\Gamma(\beta)} w - \frac{t^{2\beta-1}}{\Gamma(2\beta)} Aw + R_w(t), \tag{66}$$

where

$$\|R_w(t)\| \le C t^{3\beta-1} \left\|A^2 w\right\|, \qquad 0 < t \le 1. \tag{67}$$

Proof. For $r \ge 0$, the Mittag-Leffler series and the estimates in Appendix A imply

$$\left| E_{\beta,\beta}(-r) - \Gamma(\beta)^{-1} + r\Gamma(2\beta)^{-1} \right| \le C r^2. \tag{68}$$

Write $w = \sum_n w_n e_n$. The remainder in the nth coefficient of (66) equals

$$t^{\beta-1}\left[E_{\beta,\beta}(-\mu_n t^\beta) - \Gamma(\beta)^{-1} + \mu_n t^\beta \Gamma(2\beta)^{-1}\right] w_n.$$

By (68), its absolute value is at most $C t^{3\beta-1} \mu_n^2 |w_n|$. Parseval's identity proves (67). □

Theorem 12 (Two-term covariance asymptotics). *Let $w, z \in D(A^2)$. As $t \downarrow 0$,*

$$\begin{aligned}\mathrm{Cov}(Y_w(t), Y_z(t)) = {} & \frac{\langle \Sigma^* w, \Sigma^* z\rangle_U}{\Gamma(\beta)^2(2\beta-1)} t^{2\beta-1} \\ & - \frac{\langle \Sigma^* Aw, \Sigma^* z\rangle_U + \langle \Sigma^* w, \Sigma^* Az\rangle_U}{\Gamma(\beta)\Gamma(2\beta)(3\beta-1)} t^{3\beta-1} + O(t^{4\beta-1}).\end{aligned} \tag{69}$$

For $w = z$,

$$\begin{aligned}\mathrm{Var}(Y_w(t)) = {} & c_\beta \|\Sigma^* w\|_U^2 t^{2\beta-1} \\ & - \frac{2\langle \Sigma^* w, \Sigma^* Aw\rangle_U}{\Gamma(\beta)\Gamma(2\beta)(3\beta-1)} t^{3\beta-1} + O(t^{4\beta-1}).\end{aligned} \tag{70}$$

The constants hidden in the remainders depend continuously on $\|A^2 w\|$ and $\|A^2 z\|$.

Proof. By Proposition 8, the covariance equals

$$\int_0^t \left\langle \Sigma^* P_\beta(r) w, \Sigma^* P_\beta(r) z \right\rangle_U \mathrm{d}r.$$

Set

$$a_w = \Gamma(\beta)^{-1} \Sigma^* w, \qquad b_w = -\Gamma(2\beta)^{-1} \Sigma^* Aw, \qquad \varepsilon_w(r) = \Sigma^* R_w(r).$$

Lemma 6 gives

$$\Sigma^* P_\beta(r) w = r^{\beta-1} a_w + r^{2\beta-1} b_w + \varepsilon_w(r), \qquad \|\varepsilon_w(r)\|_U \le C r^{3\beta-1} \left\|A^2 w\right\|.$$

Multiplication of the two expansions yields

$$\langle \Sigma^* P_\beta(r)w, \Sigma^* P_\beta(r)z\rangle_U = r^{2\beta-2}\langle a_w, a_z\rangle_U + r^{3\beta-2}\big(\langle b_w, a_z\rangle_U + \langle a_w, b_z\rangle_U\big) + O(r^{4\beta-2}).$$

The remainder estimate follows from Cauchy–Schwarz and the bounds on ε_w and ε_z; every omitted exponent is at least $4\beta-2$. Integration over $(0,t)$ gives (69). Taking $z=w$ gives (70). □

5.5. Exponential Relative Values and Entropic Risk

For $w \in H$, define the exponential relative-value functional

$$\Pi_w(t) = \exp(Y_w(t)).$$

When $w = \mathbf{1}_{[0,\tau]}$, Proposition 1 gives $\Pi_w(t) = P^G(t,\tau)/P^B(t,\tau)$ pathwise.

Proposition 9 (Gaussian exponential transform)**.** *For every $\lambda \in \mathbb{R}$ and $w \in H$,*

$$\mathbb{E}\exp(\lambda Y_w(t)) = \exp\left(\lambda\langle S_\beta(t)X_0, w\rangle + \frac{\lambda^2}{2}\langle Q_t w, w\rangle\right). \tag{71}$$

If $X_0 = 0$, then

$$\log \mathbb{E}\Pi_w(t) = \frac{c_\beta}{2}\|\Sigma^* w\|_U^2 t^{2\beta-1} + o(t^{2\beta-1}). \tag{72}$$

For $w \in D(A^2)$, the refined expansion is

$$\log \mathbb{E}\Pi_w(t) = \frac{\|\Sigma^* w\|_U^2}{2\Gamma(\beta)^2(2\beta-1)}t^{2\beta-1} - \frac{\langle \Sigma^* w, \Sigma^* A w\rangle_U}{\Gamma(\beta)\Gamma(2\beta)(3\beta-1)}t^{3\beta-1} + O(t^{4\beta-1}). \tag{73}$$

Proof. A Gaussian random variable with mean m and variance V has moment-generating function $\exp(\lambda m + \lambda^2 V/2)$. Proposition 8 proves (71). Equations (72) and (73) follow from Theorems 11 and 12. □

Proposition 10 (Dominance of memory-induced convexity)**.** *Assume $X_0 \in D(A^2)$ and $w \in D(A^2)$. As $t \downarrow 0$,*

$$\log \frac{\mathbb{E}\Pi_w(t)}{\Pi_w(0)} = \frac{\|\Sigma^* w\|_U^2}{2\Gamma(\beta)^2(2\beta-1)}t^{2\beta-1} - \frac{\langle AX_0, w\rangle}{\Gamma(1+\beta)}t^\beta + o(t^\beta). \tag{74}$$

If $\Sigma^* w \neq 0$, the stochastic convexity term is asymptotically larger than the deterministic relaxation term because $2\beta - 1 < \beta$.

Proof. By spectral calculus, the scalar series for E_β gives

$$S_\beta(t)X_0 = X_0 - \frac{t^\beta}{\Gamma(1+\beta)}AX_0 + \mathcal{R}_0(t), \qquad \|\mathcal{R}_0(t)\| \le Ct^{2\beta}\left\|A^2X_0\right\|.$$

Proposition 9 then yields

$$\log \frac{\mathbb{E}\Pi_w(t)}{\Pi_w(0)} = -\frac{t^\beta}{\Gamma(1+\beta)}\langle AX_0, w\rangle + O(t^{2\beta}) + \frac{1}{2}\operatorname{Var}(Y_w(t)).$$

Theorem 12 gives

$$\frac{1}{2}\operatorname{Var}(Y_w(t)) = \frac{\|\Sigma^* w\|_U^2}{2\Gamma(\beta)^2(2\beta-1)} t^{2\beta-1} + O(t^{3\beta-1}).$$

Both 2β and $3\beta-1$ are strictly larger than β because $\beta > 1/2$. Substitution proves (74). □

The leading term in (72) applies to a discontinuous zero-coupon indicator without smoothing. The second term in (73) detects the interaction between the exposure and the maturity generator. It can change sign because the inner product $\langle \Sigma^* w, \Sigma^* A w \rangle$ depends on how the noise covariance aligns with the spectral shape of the portfolio weight.

