Resolving finite indeterminacy

A definitive constructive universal prime ideal theorem

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Abstract

Dynamical methods were designed to eliminate the ideal objects abstract algebra abounds with. Typically granted by an incarnation of Zorn's Lemma, those ideal objects often serve for proving the semantic conservation of additional non-deterministic sequents, that is, with finite but not necessarily singleton succedents. Eliminating ideal objects dynamically was possible also because (finitary) coherent or geometric logic predominates in that area: the use of a non-deterministic axiom can be captured by a finite branching of the proof tree.

Incidentally, a paradigmatic case has widely been neglected in dynamical algebra: Krull's Lemma for prime ideals. Digging deeper just about that case, which we have dealt with only recently (with Yengui), has now brought us to unearth the general phenomenon underlying dynamical algebra: Given a claim of computational nature that usually is proved by the semantic conservation of certain additional non-deterministic axioms, there is a finite labelled tree belonging to a suitable inductively generated class which tree encodes the desired computation. Our characterisation works in the fairly universal setting of a consequence relation enriched with non-deterministic axioms; uniformises many of the known instances of the dynamical method; generalises the proof-theoretic conservation criterion we have offered before (with Rinaldi); and last but not least links the syntactical with the semantic approach: every ideal object used for the customary proof of a concrete claim can be approximated by one of the corresponding tree's branches.

CCS Concepts: • **Theory of computation** → **Constructive mathematics**; **Proof theory**.

Keywords: dynamical proof, non-deterministic axiom, geometric logic, proof-theoretic conservation, finite tree, computational content, inductive generation, Krull's Lemma, prime ideals

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1 Introduction

Abstract algebra abounds with ideal objects and the invocations of transfinite methods, typically Zorn's Lemma [114], that grant those object's existence. Put under logical scrutiny, ideal objects often serve for proving the semantic conservation of additional non-deterministic sequents, that is, with finite but not necessarily singleton succedents.

By design, dynamical methods in algebra [34, 65, 113] allow to eliminate the ideal objects upon shifting focus from semantic model extension principles to their corresponding syntactic conservation theorems. This move in line with Hilbert's programme has shaped modern constructive algebra and has seen tremendous success, not least because (finitary) coherent or geometric logic [25, 58, 59, 68, 71, 72, 82, 112] predominates in that area: the use of a non-deterministic axiom can be captured by a finite branching of the proof tree [34]. Coherent theories, on the other hand, lend themselves to automated theorem proving [10, 11, 35, 42, 50, 105].

A paradigmatic case, which to a certain extent has been neglected in dynamical algebra proper, is Krull's Lemma for prime ideals. A particular form of this asserts that a multiplicative subset of a commutative ring contains the zero element if and only if the set at hand meets every prime ideal. Prompted by certain aspects in the novel treatment of valuative dimension [60], Krull's Lemma has seen a constructive treatment only recently [99]. The latter, however, has brought us to unearth the underlying general phenomenon in the present paper: Given a claim of computational nature that usually is proved by the semantic conservation of certain additional non-deterministic axioms, there is a finite labelled tree belonging to a suitable inductively generated class which tree encodes the desired computation.

Our characterisation works in the fairly universal setting of consequence relations, a cornerstone of universal logic and algebra with a long and rich history that can be traced back to Hertz [46–48, 62] and Tarski [12, 107], and, in the guise of covering relations, has played an important role in the development of formal topology [20, 21, 69, 70, 91–93].

Consequence relations serve here to capture the basic structures—ideals of commutative rings, propositional theories, and partial orders—on top of which we consider certain non-deterministic axioms that describe "ideal" refinements of those structures: prime ideals, complete theories, and linear order extensions.

A decisive aspect of our approach is the notion of a regular set for certain non-deterministic axioms over a fixed consequence relation. Abstracted from the multiplicative subsets occurring in Krull's Lemma, regular sets turn out to be the fundamental ingredient of our Universal Prime Ideal Theorem (UPIT, Proposition 3.2). They allow to calibrate precisely UPIT's gearing and account for its constructive version (Proposition 4.6).

In this manner we uniformise many of the known instances of the dynamical method [34, 65, 113]. We further generalise (Proposition 6.2) the proof-theoretic conservation criterion we have offered before [87, 88], using Scott–style entailment relations [100–102], to unify the many phenomena present in the literature, e.g. [19, 31, 64, 67, 75].

Last but not least, we thus link the syntactical with the semantic approach: every ideal object used for the traditional proof of a concrete claim can be approximated by one of the corresponding tree's branches (Proposition 4.2).

It is worth emphasizing at this point that, as compared to (propositional) dynamical algebra [27], the prime ideal theorem we offer is not only constructive but also definitive and universal. We succeed in unearthing the one common pattern of how the related trees are to be grown, by passing to the logical setting of consequence. Our approach, moreover, is ready for use in customary mathematical practice without any need to adapt first the axioms, which is not untypical for dynamical algebra. Last but not least, we identify regularity as both sufficient and necessary for the prime ideal theorem under consideration.

Structure of this paper

In Section 2 we discuss consequence relations, non-deterministic axioms, as well as regular subsets, which relate to both the former concepts. In Section 3 we present UPIT, a straightforward consequence of Zorn's Lemma but equivalent, as will be seen later on, to the Prime Ideal Theorem (for commutative rings, say). In Section 4 we introduce certain inductively defined classes of finite trees which then help us to provide the constructive counterpart (CUPIT) of the Universal Prime Ideal Theorem. Three quite different applications will be discussed in Section 5, putting emphasis on the universality of our approach: a constructive version of Krull's Lemma for commutative rings, Glivenko's theorem for propositional logic, as well as an order extension principle will all follow immediately from CUPIT. Connections with existing work on multi-conclusion entailment relations will be discussed in the final Section 6.