For $\gamma > 0$, define the entropic functional

$$\rho_\gamma(t; w) = \frac{1}{\gamma} \log \mathbb{E} \exp(-\gamma Y_w(t)). \tag{75}$$

Proposition 11 (Entropic transition functional)**.** *For every $w \in H$,*

$$\rho_\gamma(t; w) = -\langle S_\beta(t) X_0, w \rangle + \frac{\gamma}{2} \langle Q_t w, w \rangle. \tag{76}$$

If $X_0 = 0$, then

$$\rho_\gamma(t; w) = \frac{\gamma c_\beta}{2} \|\Sigma^* w\|_U^2 t^{2\beta-1} + o(t^{2\beta-1}).$$

For $w \in D(A^2)$, the second-order correction is obtained by multiplying the last two terms in (70) by $\gamma/2$.

Proof. Apply (71) with $\lambda = -\gamma$ and divide the logarithm by γ. □

5.6. Local Increments and Identification of Climate Memory

Short-time variance from the initial date and local variance around a positive date share the same exponent. The leading constants differ because an increment at a positive date contains both newly arriving noise and a reweighting of the preceding stochastic history.

Define

$$I_\beta = \int_0^\infty \left((u+1)^{\beta-1} - u^{\beta-1} \right)^2 \mathrm{d}u. \tag{77}$$

The integral is strictly positive for $1/2 < \beta < 1$. Its integrand is bounded by $Cu^{2\beta-2}$ for $0 < u \leq 1$ and by $Cu^{2\beta-4}$ for $u \geq 1$, so both improper integrals converge.

Theorem 13 (Local increment variance)**.** *Let $w \in H$ and suppose that $\Sigma^* w \neq 0$. For $t_0 = 0$,*

$$\operatorname{Var}(Y_w(h) - Y_w(0)) \sim \frac{\|\Sigma^* w\|_U^2}{\Gamma(\beta)^2(2\beta-1)} h^{2\beta-1}. \tag{78}$$

For every $t_0 > 0$,

$$\operatorname{Var}(Y_w(t_0+h) - Y_w(t_0)) \sim d_\beta \|\Sigma^* w\|_U^2 h^{2\beta-1}, \tag{79}$$

where

$$d_\beta = \frac{1}{\Gamma(\beta)^2} \left(\frac{1}{2\beta-1} + I_\beta \right). \tag{80}$$

The convergence in (79) is independent of the value of t_0 so long as $t_0 > 0$ is fixed.

Proof. Equation (78) is Theorem 11. Fix $t_0 > 0$. The increment of the stochastic convolution decomposes as

$$Z(t_0+h) - Z(t_0) = \int_0^{t_0} [P_\beta(t_0+h-s) - P_\beta(t_0-s)]\Sigma \, \mathrm{d}W(s) + \int_{t_0}^{t_0+h} P_\beta(t_0+h-s)\Sigma \, \mathrm{d}W(s).$$

The two terms are independent. Pairing with w and applying Itô's isometry gives

$$\mathrm{Var}(Y_w(t_0+h) - Y_w(t_0)) = I_h^{\mathrm{old}} + I_h^{\mathrm{new}}, \tag{81}$$

$$I_h^{\mathrm{old}} = \int_0^{t_0} \left\| \Sigma^* [P_\beta(r+h) - P_\beta(r)] w \right\|_U^2 \mathrm{d}r, \tag{82}$$

$$I_h^{\mathrm{new}} = \int_0^{h} \left\| \Sigma^* P_\beta(r) w \right\|_U^2 \mathrm{d}r. \tag{83}$$

Theorem 11 implies

$$h^{1-2\beta} I_h^{\mathrm{new}} \longrightarrow \frac{\|\Sigma^* w\|_U^2}{\Gamma(\beta)^2(2\beta-1)}.$$

For the old history term, set $r = hu$. Then,

$$h^{1-2\beta} I_h^{\mathrm{old}} = \int_0^{t_0/h} \left\| \Sigma^* h^{1-\beta} [P_\beta(h(u+1)) - P_\beta(hu)] w \right\|_U^2 \mathrm{d}u.$$

For every fixed $u > 0$, strong continuity of the rescaled resolvent gives

$$h^{1-\beta} P_\beta(hu) w = u^{\beta-1} E_{\beta,\beta}(-h^\beta u^\beta A) w \longrightarrow \frac{u^{\beta-1}}{\Gamma(\beta)} w.$$

Therefore, the pointwise limit of the integrand is

$$\frac{\|\Sigma^* w\|_U^2}{\Gamma(\beta)^2} \left((u+1)^{\beta-1} - u^{\beta-1} \right)^2.$$

A dominating function on $(0, \infty)$ follows from two estimates. For $0 < u < 1$, Lemma 2 gives

$$h^{1-\beta} \left\| P_\beta(h(u+1)) - P_\beta(hu) \right\| \le C\left((u+1)^{\beta-1} + u^{\beta-1} \right).$$

Its square is integrable near zero. For $u \ge 1$, Lemma 3 gives

$$h^{1-\beta} \left\| P_\beta(h(u+1)) - P_\beta(hu) \right\| \le h^{1-\beta} \int_{hu}^{h(u+1)} C q^{\beta-2} \, \mathrm{d}q \le C u^{\beta-2}.$$

Its square is integrable at infinity. More specifically, after extension by zero outside $(0, t_0/h)$, the squared integrand is bounded by

$$C \|\Sigma^*\|_{\mathcal{L}(H,U)}^2 \|w\|^2 \left[u^{2\beta-2} \mathbf{1}_{(0,1]}(u) + u^{2\beta-4} \mathbf{1}_{(1,\infty)}(u) \right].$$

The integrals of the two powers equal $(2\beta-1)^{-1}$ and $(3-2\beta)^{-1}$, respectively; therefore, dominated convergence gives the unambiguous identity

$$\lim_{h \downarrow 0} h^{1-2\beta} I_h^{\mathrm{old}} = \frac{I_\beta}{\Gamma(\beta)^2} \|\Sigma^* w\|_U^2. \tag{84}$$

Adding the independent new history limit proves (79) with $d_\beta = \Gamma(\beta)^{-2}\{(2\beta-1)^{-1} + I_\beta\}$. The limit concerns each fixed $t_0 > 0$; no uniform assertion as t_0 approaches zero is used. □

Definition 2 (Second-order temporal exponent). *For a square-integrable real process Y and a time $t_0 \geq 0$, define*

$$H_2(Y; t_0) = \frac{1}{2} \lim_{h \downarrow 0} \frac{\log \mathrm{Var}(Y(t_0 + h) - Y(t_0))}{\log h},$$

provided that the limit exists and the increment variance is positive for small h. The corresponding variogram exponent is

$$\zeta_2(Y; t_0) = 2H_2(Y; t_0).$$

Corollary 6 (Identification of the Caputo order). *Under the assumptions of Theorem 13,*

$$\beta = \frac{1}{2}\left(1 + \lim_{h \downarrow 0} \frac{\log \mathrm{Var}(Y_w(t_0 + h) - Y_w(t_0))}{\log h}\right) \tag{85}$$

for every $t_0 \geq 0$. Moreover,

$$H_2(Y_w; t_0) = \beta - \frac{1}{2}, \qquad \zeta_2(Y_w; t_0) = 2\beta - 1.$$

Proof. Theorem 13 gives $\mathrm{Var}(Y_w(t_0 + h) - Y_w(t_0)) = c(t_0)h^{2\beta-1}(1 + o(1))$ with $c(t_0) > 0$. Taking logarithms and dividing by $\log h$ proves the assertions. □

Proposition 12 (Two-scale identification and conditional calibration). *Fix $t_0 \geq 0$, a weight $w \in H$ with $\Sigma^* w \neq 0$, and a scale ratio $a > 1$. Put $V(h) = \mathrm{Var}\{Y_w(t_0 + h) - Y_w(t_0)\}$. Then,*

$$\lim_{h \downarrow 0} \frac{1}{2}\left[1 + \frac{\log V(ah) - \log V(h)}{\log a}\right] = \beta. \tag{86}$$

If positive variance estimates satisfy $\widehat{V}(h)/V(h) \to 1$ and $\widehat{V}(ah)/V(ah) \to 1$ in probability, the statistic

$$\widehat{\beta}_h = \frac{1}{2}\left[1 + \frac{\log \widehat{V}(ah) - \log \widehat{V}(h)}{\log a}\right] \tag{87}$$

converges in probability to β.

Proof. Theorem 13 gives $V(h) = c(t_0)h^{2\beta-1}\{1 + \varepsilon(h)\}$ with $c(t_0) > 0$ and $\varepsilon(h) \to 0$. Therefore,

$$\frac{V(ah)}{V(h)} = a^{2\beta-1}\frac{1 + \varepsilon(ah)}{1 + \varepsilon(h)} \longrightarrow a^{2\beta-1},$$

which proves (86). For the estimates, subtracting the population statistic from (87) gives

$$\frac{\log\{\widehat{V}(ah)/V(ah)\} - \log\{\widehat{V}(h)/V(h)\}}{2\log a}.$$

Continuity of the logarithm at one makes this difference converge to zero in probability. If an estimate can be non-positive, the statistic may be defined arbitrarily on that event, for which the probability tends to zero under the stated relative consistency assumptions. □

This result separates population identification from empirical estimation. The assumptions on $\widehat{V}$ are not supplied by a single observed path without further sampling arguments.

At sampled dates, additive independent measurement errors with variance $\eta^2 > 0$ (also independent of the latent surface) give the observed variogram

$$V_{\mathrm{obs}}(h) = V(h) + 2\eta^2. \tag{88}$$

Its two-scale ratio tends to one as $h \downarrow 0$, so the uncorrected statistic tends to the boundary value $1/2$ rather than the latent β; therefore, the stochastic integrability interval is a model constraint.

Theorem 14 (Sample regularity and graph-dimension bound)**.** *Let $w \in H$, and assume that $X_0 \in D(A^{\vartheta})$ for some*

$$1 - \frac{1}{2\beta} \le \vartheta \le 1. \tag{89}$$

The process Y_w has a modification for which the paths are locally Hölder continuous with every exponent

$$0 < \eta < \beta - \frac{1}{2}. \tag{90}$$

For every $T > 0$, the Hausdorff dimension of the graph of this modification satisfies

$$\dim_{\mathrm{H}}\{(t, Y_w(t)) : 0 \le t \le T\} \le \frac{5}{2} - \beta \qquad \text{almost surely.} \tag{91}$$

Proof. Write $Y_w = m_w + \widetilde{Y}_w$, where $m_w(t) = \langle S_\beta(t)X_0, w\rangle$ and $\widetilde{Y}_w$ is centred. By (23) and (89),

$$|m_w(t) - m_w(s)| \le C\|w\|\left\|A^{\vartheta}X_0\right\||t-s|^{\beta\vartheta} \le C_T|t-s|^{\beta-1/2}.$$

For the centred part, the decomposition used in Lemma 5 together with (25) gives

$$\mathrm{Var}(\widetilde{Y}_w(t) - \widetilde{Y}_w(s)) \le C_T\|\Sigma^*\|^2_{\mathcal{L}(H,U)}\|w\|^2|t-s|^{2\beta-1}.$$

Gaussian moment equivalence and the elementary inequality for the sum of the deterministic and centred increments imply that for every $2 \le p < \infty$,

$$\mathbb{E}|Y_w(t) - Y_w(s)|^p \le C_{p,T}|t-s|^{p(\beta-1/2)}.$$

For each exponent $\eta < \beta - 1/2$, choose p to be sufficiently large that $\eta < \beta - 1/2 - 1/p$. Kolmogorov's theorem produces an η-Hölder modification. A countable increasing sequence of such exponents yields one modification with all the regularities in (90). A real function that is Hölder continuous of order $\eta \in (0,1)$ has graph dimension at most $2-\eta$. Letting a countable sequence η increase to $\beta - 1/2$ proves (91). □

Remark 6 (Fractal terminology)**.** *The quantity $\beta - 1/2$ is the second-order temporal exponent. The number $2\beta - 1$ is its variogram exponent. Formula (91) gives the graph-dimension upper bound $5/2 - \beta$. These three quantities have different definitions and are distinct.*

5.7. Maturity Spectrum of the Limiting Covariance

The temporal and maturity orders interact explicitly when the noise covariance shares the eigenbasis of A. Suppose that $U = H$ and

$$\Sigma\Sigma^* e_n = \sigma_n^2 e_n, \qquad n \ge 1, \tag{92}$$

with $\sum_n \sigma_n^2 < \infty$. Define

$$K_\beta = \int_0^\infty r^{2\beta-2} E_{\beta,\beta}(-r^\beta)^2 \, \mathrm{d}r. \tag{93}$$

The integral is finite and strictly positive.

Theorem 15 (Exact limiting covariance spectrum)**.** *Under* (92)*, the limiting covariance* Q_∞ *is diagonal in* (e_n) *and*

$$Q_\infty e_n = q_n e_n, \qquad q_n = K_\beta \sigma_n^2 \mu_n^{-(2\beta-1)/\beta}. \tag{94}$$

If

$$\sigma_n^2 \sim c_\sigma n^{-2\delta} \qquad \text{for some } c_\sigma > 0 \text{ and } \delta > \frac{1}{2}, \tag{95}$$

then

$$q_n \sim C_{\alpha,\beta,\kappa,L,\sigma} n^{-2\delta-\alpha(2\beta-1)/\beta}, \tag{96}$$

where

$$C_{\alpha,\beta,\kappa,L,\sigma} = K_\beta c_\sigma \left[\kappa \left(\frac{\pi}{L}\right)^\alpha\right]^{-(2\beta-1)/\beta}.$$

Consequently,

$$\alpha = \frac{\beta}{2\beta-1} \lim_{n\to\infty} \frac{-\log(q_n/\sigma_n^2)}{\log n}. \tag{97}$$

Proof. Because $P_\beta(t)$ and $\Sigma\Sigma^*$ are diagonal in the same basis,

$$q_n = \sigma_n^2 \int_0^\infty t^{2\beta-2} E_{\beta,\beta}(-\mu_n t^\beta)^2 \, \mathrm{d}t.$$

Set $r = \mu_n^{1/\beta} t$. Then,

$$q_n = \sigma_n^2 \mu_n^{-(2\beta-1)/\beta} \int_0^\infty r^{2\beta-2} E_{\beta,\beta}(-r^\beta)^2 \, \mathrm{d}r,$$

which proves (94). Near zero, the integrand in (93) behaves as $r^{2\beta-2}/\Gamma(\beta)^2$, which is integrable. At infinity, the estimate $\left|E_{\beta,\beta}(-r^\beta)\right| \le C(1+r^\beta)^{-2}$ gives an integrable upper bound of order $r^{-2\beta-2}$. Positivity follows because the integrand is not identically zero.