On method and foundations

Unless specified otherwise, we work in a suitable fragment of Aczel's *Constructive Zermelo–Fraenkel Set Theory* (**CZF**) [2–6] based on intuitionistic first-order predicate logic. When we occasionally need to invoke a fragment of the principle of Excluded Middle or even a form of the Axiom of Choice (AC), and thus go beyond **CZF**, we simply switch to **ZF** and **ZFC**, respectively, and indicate this accordingly.

By a *finite* set we understand a set that can be written as $\{ a_1, \dots, a_n \}$ for some $n0$. Given any set S, let $\mathrm{Pow}(S)$ (respectively, $\mathrm{Fin}(S)$) consist of the (finite) subsets of S. We refer to [87, 88] for further provisos to carry over to this note.[1]

From formal topology [92] we borrow the *overlap* symbol: the notation UV is to say that the sets U and V have an element in common.

2 Key notions

2.1 Consequence relations

By a *consequence relation* or a *single-conclusion entailment relation* we understand a relation

$$\rhd \subseteq \mathrm{Fin}(S) \times S$$

[1] For example, we deviate from the terminology prevalent in constructive mathematics and set theory [5, 6, 13, 14, 65, 66]: to reserve the term 'finite' to sets which are in *bijection* with $\{1, \dots, n\}$ for a necessarily unique $n0$. Those exactly are the sets which are finite in our sense and are *discrete* too, i.e. have decidable equality [66].

which is *reflexive, monotone* and *transitive* in the following sense:

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{U \rhd a}{U, V \rhd a}\ (\mathrm{M}) \qquad \frac{U \rhd b \quad U, b \rhd a}{U \rhd a}\ (\mathrm{T})$$

where as usual $U, V \equiv U \cup V$ and $U, b \equiv U \cup \{b\}$. We also sometimes write $a_1, \dots, a_n$ in place of $\{a_1, \dots, a_n\}$ even if $n = 0$.

It is of course well-known that every consequence relation $\rhd$ gives way to an algebraic closure operator

$$\langle - \rangle : \mathrm{Pow}(S) \to \mathrm{Pow}(S)$$

defined by

$$a \in \langle T \rangle \ \equiv\ (\exists U \in \mathrm{Fin}(T))\, U \rhd a.$$

Conversely, given $\langle - \rangle$, by stipulating

$$U \rhd a \ \equiv\ a \in \langle U \rangle$$

we gain back a consequence relation from an algebraic closure operator.

The *ideals* of a consequence relation $\rhd$ are the subsets $\mathfrak{a}$ of S which are closed with respect to the corresponding closure operator, which is to say that $\mathfrak{a} = \langle \mathfrak{a} \rangle$. These are precisely the subsets $\mathfrak{a}$ of S such that if $\mathfrak{a} \supseteq U$ and $U \rhd a$, then $a \in \mathfrak{a}$. We say that $\mathfrak{a}$ is *finitely generated* if $\mathfrak{a} = \langle U \rangle$ for some $U \in \mathrm{Fin}(S)$.

2.2 Non-deterministic axioms

By a *non-deterministic axiom*[2] *on* S we understand a pair $(A, B) \in \mathrm{Fin}(S) \times \mathrm{Fin}(S)$, which we often put in turnstile notation:

$$A \vdash B.$$

A subset $\mathfrak{p}$ of S is *closed* for (A, B) if $A \subseteq \mathfrak{p}$ implies $\mathfrak{p}B$.

Let $\mathcal{E}$ be a set of non-deterministic axioms. An ideal of $\rhd$ that is closed for every axiom of $\mathcal{E}$ will be called a *prime ideal*.[3] We denote with

$$\mathrm{Spec}(\mathcal{E})$$

the class of prime ideals of $\mathcal{E}$. Given an ideal $\mathfrak{a}$ of $\rhd$, let

$$\mathrm{Spec}(\mathcal{E})/\mathfrak{a} = \{\, \mathfrak{p} \in \mathrm{Spec}(\mathcal{E}) \mid \mathfrak{p} \supseteq \mathfrak{a} \,\}.$$

2.3 Regular subsets

Convention. *From now on, and throughout the following Sections 3 and 4, let* S *be a set with consequence relation* $\rhd$*, and let* $\mathcal{E}$ *be a set of non-deterministic axioms on* S.

[2]Our terminology borrows from van den Berg's principle of *non-deterministic inductive definitions* [108], variants of which have recently come to play a role in constructive reverse mathematics [49, 56].

[3]We say "prime ideal" to stress that variants of the prime ideal theorem (e.g., for commutative rings, distributive lattices, Boolean algebras) form the ground for our abstract version (Proposition 3.2).

We say that a subset R of S is *regular* if, for all $U \in \mathrm{Fin}(S)$ and $(A, B) \in \mathcal{E}$,

$$\frac{(\forall b \in B)\, \langle U, b \rangle R}{\langle U, A \rangle R}$$

An element r of S is said to be *regular* if so is $\{\, r \,\}$. Hence r is regular precisely when, for all $U \in \mathrm{Fin}(S)$ and $(A, B) \in \mathcal{E}$,

$$\frac{(\forall b \in B)\, U, b \rhd r}{U, A \rhd r}$$

Regularity of an element r of S thus means to require *disjunction elimination* [87, 88]

$$\frac{U, b_1 \rhd r \quad \dots \quad U, b_\ell \rhd r}{U, a_1, \dots, a_k \rhd r}$$

for the succedent of every $a_1, \dots, a_k \vdash b_1, \dots, b_\ell$ in $\mathcal{E}$.

3 A universal prime ideal theorem

The following is an abstraction of the usual proof of Krull's Lemma [61] and related prime ideal principles [86].

Lemma 3.1 (**ZFC**)**.** *Let* $R \subseteq S$ *be regular and let* $\mathfrak{a}$ *be an ideal. If* $R \cap \mathfrak{a} = \emptyset$*, then there is a prime ideal* $\mathfrak{p} \supseteq \mathfrak{a}$ *such that* $R \cap \mathfrak{p} = \emptyset$.

Proof. Zorn's Lemma yields an ideal $\mathfrak{p}$ over $\mathfrak{a}$ which is maximal with respect to the property that $R \cap \mathfrak{p} = \emptyset$. Every such $\mathfrak{p}$ is a prime ideal! To see this, let $(A, B) \in \mathcal{E}$ be such that $A \subseteq \mathfrak{p}$ yet $\mathfrak{p} \cap B = \emptyset$. Maximality implies that $\langle \mathfrak{p}, b \rangle R$ for every $b \in B$. Since R is regular, it follows that $\mathfrak{p} = \langle \mathfrak{p}, A \rangle R$, a contradiction. ☐

Notice that the proof of Lemma 3.1 establishes that if R is regular, then every ideal which is maximal among those avoiding R is prime. It is necessary for this that R be regular.

Here is our semantic classical *Universal Prime Ideal Theorem* (UPIT):

Proposition 3.2 (**ZFC**)**.** *Let* $R \subseteq S$*. The following are equivalent.*

1. R *is regular.*
2. *For every (finitely generated) ideal* $\mathfrak{a}$*, the following are equivalent:*
   a. $R\mathfrak{a}$.
   b. $(\forall \mathfrak{p} \in \mathrm{Spec}(\mathcal{E})/\mathfrak{a})\, R\mathfrak{p}$.

Proof. Suppose that R is regular. For each ideal $\mathfrak{a}$, every element witnessing item 2.a witnesses item 2.b just as well. The reverse implication is the contrapositive of Lemma 3.1.

Conversely, suppose that for every finitely generated ideal the equivalence of item 2 holds, let $U \in \mathrm{Fin}(S)$ and $(A, B) \in \mathcal{E}$ such that $\langle U, b \rangle R$ for every $b \in B$. To show that $\langle U, A \rangle R$ it now suffices to check that every prime ideal over $\langle U, A\rangle$ meets R. In fact, if $\mathfrak{p}$ is prime and $\mathfrak{p} \supseteq \langle U, A\rangle \supseteq A$, then there is $b \in \mathfrak{p} \cap B$, for which $\langle U, b\rangle R$ and thus $\mathfrak{p}R$. ☐

UPIT is a weak form of the Axiom of Choice, equivalent to the prime ideal theorem for distributive lattices (cf. Proposition 6.8). There is no way around excluded middle to prove Proposition 3.2, see Proposition 6.6 below.

Corollary 3.3 (**ZFC**). *The following are equivalent.*

1. *Every element of S is regular.*
2. *For every (finitely generated) ideal $\mathfrak{a}$,*

$$\mathfrak{a} = \bigcap \mathrm{Spec}(\mathcal{E})/\mathfrak{a}.$$

4 Trees for prime ideals

Given an ideal $\mathfrak{a}$, we consider next a certain collection $T_\mathfrak{a}$ of finite labelled trees, generated in such a manner that the root of every $t \in T_\mathfrak{a}$ be labelled with a finite subset U of $\mathfrak{a}$, and the nodes be labelled with elements of S. The latter will be determined successively by consequences of U along the additional axioms of $\mathcal{E}$.

Given a path π of such a tree $t \in T_\mathfrak{a}$, we write

$$\pi \rhd r$$

to say that r is a consequence of the set labelling the root of t together with the labels occurring at the nodes of π. Set

$$\langle \pi\rangle = \{\, r \in S \mid \pi \rhd r \,\}\,.$$

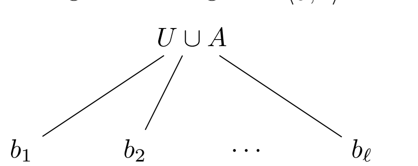
Note that *we understand paths to lead from the root of a tree to one of its leaves.*

Definition 4.1. Let $\mathfrak{a}$ be an ideal. We generate $T_\mathfrak{a}$ inductively according to the following rules:

1. For every $U \in \mathrm{Fin}(\mathfrak{a})$, the trivial tree (i.e., the root-only tree) labelled with U belongs to $T_\mathfrak{a}$.
2. If $(A, B) \in \mathcal{E}$ and if $t \in T_\mathfrak{a}$ has a path π such that $\langle \pi\rangle \supseteq A$, i.e., $\pi \rhd a$ for every $a \in A$, then add, for every $b \in B$, a child labelled with b at the leaf of π.

This is a so-called *generalised inductive definition* [1, 81, 103].

For instance, if $(A, \{\, b_1, \ldots, b_\ell \,\}) \in \mathcal{E}$ and $U \in \mathrm{Fin}(S)$, then the following tree belongs to $T_{\langle U,A\rangle}$:



(1)

By a slight abuse of notation, we say that a tree $t \in T_\mathfrak{a}$ is *trivial* if it results from an application of the base rule in Definition 4.1 only. The trivial trees in $T_\mathfrak{a}$ thus correspond with the elements of $\mathrm{Fin}(\mathfrak{a})$.

It is instructive to think of the given ideal $\mathfrak{a}$ as a set of initial data, of which just a finite amount U be used for computation; with this we label the root. The paths of a tree $t \in T_\mathfrak{a}$ then represent the possible courses of a computation *as if* the ideal $\mathfrak{a}$ were prime.