From (11),

$$\mu_n \sim \kappa \left(\frac{\pi}{L}\right)^\alpha n^\alpha.$$

Combining this relation with (95) and (94) proves (96). Finally,

$$-\log(q_n/\sigma_n^2) = \frac{2\beta-1}{\beta} \log \mu_n - \log K_\beta,$$

and division by $\log n$ gives (97). □

Corollary 7 (Regularity of the limiting Gaussian surface)**.** *Assume* (92) *and* (95)*, and let* Z_∞ *have law* $\mathcal{N}(0, Q_\infty)$*. Then,*

$$\mathbb{E}\left\| A^\theta Z_\infty \right\|^2 < \infty \tag{98}$$

if and only if

$$\theta < \frac{2\delta - 1 + \alpha(2\beta-1)/\beta}{2\alpha}. \tag{99}$$

Proof. The Karhunen–Loève expansion gives

$$\mathbb{E}\left\|A^{\theta}Z_{\infty}\right\|^{2}=\sum_{n\geq 1}\mu_{n}^{2\theta}q_{n}.$$

By (96), the nth summand is asymptotic to a positive constant times

$$n^{2\alpha\theta-2\delta-\alpha(2\beta-1)/\beta}.$$

The series converges exactly when the exponent is smaller than -1, which is equivalent to (99). At equality, the terms are asymptotic to a positive multiple of n^{-1}, so the series diverges. □

Corollary 8 (Continuity of the limiting maturity surface)**.** *Under* (92) *and* (95)*, suppose that*

$$2\delta+\frac{\alpha(2\beta-1)}{\beta}>2. \tag{100}$$

Then, Z_{∞} admits a $C([0,L])$-valued version and

$$\mathbb{E}\|Z_{\infty}\|^{2}_{C([0,L])}<\infty.$$

Proof. Condition (100) is equivalent to the existence of a number θ such that

$$\frac{1}{2\alpha}<\theta<\frac{2\delta-1+\alpha(2\beta-1)/\beta}{2\alpha}.$$

Corollary 7 then gives $Z_{\infty}\in L^{2}(\Omega;D(A^{\theta}))$. The embedding $D(A^{\theta})\hookrightarrow C([0,L])$ from Appendix C completes the proof. □

The spectrum in (96) separates two sources of maturity smoothness. The factor $n^{-2\delta}$ comes from the spatial colour of the noise, while the factor generated by $\alpha(2\beta-1)/\beta$ comes from the fractional resolvent. Estimation of β from temporal increments can be combined with the maturity covariance slope to infer α, provided that the high-mode behaviour of the noise is either known or estimated independently.

5.8. Noise Colour and the Non-Diagonal Inverse Problem

The diagonal assumption in (92) identifies the covariance eigenvectors with the known maturity modes. It is a benchmark assumption, not a generic property of bond data; without it, modal variances remain meaningful, but are not generally the ordered eigenvalues of the covariance operator.

Proposition 13 (Covariance matrix without simultaneous diagonalisation)**.** *Let $C=\Sigma\Sigma^{*}$ and $C_{mn}=\langle Ce_{n},e_{m}\rangle$. Define*

$$p_{n}(r)=r^{\beta-1}E_{\beta,\beta}(-\mu_{n}r^{\beta}),\qquad \mathcal{J}_{mn}=\int_{0}^{\infty}p_{m}(r)p_{n}(r)\,\mathrm{d}r,\qquad \vartheta_{\beta}=\frac{2\beta-1}{\beta}.$$

Then,

$$\langle Q_{\infty}e_{n},e_{m}\rangle=C_{mn}\mathcal{J}_{mn},\qquad |\mathcal{J}_{mn}|\leq K_{\beta}(\mu_{m}\mu_{n})^{-\vartheta_{\beta}/2}. \tag{101}$$

In particular,

$$\langle Q_{\infty}e_{n},e_{n}\rangle=K_{\beta}C_{nn}\mu_{n}^{-\vartheta_{\beta}} \tag{102}$$

without any commutation assumption on C and A.

Proof. Since $P_\beta(r)e_n = p_n(r)e_n$ and $P_\beta(r)$ is self-adjoint,

$$\langle P_\beta(r)CP_\beta(r)e_n, e_m\rangle = p_m(r)p_n(r)C_{mn}.$$

The trace norm integrability in Theorem 9 permits pairing the integral with e_n, e_m. Each p_n belongs to $L^2(0,\infty)$, and the change of variables used in Theorem 15 gives

$$\int_0^\infty p_n(r)^2\,\mathrm{d}r = K_\beta \mu_n^{-\vartheta_\beta}.$$

Cauchy–Schwarz proves the bound in (101). Taking $m = n$ gives (102). □

If C_{nn} is independently known and positive for large n, then the logarithmic identification formula still applies to the ratio of the modal variance to C_{nn}. Without this information, even a diagonal limiting covariance need not identify α. To see this more clearly, fix two different orders $\alpha_1, \alpha_2 \in (1,2]$ and their eigenvalues $\mu_n^{(1)}, \mu_n^{(2)}$. Choose $M > 1 + \vartheta_\beta \max\{\alpha_1, \alpha_2\}$ and set

$$q_n = n^{-M}, \qquad c_n^{(i)} = K_\beta^{-1} q_n \big(\mu_n^{(i)}\big)^{\vartheta_\beta}, \qquad i = 1,2. \tag{103}$$

Both sequences $(c_n^{(i)})$ are positive and summable; therefore, the operators with diagonal noise covariance $c_n^{(i)}$ satisfy the standing Hilbert–Schmidt requirement, yet each gives the same limiting covariance with eigenvalues q_n. A long-horizon covariance alone cannot distinguish these two maturity orders.

For non-diagonal noise, (101) is a matrix-valued inverse problem with the positivity and trace constraints on C retained. Estimating eigenvalue decay of Q_∞ without controlling its eigenvectors and the noise colour does not implement Theorem 15. Additional short-time information can change the identification problem: Theorem 11 determines C at the population level through $c_\beta^{-1} \lim_{t\downarrow 0} t^{1-2\beta} Q_t$. If both this limit and Q_∞ are known, then every mode with $C_{nn} > 0$ satisfies

$$\mu_n = \left[\frac{K_\beta C_{nn}}{\langle Q_\infty e_n, e_n\rangle}\right]^{1/\vartheta_\beta}. \tag{104}$$

This conditional identity does not establish stable estimation from a noisy finite panel, and unforced modes are not identified by it.