Proposition 4.2. *Let $\mathfrak{a}$ be an ideal, and $t \in T_\mathfrak{a}$ a tree. For every prime ideal $\mathfrak{p} \supseteq \mathfrak{a}$ there is a path π through t such that $\mathfrak{p} \supseteq \langle \pi\rangle$.*

Proof. The path can be constructed by induction as follows. To begin with, $\mathfrak{p}$ contains the finite subset of $\mathfrak{a}$ that labels the root of t. Now suppose that the path has been constructed up to a node ν at which t branches with respect to $(A, B) \in \mathcal{E}$. By induction, $\mathfrak{p} \supseteq A$ and thus $\mathfrak{p} \ni b$ for some $b \in B$. We then add to the path the child of ν labelled by this b. ☐

The paths of every $t \in T_\mathfrak{a}$ may thus be considered finite approximations of the prime ideals that contain $\mathfrak{a}$.

Definition 4.3. Let $\mathfrak{a}$ be an ideal. We say that a tree $t \in T_\mathfrak{a}$ *terminates* in a subset $R \subseteq S$, in short

$$t \downarrow R,$$

if for every path π of t there is $r \in R$ such that $\pi \rhd r$, that is, $\langle \pi\rangle R$. We say that a tree $t \in T_\mathfrak{a}$ *terminates* in an element r of S if t terminates in the singleton set $\{\, r \,\}$.

Example 4.4. $R\mathfrak{a}$ if and only if there is a trivial tree in $T_\mathfrak{a}$ terminating in R.

The main result of this paper, Proposition 4.6 below, boils down to the following key observation.

Lemma 4.5. *Let R be a regular subset of S and let $\mathfrak{a}$ be an ideal. If some $t \in T_\mathfrak{a}$ terminates in R, then $R\mathfrak{a}$.*

Proof. By induction on the construction of $t \in T_\mathfrak{a}$. The base case is trivial. Consider next the case in which a tree in $T_\mathfrak{a}$ has been extended at the leaf of one of its paths π with children labelled with $b \in B$, where $(A, B) \in \mathcal{E}$ and $\langle \pi\rangle \supseteq A$. Suppose then that for every $b \in B$ there is $r \in R$ such that $\pi, b \rhd r$. Regularity implies that there is $r_0 \in R$ such that $\pi, A \rhd r_0$. Since $\langle \pi\rangle \supseteq A$, it follows that $\pi \rhd r_0$, whence the induction hypothesis applies. ☐

Here is our *Constructive Universal Prime Ideal Theorem* (CUPIT), the constructive counterpart of Proposition 3.2.

Proposition 4.6. *Let $R \subseteq S$. The following are equivalent.*

1. *R is regular.*
2. *For every (finitely generated) ideal $\mathfrak{a}$, the following are equivalent:*
   a. *$R\mathfrak{a}$.*
   b. *There is a tree $t \in T_\mathfrak{a}$ which terminates in R.*

Proof. Suppose that R is regular. If $a \in R \cap \mathfrak{a}$, then the trivial tree, labelled with a, terminates in R; conversely, if $t \in T_\mathfrak{a}$ terminates in R, then $R\mathfrak{a}$ by Lemma 4.5.

As regards the converse, to show that R is regular let $U \in \mathrm{Fin}(S)$ and $(A, B) \in \mathcal{E}$ such that $\langle U, b\rangle R$ for every $b \in B$. To see that $\langle U, A\rangle R$, by item 2 it is enough to observe that the tree displayed in (1) terminates in R. ☐

CUPIT is fairly versatile, as will be emphasised by means of three applications in Section 5.

Here is the constructive counterpart of Corollary 3.3.

Corollary 4.7. *The following are equivalent.*

1. *Every element of S is regular.*
2. *For every (finitely generated) ideal $\mathfrak{a}$,*

$$\mathfrak{a} = \{\, r \in S \mid (\exists t \in T_\mathfrak{a})\, t \downarrow r \,\}\,.$$

Corollary 4.7 corresponds to a certain conservation result of [87, 88] as will briefly be outlined in Section 6.

5 Applications

5.1 Krull's Lemma

Our first case study concerns prime ideals of commutative rings. These have already been considered from a similar angle [99], into which we have recently been led by certain aspects of the novel treatment of valuative dimension [60]. Our concept of regular subset now allows us to go beyond. For an algorithmic approach via proof mining to the existence of ideal objects in commutative algebra we refer to [80].

Let $\mathbf{A}$ be a commutative ring with 1. On $S = \mathbf{A}$ we consider the entailment relation of *radical ideal*:

$$a_1, \ldots, a_k \rhd a \;\equiv\; (\exists n \in \mathbb{N})\, a^n \in \sum_{i=1}^{k} \mathbf{A} a_i .$$

Note that an ideal for $\rhd$ is nothing but a radical ideal of $\mathbf{A}$. On top of $\rhd$ we consider the non-deterministic axiom of *prime ideal*, i.e., we let $\mathcal{E}$ consist of all the instances, with $a, b \in \mathbf{A}$, of

$$ab \vdash a, b\,.$$

We say that a subset M of $\mathbf{A}$ is *weakly multiplicative* if for all $a, b \in M$ there is $x \in M$ such that $ab \in \mathrm{Fil}(x)$, where $\mathrm{Fil}(x)$ denotes the principal filter generated by x. In other words, M is weakly multiplicative if for every pair of elements $a, b \in M$ there is $x \in M$ along with $n \in \mathbb{N}$ and $c \in \mathbf{A}$ such that $x^n = abc$. In particular, every multiplicative subset is weakly multiplicative.