6. Conclusions

The greenium has been formulated as an infinite-dimensional random surface for which the maturity integral gives the logarithm of a green–brown zero-coupon ratio. This representation places the economic comparison between labelled and conventional debt inside a stochastic evolution equation without reducing the term structure to a single spread. The financial interpretation remains pathwise at the level of bond ratios and statistical at the level of moments, preserving the distinction between observable relative values and prices generated under an equivalent martingale measure.

The Caputo derivative gives a precise mechanism for climate memory. Its resolvent replaces exponential relaxation by a Mittag-Leffler decay, and Theorem 8 translates this change into a polynomial persistence law for the expected greenium curve. In this way, a transition shock leaves a profile proportional to $A^{-1}X_0$ at long horizons, with decay order $t^{-\beta}$. The spatial inverse determines how the initial shock is redistributed across maturities, while the temporal order determines how slowly the redistributed profile disappears.

The nonlinear theory establishes that the surface remains mathematically well defined when a maturity-weighted transition factor feeds back into the curve and when the noise depends on the current state. The fixed-point argument works in every L^p space with $2 \leq p < \infty$, and its contraction exponents display the separate roles of deterministic memory and stochastic singularity. The temporal estimate in Theorem 4 identifies $\beta - 1/2$ as the limiting stochastic regularity. The fractional-domain estimate identifies the complementary maturity condition. Their combination shows when a greenium trajectory has enough time regularity for pathwise monitoring and enough maturity regularity for interpretation as a continuous curve.

The Gaussian benchmark yields operator-level asymptotics rather than a collection of isolated scalar calculations. Trace norm convergence of the rescaled covariance proves the short-time variance law simultaneously for every square-integrable maturity exposure. The zero-coupon indicator is covered directly, so no smoothing argument is needed for the economically exact bond ratio. The second-order expansion for regular weights adds the action of A and reveals how cross-maturity propagation modifies the leading convexity effect. The long-horizon covariance converges to a trace-class operator, and its diagonal form shows how the Caputo order changes the decay of maturity modes.

The local increment theorem gives the same temporal exponent at the initial date and at every positive observation date. The coefficient at a positive date includes an additional term generated by the reweighting of the preceding stochastic history. The resulting log-variance slope identifies β from any non-degenerate weighted exposure. Gaussian moment estimates then provide pathwise Hölder regularity and the graph-dimension upper bound $5/2 - \beta$. The Hölder exponent is defined separately from the variogram exponent. Neither quantity is identified with graph dimension, which is controlled through an independent pathwise estimate.

The limiting covariance spectrum provides a second identification channel. When the noise covariance is diagonal in the maturity basis, the mode variance is proportional to $\sigma_n^2 \mu_n^{-(2\beta-1)/\beta}$. Temporal data identify β, while the high-mode maturity slope identifies α after accounting for the colour of the noise. Therefore, climate memory and cross-maturity propagation enter observable statistics through different asymptotic directions, although they remain coupled in the stationary maturity spectrum.

Within sustainable fixed income, these conclusions imply that a persistent greenium need not be represented by a slowly mean-reverting finite-dimensional factor; the persistence can arise from a fractional response kernel for which the order is visible in high-frequency dispersion, while the maturity profile is governed by a separate nonlocal operator. The covariance evolution controls relative-value convexity, while its exponential tilt yields the entropic transition functional and projection against an indicator weight gives the exact zero-coupon variance. In this way, the mathematical framework connects the economic language of climate memory with testable implications for green–brown term structures.

7. Future Developments

History States and a Conditional Pricing Equation

A Markovian lift records memory as an additional state variable. Its purpose is to make the conditional evolution depend on the enlarged present state, although it does not by itself provide a martingale measure for a traded bond. The Hilbert-space lifting of completely monotone Volterra kernels is developed in [44], and Proposition 2 exhibits the relevant exponential mixture for k_β.

A modal construction makes the role of this enlargement explicit. For a finite spectral system, write $x_n(t) = \langle X_N(t), e_n\rangle$, $f_n(t) = \langle F(X_N(t)), e_n\rangle$ and $g_n(t) = G(X_N(t))^* e_n$. The scalar resolvent identity transforms the mild equation into

$$x_n(t) = x_n(0) + \int_0^t k_\beta(t-s)[-\mu_n x_n(s) + f_n(s)]\,\mathrm{d}s + \int_0^t k_\beta(t-s)\langle g_n(s),\, \mathrm{d}W(s)\rangle_U. \quad (105)$$

Indeed, Laplace transforms give $p_n = k_\beta - \mu_n k_\beta * p_n$ and $E_\beta(-\mu_n t^\beta) = 1 - \mu_n k_\beta * E_\beta(-\mu_n \cdot^\beta)$. Inserting these identities into the mild equation proves (105). Deterministic Fubini uses $k_\beta \in L^1(0,T)$, and stochastic Fubini is justified by the local square integrability of p_n and the finite moment bound for g_n.

Define the history field

$$\begin{aligned} z_n(t,r) = &\int_0^t e^{-r(t-s)}[-\mu_n x_n(s) + f_n(s)]\,\mathrm{d}s \\ &+ \int_0^t e^{-r(t-s)}\langle g_n(s),\, \mathrm{d}W(s)\rangle_U. \end{aligned} \quad (106)$$

Then, $x_n(t) = x_n(0) + \int_0^\infty z_n(t,r)\nu_\beta(\,\mathrm{d}r)$ and

$$\mathrm{d}z_n(t,r) = [-r z_n(t,r) - \mu_n x_n(t) + f_n(t)]\,\mathrm{d}t + \langle g_n(t),\, \mathrm{d}W(t)\rangle_U. \quad (107)$$

For the stochastic interchange, the required integrability follows from

$$\int_0^\infty \left(\int_0^t e^{-2r(t-s)}\,\mathrm{d}s\right)^{1/2} \nu_\beta(\,\mathrm{d}r) < \infty.$$

Its integrand is bounded by a constant times $r^{-\beta}$ near zero and by a constant times $r^{-\beta-1/2}$ at infinity. These bounds use precisely $1/2 < \beta < 1$. The full infinite-dimensional lift additionally requires a state space on which the maturity generator and the observation of the history field both have their proper domains.

Suppose that a predictable U-valued process λ defines a true exponential martingale

$$\mathcal{E}_t = \exp\left\{-\int_0^t \langle \lambda(s),\, \mathrm{d}W(s)\rangle_U - \frac{1}{2}\int_0^t \|\lambda(s)\|_U^2\,\mathrm{d}s\right\}.$$

Under the equivalent measure with density $\mathcal{E}_T$, the process $W^{\mathbb{Q}}(t) = W(t) + \int_0^t \lambda(s)\,\mathrm{d}s$ is cylindrical Brownian motion. Whenever the transformed integrals are well defined, the mild drift becomes

$$F(X(s)) - G(X(s))\lambda(s), \quad (108)$$

inside the same P_β convolution. This change of measure leaves the memory kernel unchanged and does not automatically impose the HJM restriction.