Lemma 5.1. *A subset R of $\mathbf{A}$ is regular if and only if it is weakly multiplicative.*

Proof. Suppose that R is regular and let $a, b \in R$. By regularity, there is $r \in R$ such that $ab \rhd r$ which translates as claimed. Conversely, if R is weakly multiplicative, suppose that $U, a \rhd x$ and $U, b \rhd y$ for certain $x, y \in R$. This is to say that there are $k, \ell \in \mathbb{N}$ and $r, s \in \mathbf{A}$ as well as $u, v \in \langle U\rangle$ such that

$$x^k = ra + u \quad \text{and} \quad y^\ell = sb + v.$$

It follows that

$$(xy)^{k+\ell} = tab + w$$

for certain $t \in \mathbf{A}$ and $w \in \langle U\rangle$. Since R is weakly multiplicative there is $z \in R$ along with $n \in \mathbb{N}$ and $c \in \mathbf{A}$ such that $z^n = xyc$. Thus,

$$z^{n(k+\ell)} = c^{k+\ell}(tab + w),$$

which witnesses $U, ab \rhd z \in R$, whence R is regular. ☐

The following is a constructive version of Krull's Lemma that every radical ideal is the intersection of all containing prime ideals [61]. It is a direct consequence of CUPIT and the preceding Lemma 5.1.

Proposition 5.2. *Let $M \subseteq \mathbf{A}$. The following are equivalent.*

1. *M is weakly multiplicative.*
2. *For every radical ideal $\mathfrak{a}$ of $\mathbf{A}$, the following are equivalent:*
   a. *$M\mathfrak{a}$.*
   b. *There is a tree $t \in T_\mathfrak{a}$ which terminates in M.*

Corollary 5.3. *For every $a \in \mathbf{A}$, the following are equivalent.*

1. *a is nilpotent, i.e., there is $n \in \mathbb{N}$ such that $a^n = 0$.*
2. *There is a tree $t \in T_0$ terminating in $\{\, a^n \mid n \in \mathbb{N} \,\}$.*

As an application of Corollary 5.3 we consider the well-known theorem that every non-constant coefficient of an invertible polynomial is nilpotent. This result has an elegant proof by reduction to the integral case, and has already seen many a constructive treatment—see, e.g., [8, 27, 65, 79, 83, 96, 97]. Thus, suppose that

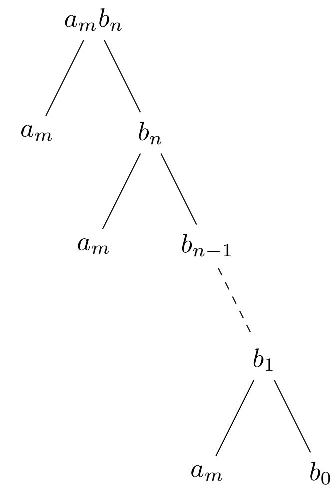
$$f = \sum_{i=0}^{m} a_i X^i \quad \text{and} \quad g = \sum_{j=0}^{n} b_j X^j$$

are such that

$$1 = fg = \sum_{k=0}^{m+n} c_k X^k, \quad \text{where } c_k = \sum_{i+j=k} a_i b_j . \tag{2}$$

It suffices to check by induction on $m > 1$ that a_m is nilpotent. This is enough because in every commutative ring the sum of an invertible and a nilpotent element is in turn invertible.

From (2) we infer that $a_m b_n = 0$, $a_0 b_0 = 1$ as well as that $a_m b_j \in \langle b_{j+1}, \ldots, b_n\rangle$ for $0j < n$. With this information we construct a tree which terminates in a_m, thus witnessing the latter's nilpotency:



In fact, the rightmost path entails a_m since b_0 is invertible; all other paths trivially entail a_m.

5.2 Glivenko's Theorem

Let $\vdash_i$ and $\vdash_c$ stand for intuitionistic and classical logic in a propositional language S. It is known [55, 74] that

$$\Gamma \vdash_c \varphi \quad \text{if and only if} \quad \Gamma, \Delta \vdash_i \varphi$$

for a suitable finite set Δ of formulas $\psi \vee \neg\psi$, where ψ is a propositional variable occurring in Γ or φ.

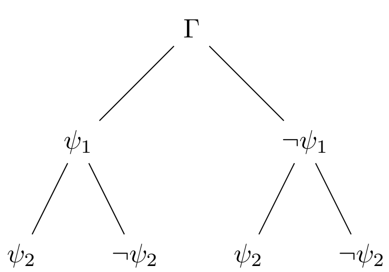
Let $\rhd = \vdash_i$ on S and consider on top of $\rhd$ the non-deterministic axiom of excluded middle, i.e., let $\mathcal{E}$ consist of all the instances, with $\varphi \in S$, of

$$\vdash \varphi, \neg\varphi$$

Let us consider an example. If, say,

$$\Gamma, \psi_1 \vee \neg\psi_1, \psi_2 \vee \neg\psi_2 \vdash_i \varphi,$$

then the tree



belongs to $T_{\langle\Gamma\rangle}$ and terminates in φ. Proposition 4.6 implies that if φ is regular, then $\Gamma \rhd \varphi$, which is to say that $\Gamma \vdash_i \varphi$.

Lemma 5.4. *A formula φ is regular if and only if it is stable, i.e., such that $\neg\neg\varphi \rhd \varphi$.*

Proof. Clearly, if φ is regular, then $\neg\neg\varphi \rhd \varphi$ follows from $\neg\neg\varphi, \varphi \rhd \varphi$ and $\neg\neg\varphi, \neg\varphi \rhd \varphi$. For the converse use the intuitionistic properties of (double) negation, due to Brouwer [16, 17], that $\rhd \neg\neg(\psi \vee \neg\psi)$, and that $\Gamma, \psi \rhd \neg\chi$ implies $\Gamma, \neg\neg\psi \rhd \neg\chi$. □

Since every negated formula $\neg\chi$ is stable, we regain the following version of Glivenko's theorem [44] from Proposition 4.6 with $\mathfrak{a} = \langle\Gamma\rangle$ and the above observations for $\neg\chi$ as φ.