Proposition 14 (Conditional martingale criterion on a lifted state space)**.** *Let Z be a Markov state under $\mathbb{Q}$ with generator $\mathcal{G}^{\mathbb{Q}}$, and let $r(Z_t)$ define the money market account $M_t = \exp\{\int_0^t r(Z_s)\,\mathrm{d}s\}$. Suppose that a positive function $b(\tau, z)$ is continuously differentiable in τ and belongs to an Itô domain for the state process. If*

$$[\mathcal{G}^{\mathbb{Q}} - \partial_\tau]b(\tau, z) = r(z)b(\tau, z), \qquad b(0,z) = 1, \quad (109)$$

then $M_t^{-1}b(T-t, Z_t)$ is a local martingale for each fixed maturity T. Conversely, local martingality forces the drift in (109) to vanish along the state trajectories. True martingality and the terminal payoff condition yield the corresponding expectation valuation.

Proof. Itô's formula for the time-dependent function $b(T-t, Z_t)$ gives drift $[\mathcal{G}^{\mathbb{Q}} b - \partial_\tau b](T-t, Z_t)\,\mathrm{d}t$. Since $\mathrm{d}(M_t^{-1}) = -r(Z_t)M_t^{-1}\,\mathrm{d}t$, the drift of the discounted process is

$$M_t^{-1}[\mathcal{G}^{\mathbb{Q}} b - \partial_\tau b - rb](T-t, Z_t)\,\mathrm{d}t.$$

Equation (109) makes this term zero. Conversely, the uniqueness of the predictable finite-variation part of the semimartingale decomposition makes the drift zero along the trajectories of every local martingale price. If the discounted process is a true martingale, then its terminal value M_T^{-1} gives $b(T-t, Z_t) = M_t \mathbb{E}^{\mathbb{Q}}[M_T^{-1} \mid \mathcal{F}_t]$. □

The classical forward rate restriction is recovered in a sufficiently regular default-free reference model. Suppose

$$\mathrm{d}\widehat{f}(t,T) = a(t,T)\,\mathrm{d}t + \left\langle \sigma(t,T),\, \mathrm{d}W^{\mathbb{Q}}(t) \right\rangle_U, \qquad r_t = \widehat{f}(t,t).$$

Assuming the deterministic and stochastic Fubini conditions needed to integrate over maturity, set $B(t,T) = \int_t^T \sigma(t,u)\,\mathrm{d}u$. Itô's formula for $P(t,T) = \exp\{-\int_t^T \widehat{f}(t,u)\,\mathrm{d}u\}$ gives

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \left[r_t - \int_t^T a(t,u)\,\mathrm{d}u + \frac{1}{2}\|B(t,T)\|_U^2 \right] \mathrm{d}t - \left\langle B(t,T),\, \mathrm{d}W^{\mathbb{Q}}(t) \right\rangle_U. \tag{110}$$

Zero discounted drift and differentiation in T imply

$$a(t,T) = \left\langle \sigma(t,T),\, \int_t^T \sigma(t,u)\,\mathrm{d}u \right\rangle_U. \tag{111}$$

The HJM condition [12] is recovered under the stated regularity assumptions. The observation $x_n(t)$ in (105) need not be a semi-martingale merely because the history field has Markovian dynamics; the observation integrates against an infinite measure, and may be unbounded on the lifted state space. Its pricing use must satisfy the regularity assumptions of Proposition 14 or arise as a factor in a separate semi-martingale price model. Credit-sensitive green and conventional claims also require their own payoff specification; two default-free unit claims with the same maturity and the same pricing account cannot have different arbitrage-free values. The present statistical greenium model makes no such identification.

A coupled system can incorporate a carbon allowance curve or a renewable electricity forward curve. Cross-kernels between the components would distinguish a policy shock transmitted through carbon prices from a shock transmitted through financing demand. The covariance operator methods developed here extend to such a system through block operators, while the spectral identification problem becomes a matrix-valued inverse problem.

The local construction in Proposition 5 permits superlinear transition feedback up to the explosion time. Its extension to a global process depends on uniform stopped supremum estimates such as (37). A drift coercivity inequality is a possible ingredient, but the memory terms and the diffusion must also enter the estimate. A nonlinear Gaussian approximation would require a separate error analysis before the covariance asymptotics could be transferred to it.

Statistical implementation requires an observation model for irregular bond maturities and curve construction error. Bounded finite-rank observations preserve the trace norm covariance limit, but a sampling theorem remains necessary in order to obtain the relative consistency used in Proposition 12. Identification of the maturity order additionally requires control of the forcing covariance. Under the diagonal benchmark, high-mode variances can be used after accounting for the noise colour. Without simultaneous diagonalisation, the

unknown matrix in (101) must be estimated jointly with the maturity parameters; correcting observation noise alone does not resolve that inverse problem.

A complete sample path fractal analysis would add lower bounds for the graph dimension. Such bounds require a quantitative non-degeneracy condition for conditional variances, as Hölder regularity alone only provides the upper estimate proved here. Strong local nondeterminism or an appropriate sectorial local nondeterminism property would yield matching lower bounds for weighted observations, and could then be extended to sections of the full time-maturity surface.

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Appendix A. Mittag-Leffler and Fractional Resolvent Tools

For $0 < \beta < 1$, the Caputo derivative of an absolutely continuous scalar function u is

$$ {}^{C}D_t^{\beta} u(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} u'(s)\, \mathrm{d}s. $$

Its Laplace transform satisfies

$$ \mathcal{L}[{}^{C}D^{\beta} u](\lambda) = \lambda^{\beta} \widehat{u}(\lambda) - \lambda^{\beta-1} u(0). $$

Applying this identity to ${}^{C}D_t^{\beta} u = -\mu u + f$ gives

$$ u(t) = E_{\beta}(-\mu t^{\beta}) u(0) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-\mu (t-s)^{\beta}) f(s)\, \mathrm{d}s. $$

The operator formulas in (15) and (16) follow by applying the scalar identity to each spectral mode.

The scalar estimates used throughout the paper are collected in the next lemma. They are classical; see [15,16,18,19].

Lemma A1 (Scalar Mittag-Leffler bounds)**.** *Let $\beta \in (0,1)$. A constant C_β exists such that for every $r \geq 0$,*

$$ \left| E_{\beta}(-r) \right| \leq \frac{C_\beta}{1+r}, \tag{A1} $$

$$ \left| E_{\beta,\beta}(-r) \right| \leq \frac{C_\beta}{(1+r)^2}, \tag{A2} $$

$$ \left| E_{\beta,\beta-1}(-r) \right| \leq C_\beta. \tag{A3} $$

For $0 \leq \theta \leq 1$,

$$ \sup_{r \geq 0} r^{\theta} \left| E_{\beta}(-r) \right| < \infty, \qquad \sup_{r \geq 0} r^{\theta} \left| E_{\beta,\beta}(-r) \right| < \infty. $$

The short-argument remainders satisfy

$$ \left| E_{\beta,\beta}(-r) - \Gamma(\beta)^{-1} \right| \leq C_\beta r, \tag{A4} $$

$$ \left| E_{\beta,\beta}(-r) - \Gamma(\beta)^{-1} + r\Gamma(2\beta)^{-1} \right| \leq C_\beta r^2. \tag{A5} $$

Finally,

$$E_\beta(-r) = \frac{1}{r\Gamma(1-\beta)} + O(r^{-2}) \qquad \text{as } r \to \infty. \tag{A6}$$

Proof. A Hankel contour representation gives the following for $r > 0$:

$$E_{\beta,\gamma}(-r) = \frac{1}{2\pi i}\int_{\mathcal{H}} \frac{\zeta^{\beta-\gamma}e^{\zeta}}{\zeta^\beta + r}\,\mathrm{d}\zeta$$

where $\mathcal{H}$ surrounds the negative real axis and the principal branch is used. Choose fixed rays with angles $\pm\varphi$, where $\pi/2 < \varphi < \pi$, joined by a fixed small arc around zero. Since $\beta\varphi < \pi$, the contour satisfies $\left|\zeta^\beta + r\right| \geq c(|\zeta|^\beta + r)$ for $r > 0$. Use the finite algebraic identity

$$\frac{1}{r+\zeta^\beta} = \sum_{j=0}^{m-1} \frac{(-\zeta^\beta)^j}{r^{j+1}} + \frac{(-\zeta^\beta)^m}{r^m(r+\zeta^\beta)}.$$

The integrated remainder is bounded by $Cr^{-m-1}\int_{\mathcal{H}} \left|e^\zeta\right| |\zeta|^{\beta(m+1)-\gamma}|\,\mathrm{d}\zeta|$. This integral is finite: the arc stays away from zero, and the exponential decays on both rays. Applying the Hankel formula for the reciprocal gamma function to the finite sum gives

$$E_{\beta,\gamma}(-r) = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{r^k\Gamma(\gamma-\beta k)} + O(r^{-m-1}). \tag{A7}$$

The convention $1/\Gamma(-j) = 0$ for $j \in \mathbb{N}_0$ applies because $1/\Gamma$ is entire. Taking $\gamma = 1$ and $m = 1$ proves (A6). Taking $\gamma = \beta$ makes the $k = 1$ coefficient vanish, so $E_{\beta,\beta}(-r) = O(r^{-2})$. Taking $\gamma = \beta - 1$ shows that $E_{\beta,\beta-1}(-r)$ decays at infinity. Continuity of the entire functions on bounded intervals completes (A1)–(A3), and the bounds with r^θ follow immediately.

For $0 \leq r \leq 1$, the defining series gives

$$E_{\beta,\beta}(-r) = \frac{1}{\Gamma(\beta)} - \frac{r}{\Gamma(2\beta)} + \sum_{k=2}^{\infty} \frac{(-r)^k}{\Gamma(\beta k+\beta)}.$$

Absolute convergence is uniform on $[0,1]$, and the tail is bounded by Cr^2. This proves both remainder estimates on that interval. For $r \geq 1$, Equation (A2) implies that the left-hand side of (A4) is bounded by a constant, which is at most Cr. The left-hand side of (A5) is bounded by $C(1+r)$, which is at most Cr^2, which completes the proof. □

Lemma A2 (Strong rescaled limit)**.** *For every $x \in H$,*

$$t^{1-\beta}P_\beta(t)x \longrightarrow \Gamma(\beta)^{-1}x \qquad \text{in } H$$

as $t \downarrow 0$.

Proof. The nth spectral multiplier is $E_{\beta,\beta}(-\mu_n t^\beta)$, which converges to $\Gamma(\beta)^{-1}$. Lemma A1 provides a uniform bound. Dominated convergence in the square-summable spectral coefficients of x proves the assertion. □

Appendix B. Stochastic Integration and Operator Ideals

For separable Hilbert spaces U and H, the Hilbert–Schmidt norm is

$$\|B\|^2_{\mathcal{L}_2(U,H)} = \sum_{k\geq 1} \|Bu_k\|^2_H,$$

where (u_k) is an orthonormal basis of U. If $R \in \mathcal{L}(H)$ and $B \in \mathcal{L}_2(U,H)$, then

$$\|RB\|_{\mathcal{L}_2(U,H)} \le \|R\|_{\mathcal{L}(H)} \|B\|_{\mathcal{L}_2(U,H)}.$$

If $B_1, B_2 \in \mathcal{L}_2(U,H)$, then $B_1 B_2^*$ is trace-class and

$$\|B_1 B_2^*\|_{\mathcal{L}_1(H)} \le \|B_1\|_{\mathcal{L}_2(U,H)} \|B_2\|_{\mathcal{L}_2(U,H)}. \tag{A8}$$

These ideal properties may be found in [45].

For a predictable Φ satisfying $\int_0^T \mathbb{E}\|\Phi(s)\|_{\mathcal{L}_2}^2 \, \mathrm{d}s < \infty$, the stochastic integral obeys Itô's isometry

$$\mathbb{E}\left\| \int_0^T \Phi(s) \, \mathrm{d}W(s) \right\|^2 = \int_0^T \mathbb{E}\|\Phi(s)\|_{\mathcal{L}_2}^2 \, \mathrm{d}s. \tag{A9}$$

For $2 \le p < \infty$, the Hilbert-space Burkholder–Davis–Gundy inequality gives

$$\left(\mathbb{E}\left\| \int_0^T \Phi(s) \, \mathrm{d}W(s) \right\|^p \right)^{1/p} \le C_p \left(\mathbb{E}\left[\int_0^T \|\Phi(s)\|_{\mathcal{L}_2}^2 \, \mathrm{d}s \right]^{p/2} \right)^{1/p}. \tag{A10}$$

When a deterministic operator kernel $R(T-s)$ multiplies a process $B(s)$, Minkowski's integral inequality in $L^{p/2}(\Omega)$ yields

$$\left(\mathbb{E}\left\| \int_0^T R(T-s) B(s) \, \mathrm{d}W(s) \right\|^p \right)^{1/p} \le C_p \left(\int_0^T \|R(T-s)\|^2 \left(\mathbb{E}\|B(s)\|_{\mathcal{L}_2}^p \right)^{2/p} \mathrm{d}s \right)^{1/2}. \tag{A11}$$

Equations (A9)–(A11) are standard; see [25,46].

Lemma A3 (Closed fractional powers on Bochner spaces)**.** *Let A^θ be a closed densely defined operator on H, and define*

$$\mathfrak{A}_\theta Z(\omega) = A^\theta Z(\omega)$$

on the natural domain in $L^2(\Omega; H)$. Then, $\mathfrak{A}_\theta$ is closed.

Proof. Suppose that $Z_n \to Z$ and $\mathfrak{A}_\theta Z_n \to Y$ in $L^2(\Omega; H)$. A subsequence can be selected such that both convergences hold pointwise almost surely. For almost every ω, one has $Z_{n_k}(\omega) \to Z(\omega)$ and $A^\theta Z_{n_k}(\omega) \to Y(\omega)$ in H. Closedness of A^θ implies $Z(\omega) \in D(A^\theta)$ and $A^\theta Z(\omega) = Y(\omega)$. Hence, Z belongs to the domain of $\mathfrak{A}_\theta$ and $\mathfrak{A}_\theta Z = Y$. □

Appendix C. Spectral Sobolev Embeddings

For the Dirichlet eigenbasis, the spectral Sobolev norm of order $s \ge 0$ is

$$\|x\|_{\mathbb{H}_D^s}^2 = \sum_{n \ge 1} (1+n^2)^s |x_n|^2.$$

Since $\mu_n \asymp 1 + n^\alpha$,

$$\left\| A^\theta x \right\|^2 \asymp \sum_{n \ge 1} (1+n^2)^{\alpha\theta} |x_n|^2 = \|x\|_{\mathbb{H}_D^{\alpha\theta}}^2.$$