Proposition 5.5 (Glivenko)**.** *If $\Gamma \vdash_c \neg\chi$, then $\Gamma \vdash_i \neg\chi$.*

We hasten to say that proofs of Glivenko's theorem usually go along similar lines. Recent literature about Glivenko's result includes [37, 38, 45, 57, 63, 76, 78].[4]

But what has Glivenko's Theorem to do with transfinite methods? In fact Proposition 5.5 is the syntactical underpinning of the following special case of Proposition 3.2, which in turn is a variant [38] of Lindenbaum's Lemma [107].

Proposition 5.6 (**ZFC**)**.** *The intersection of all complete theories over Γ equals $\{ \varphi \in S \mid \Gamma \vdash_i \neg\neg\varphi \}$.*

As usual, by a *complete theory* we mean a deductively closed subset Γ of S such that for every $\varphi \in S$ either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. By the latter condition it is irrelevant whether deductive closure is understood for $\vdash_i$ or $\vdash_c$, but of course the theories of the former are exactly the ideals of $\rhd = \vdash_i$.

5.3 Order extension

Here is another application, this one in the context of order relations. Let E be a set. We say that a binary relation R on S is *order-regular* if, for all $a, b, c, d \in E$,

$$(a, b) \in R \wedge (c, d) \in R \implies (a, d) \in R \vee (c, b) \in R. \quad (3)$$

An *interval order* is an order-regular relation which, in addition, is irreflexive.[5] Hence every interval order is transitive. By a *quasiorder* we understand a reflexive transitive relation. A *linear order* L is a transitive relation such that, for all $a, b \in E$ either $(a, b) \in L$ or $(b, a) \in L$, by way of which it is reflexive. A quasiorder is order-regular precisely when it is linear.

If P and P' are quasiorders such that $P \subseteq P'$, then we say that the latter *extends* the former. With this terminology, the problem of finding a *linear extension* of a quasiorder becomes trivial: the cartesian product $E \times E$ will do. Thus a more restrictive concept of extension is at

[4] This list of references is by no means meant exhaustive.

[5] The concept of interval order can be traced back to early work of Wiener's on the theory of measurement [40]. Fishburn coined the term "interval order" for what Wiener had called "relation of complete sequence" [39–41]. Yet another occurrence of this concept goes under the name of an irreflexive "Ferrers relation" due to Riguet [84].

work, e.g., in social choice theory [15, 18], yet to achieve linearity often requires transfinite methods [98, 106]. The following results force non-triviality by demanding that a linear order be in the complement of a given order-regular relation.

Let $S = E \times E$ and let $\rhd$ correspond to *transitive closure*, i.e., put

$$U \rhd (a,b) \;\equiv\; (a,b) \in \bigcup_{i1} U^i$$

where $U^1 = U$ and $U^{i+1} = U^i \circ U$, where

$$U \circ V = \{\, (a,c) \mid (\exists b \in E)((a,b) \in U \wedge (b,c) \in V) \,\} \, .$$

On top of this we now consider the non-deterministic axiom of order-regularity, i.e. we let $\mathcal{E}$ consist of all the instances, with $a,b,c,d \in E$, of

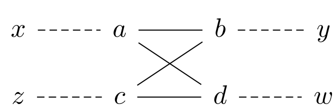
$$(a,b),(c,d) \vdash (a,d),(c,b) \, .$$

Lemma 5.7. *Every order-regular relation is regular.*

Proof. Suppose that R is order-regular. We need to show that

$$\frac{\langle U,(a,b)\rangle R \quad \langle U,(c,d)\rangle R}{\langle U,(a,d),(c,b)\rangle R}$$

where $U \in \mathrm{Fin}(S)$. It suffices to settle the case in which the two assumptions are witnessed by chains through (a,b) and (c,d), respectively. Invoking order-regularity immediately yields the result. At one glance:



Here $(x,y),(z,w) \in R$, and dashed lines indicate chains through U. Depending on whether $(x,w) \in R$ or $(z,y) \in R$, glue along (a,d) or (c,b), accordingly. □

In **ZFC**, UPIT implies that if R is an order-regular relation, and P is a quasiorder on E, then R and P are disjoint if and only if there is a linear order L that extends P yet keeps off R. The particular case where P is the diagonal relation yields that an order-regular relation R is an interval order if and only if its complement contains a linear order. In fact, a prime quasiorder is nothing but a linear one.

We further note the following consequence of CUPIT, which is the constructive counterpart of the aforesaid.

Proposition 5.8. *Let R be an order-regular relation and let P be a quasiorder on E. The following are equivalent.*

1. *RP.*
2. *There is a tree $t \in T_P$ which terminates in R.*

6 Multi-conclusion entailment

In this final section we shed some light on certain aspects of multi-conclusion entailment relations as extending their single-conclusion counterparts. The relevance of the notion of entailment relation to constructive algebra and point-free topology has been pointed out in [19], and has been used very widely, e.g. in [22–24, 26, 28, 31, 33, 75, 85, 89, 95, 109, 110]. Lorenzen's precedence is currently under scrutiny [29, 30]. Consequence and entailment have further caught interest from various angles [7, 33, 36, 43, 52–54, 77, 90, 94, 104, 111].

We begin with a brief summary, closely referring to [87, 88] for a thorough account which builds on ideas and results from proof theory [75] and dynamical algebra [31, 34, 65].

Let S be a set. Recall that a *multi-conclusion entailment relation* [100–102] is a relation

$$\vdash \;\subseteq \mathrm{Fin}(S) \times \mathrm{Fin}(S)$$

between finite subsets U and V of S which is *reflexive* and *monotone*:

$$\frac{UV}{U \vdash V}\ (\mathrm{R}) \qquad\qquad \frac{U \vdash V}{U,U' \vdash V,V'}\ (\mathrm{M})$$

as well as *transitive*:

$$\frac{U \vdash V,a \quad U,a \vdash V}{U \vdash V}\ (\mathrm{T})$$

where we make use of the usual shorthand notations.

Given $\vdash$, its *trace* $\rhd_\vdash$ is defined by

$$U \rhd_\vdash a \;\equiv\; U \vdash a$$

and in fact is a consequence relation.

A *model* of $\vdash$ is a subset $\mathfrak{p}$ of S such that if $\mathfrak{p} \supseteq U$ and $U \vdash V$ then $\mathfrak{p}V$. The class of all models of $\vdash$ will be denoted by

$$\mathrm{Mod}(\vdash).$$

It is a consequence of the prime ideal theorem for distributive lattices (PIT) that every multi-conclusion entailment relation is determined by its models [19], which is to say that

$$(\forall \mathfrak{p} \in \mathrm{Mod}(\vdash))(\, U \subseteq \mathfrak{p} \implies \mathfrak{p}V \,) \implies U \vdash V. \qquad (4)$$

Every set $\mathcal{E}$ of non-deterministic axioms gives rise to a multi-conclusion entailment relation $\vdash_{\mathcal{E}}$ which is least among those $\vdash$ for which $A \vdash B$ for all $(A,B) \in \mathcal{E}$. Following [90], this $\vdash_{\mathcal{E}}$ can be generated inductively by the rules for reflexivity and transitivity on axioms:

$$\frac{UV}{U \vdash_{\mathcal{E}} V}\ (\mathrm{R}) \qquad \frac{(A,B) \in \mathcal{E} \quad (\forall b \in B)\, U,b \vdash_{\mathcal{E}} V}{U,A \vdash_{\mathcal{E}} V}\ (\mathrm{Ax}) \qquad (5)$$

much akin to [32, 73]. Every model of $\vdash_{\mathcal{E}}$ is apparently closed for every member of $\mathcal{E}$, keeping in mind that the elements of the latter are turned into entailments. Conversely, by induction it is easy to see that if a subset of S is closed for every member of $\mathcal{E}$, then it is in fact a model of $\vdash_{\mathcal{E}}$.

6.1 Regularity as conservation

A multi-conclusion entailment relation $\vdash$ *extends* a single-conclusion entailment relation $\rhd$ on S if, for every $U \in \mathrm{Fin}(S)$ and $a \in S$, $U \rhd a$ implies $U \vdash a$, which is to say that $\rhd \subseteq \rhd_{\vdash}$. In case the converse holds as well, i.e. altogether $\rhd = \rhd_{\vdash}$, any such extension is said to be *conservative*. An extension $\vdash$ of $\rhd$ is conservative if and only if [87, 88], for all entailments $a_1, \ldots, a_k \vdash b_1, \ldots, b_\ell$ and $U \in \mathrm{Fin}(S)$,

$$\frac{U, b_1 \rhd c \quad \ldots \quad U, b_\ell \rhd c}{U, a_1, \ldots, a_k \rhd c} \tag{6}$$

It suffices to consider only initial entailments in place of $a_1, \ldots, a_k \vdash b_1, \ldots, b_\ell$ whenever $\vdash$ is inductively generated.

Consider again our default set $\mathcal{E}$ of non-deterministic axioms on top of a single-conclusion entailment relation $\rhd$. Passing to the union of $\mathcal{E}$ and $\rhd$, we may assume that $(U, \{ a \}) \in \mathcal{E}$ whenever $U \rhd a$. Next let $\vdash_{\mathcal{E}}$ denote the multi-conclusion entailment relation which is inductively generated by $\mathcal{E}$ according to the rules (5) laid out before. In other words, this $\vdash_{\mathcal{E}}$ is the least multi-conclusion entailment relation which extends $\rhd$ such that $A \vdash_{\mathcal{E}} B$ for all $(A, B) \in \mathcal{E}$. By the above remarks on the models $\vdash_{\mathcal{E}}$, we immediately know about the semantics of this entailment relation:

Proposition 6.1. $\mathrm{Spec}(\mathcal{E}) = \mathrm{Mod}(\vdash_{\mathcal{E}})$.

The following is a mere rephrasing of the conservation criterion (6) recalled before.

Proposition 6.2. *The following are equivalent.*

1. *Every element of S is regular.*
2. *$\vdash_{\mathcal{E}}$ is conservative over $\rhd$.*

6.2 Intersecting prime ideals

Next we aim at identifying those elements which are common to every prime ideal over a given one. If $t \in T_{\mathfrak{a}}$ for a certain ideal $\mathfrak{a}$, let $\mathsf{paths}(t)$ denote the set of all paths of t. Next we introduce an auxiliary relation

$$U \rhd_T a \equiv (\exists t \in T_{\langle U \rangle})(\forall \pi \in \mathsf{paths}(t))\, \pi \rhd_0 a,$$

where $\rhd_0$ is the least consequence relation that extends $\rhd$ with additional axioms

$$A \rhd_0 a \qquad \text{for } (A, \emptyset) \in \mathcal{E},$$

and where we understand $\pi \rhd_0 a$ according to the conventions laid out in the first paragraphs of Section 4. It will turn out that $\rhd_T$ is a consequence relation. Notice that

$$U \rhd_0 a \quad \text{implies} \quad U \vdash_{\mathcal{E}} a, \tag{7}$$

simply because $\vdash_{\mathcal{E}}$ contains the generating axioms of $\rhd_0$.