The space $\mathbb{H}_D^s$ embeds continuously into the usual Sobolev space $H^s(0,L)$; the distinction concerns the boundary trace when $s > 1/2$. Therefore, the one-dimensional Sobolev embedding gives

$$\alpha\theta > \frac{1}{2} \quad \Longrightarrow \quad D(A^\theta) \hookrightarrow C([0,L]). \tag{A12}$$

A direct proof follows from the sine expansion. For $x = \sum_n x_n e_n$ and $\alpha\theta > 1/2$,

$$\begin{aligned}\sup_{\xi\in[0,L]}|x(\xi)| &\le \sqrt{\frac{2}{L}}\sum_{n\ge1}|x_n| \\ &\le \sqrt{\frac{2}{L}}\left(\sum_{n\ge1}\mu_n^{-2\theta}\right)^{1/2}\left(\sum_{n\ge1}\mu_n^{2\theta}|x_n|^2\right)^{1/2} \\ &\le C\left\|A^\theta x\right\|.\end{aligned}$$

The series $\sum_n \mu_n^{-2\theta}$ converges exactly when $2\alpha\theta > 1$. Uniform convergence of the sine series then yields continuity. General Sobolev formulations are given in [47].

Appendix D. Gaussian Convergence and Deterministic Graph Estimates

A centred Gaussian measure on a separable Hilbert space is determined by its positive trace-class covariance operator; see [48]. Suppose that $m_n \to m$ in H and $Q_n \to Q$ in the trace norm. Let R_N denote the spectral projection onto the first N eigenvectors of Q. Then,

$$\operatorname{Tr}\big((\mathrm{Id}-R_N)Q_n\big) \le \|Q_n - Q\|_{\mathcal{L}_1(H)} + \operatorname{Tr}\big((\mathrm{Id}-R_N)Q\big).$$

After N is fixed, the right-hand side is small for all sufficiently large n. The finitely many remaining covariance operators admit a common finite-rank projection with arbitrarily small trace outside its range.

Let Z_n have law $\mathcal{N}(m_n, Q_n)$ and fix $\varepsilon > 0$. Convergence of the means and the preceding trace estimates give finite-rank orthogonal projections P_j with increasing ranges such that

$$\sup_n \mathbb{E}\big\|(\mathrm{Id}-P_j)Z_n\big\|^2 \le \varepsilon 2^{-3j-2}, \qquad j \ge 1.$$

The projections may be enlarged to make their ranges increasing, since orthogonal projection onto a larger subspace decreases the residual norm. Moreover, $M := \sup_n \mathbb{E}\|Z_n\|^2 < \infty$. Choose R with $M/R^2 < \varepsilon/2$ and define

$$K = \{x \in H : \|x\| \le R,\ \big\|(\mathrm{Id}-P_j)x\big\| \le 2^{-j} \text{ for every } j \ge 1\}.$$

This set is closed. It is totally bounded, since at any prescribed accuracy, every element is uniformly close to a bounded subset of one finite-dimensional range of P_j; therefore, completeness of H makes K compact. Chebyshev's inequality and the union bound give

$$\sup_n \mathbb{P}\{Z_n \notin K\} \le M/R^2 + \sum_{j\ge1} 2^{2j} \sup_n \mathbb{E}\big\|(\mathrm{Id}-P_j)Z_n\big\|^2 < \varepsilon.$$

Thus, the Gaussian laws are tight. Their characteristic functionals satisfy

$$\exp(i\langle m_n, h\rangle - \frac{1}{2}\langle Q_n h, h\rangle) \longrightarrow \exp(i\langle m, h\rangle - \frac{1}{2}\langle Qh, h\rangle)$$

for every $h \in H$. Every weakly convergent subsequence has the displayed characteristic functional, and uniqueness of Gaussian characteristic functionals yields convergence of the full sequence to $\mathcal{N}(m, Q)$.

For completeness, consider a function $f{:}[0,T] \to \mathbb{R}$ satisfying $|f(t) - f(s)| \le C|t-s|^\eta$ with $0 < \eta < 1$. Divide $[0,T]$ into intervals of length ε. Over each interval, the vertical oscillation is at most $C\varepsilon^\eta$, so the part of the graph above that interval can be covered by at most $C\varepsilon^{\eta-1}$ squares of side ε. Since the number of horizontal intervals is $O(\varepsilon^{-1})$, the full graph can be covered by $O(\varepsilon^{\eta-2})$ squares. Therefore, its upper box dimension is at most $2-\eta$, and its Hausdorff dimension is no larger. This deterministic argument underlies Theorem 14; a general treatment appears in [49].

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Appendix E. Elementary Integral Estimates

For $a > 0$ and $\eta > 0$,

$$\int_0^\infty r^{a-1} e^{-\eta r}\, \mathrm{d}r = \Gamma(a)\eta^{-a}. \tag{A13}$$

For $a > -1$ and a bounded function g satisfying $g(r) \to g_0$ as $r \downarrow 0$,

$$\int_0^t r^a g(r)\, \mathrm{d}r = \frac{g_0}{a+1} t^{a+1} + o(t^{a+1}). \tag{A14}$$

Indeed, the substitution $r = tu$ gives

$$t^{-a-1} \int_0^t r^a g(r)\, \mathrm{d}r = \int_0^1 u^a g(tu)\, \mathrm{d}u,$$

and dominated convergence proves (A14).

For $-1 < a < 0$ and $0 < h < T$,

$$\int_0^h r^a\, \mathrm{d}r = \frac{h^{a+1}}{a+1}, \qquad h \int_h^T r^{a-1}\, \mathrm{d}r = \frac{h(h^a - T^a)}{-a} \le \frac{h^{a+1}}{-a}. \tag{A15}$$

The second bound uses $a < 0$. For $0 < a < 1$, the same integral is of order h at a fixed T, rather than h^{a+1}. The exponent required by Lemma 4 is $a = \beta - 1 \in (-1/2, 0)$, which belongs to the valid range above.

Lemma A4 (Beta convolution identity). *If $a, b > -1$ and $t > 0$, then*

$$\int_0^t (t-s)^a s^b\, \mathrm{d}s = t^{a+b+1} \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}. \tag{A16}$$

Consequently, for $\gamma > 0$ and

$$k_\gamma(t) = \mathbf{1}_{(0,\infty)}(t) \frac{t^{\gamma-1}}{\Gamma(\gamma)},$$

the m-fold time convolution satisfies

$$k_\gamma^{*m}(t) = \frac{t^{m\gamma-1}}{\Gamma(m\gamma)}, \qquad m \in \mathbb{N}. \tag{A17}$$

Proof. The substitution $s = tu$ gives

$$\int_0^t (t-s)^a s^b\, \mathrm{d}s = t^{a+b+1} \int_0^1 (1-u)^a u^b\, \mathrm{d}u.$$

The remaining integral is the beta function $\mathrm{B}(a+1, b+1)$, for which the gamma representation proves (A16). In particular, $k_\gamma * k_\eta = k_{\gamma+\eta}$ for positive γ and η. Induction on m proves (A17). The identity is the kernel-level counterpart of the Mittag-Leffler series generated by repeated Volterra convolution. □

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