Lemma 6.3. *For all $U \in \mathrm{Fin}(S)$ and $c \in S$ and $(A, B) \in \mathcal{E}$,*

$$\frac{(\forall b \in B)\, U, b \rhd_T c}{U, A \rhd_T c}$$

Proof. Let $B = \{ b_1, \ldots, b_n \}$. If $n = 0$, then the trivial tree labelled with U, A witnesses $U, A \rhd_T c$. Suppose next that $n1$, and that trees $t_i \in T_{\langle U, b_i \rangle}$ witness $U, b_i \rhd_T c$ for $1in$. These trees can be grafted, correspondingly, at the leaves of the tree displayed in (1) so as to obtain a witness for $U, A \rhd_T c$. ☐

Proposition 6.4. *For all $U \in \mathrm{Fin}(S)$ and $a \in S$,*

$$U \vdash_{\mathcal{E}} a \quad \textit{if and only if} \quad U \rhd_T a\,,$$

i.e., $\rhd_T$ is the trace of $\vdash_{\mathcal{E}}$. In particular, $\rhd_T$ is a consequence relation.

Proof. For the left-to-right implication we argue by induction on the generation of $U \vdash_{\mathcal{E}} a$. Reflexivity (R) is clear, and the case of transitivity (Ax) is taken care of by Lemma 6.3.

As regards the converse, we argue by induction on the tree $t \in T_{\langle U \rangle}$ witnessing $U \rhd_T a$. The base case boils down to (7). Consider next the case in which t has been extended at the leaf of one of its paths π with children labelled with the $b \in B$, where $(A, B) \in \mathcal{E}$ and $\langle \pi \rangle \supseteq A$. For every $b \in B$, relabelling the root of t with $U \cup \{ b \}$ yields a witness for $U, b \rhd_T a$ for which $U, b \vdash_{\mathcal{E}} a$ by induction, whence $U, A \vdash_{\mathcal{E}} a$ by (Ax). It remains to observe that the assumption that $\langle \pi \rangle \supseteq A$ lifts along the generation of t with successive cuts (T), so that we may assume $\langle U \rangle \supseteq A$ and conclude $U \vdash_{\mathcal{E}} a$. ☐

By Proposition 6.4, $\vdash_{\mathcal{E}}$ is conservative at least over $\rhd_T$. This means (Proposition 6.2) that every element of S is regular for $\vdash_{\mathcal{E}}$ over $\rhd_T$ (Lemma 6.3). In particular, the trace of $\vdash_{\mathcal{E}}$ equals $\rhd_T$ and thus can be computed in terms of the inductively defined tree class.

Corollary 6.5 (ZFC). *For every ideal $\mathfrak{a}$,*

$$\bigcap \mathrm{Spec}(\mathcal{E})/\mathfrak{a} = \{ a \in S \mid (\exists U \in \mathrm{Fin}(\mathfrak{a}))\, U \rhd_T a \}\,. \tag{8}$$

Proof. Combine completeness (4) with Proposition 6.1 and Proposition 6.4. ☐

If every element of S is regular for $\mathcal{E}$ over $\rhd$, then already $\rhd$ is the trace of $\vdash_{\mathcal{E}}$, and thus equals $\rhd_T$ by Proposition 6.4. So (8) boils down to Corollary 3.3.2.

6.3 Some reverse mathematics

The *Restricted Law of Excluded Middle* (REM) is not part of **CZF**. This REM means $\varphi \vee \neg\varphi$ for every set-theoretic formula φ that is *bounded* in the sense that only set-bounded quantifiers $\forall x \in y$ and $\exists x \in y$ occur in φ.

The following essentially rests on an argument of Bell's [9] and has already been used to show that completeness (4) implies REM [88].

Proposition 6.6. UPIT *implies* REM.

Proof. Given a Boolean algebra B, consider on $S = B$ the entailment relation $\rhd$ of *filter*:

$$a_1, \ldots, a_k \rhd a \equiv a_1 \wedge \cdots \wedge a_k a.$$

On top of $\rhd$ we consider the non-deterministic axioms of *proper prime filter*, viz.

$$0 \vdash$$

$$a \vee b \vdash a, b$$

for all $a, b \in B$. Distributivity ensures that every element is regular. Corollary 3.3 (which is a direct consequence of UPIT) thus implies that *in each Boolean algebra the intersection of the family of all its prime filters is* $\{ 1 \}$. Bell has shown that over intuitionistic set theory **IZ** this statement implies excluded middle [9, p. 161ff.]. The argument goes through over **CZF** if restricted to deal with bounded formulas only. □

Lemma 6.7. *Suppose that $R \subseteq S$ is regular. If $U \vdash_{\mathcal{E}} V$ and $V \subseteq R$, then $\langle U \rangle R$.*

Proof. By induction on $U \vdash_{\mathcal{E}} V$. The case of transitivity (Ax) boils down to regularity. □

In particular, we see once again that if a is regular, then $U \rhd a$ if and only if $U \vdash_{\mathcal{E}} a$, and so $\vdash_{\mathcal{E}}$ is conservative over $\rhd$ if (and only if) every element of S is regular.

As indicated in Section 5.1, Krull's Lemma, which is equivalent to the prime ideal theorem for distributive lattices (PIT) [51], is a consequence of UPIT. The reader has perhaps anticipated the following result which calibrates UPIT.

Proposition 6.8. *Over* **ZF**, UPIT *is equivalent to* PIT.

Proof. While it is evident that UPIT implies PIT, to show the converse, let $\mathfrak{a}$ be an ideal and let R be regular. Put

$$U \vdash' V \equiv (\exists U_0 \in \mathrm{Fin}(\mathfrak{a}))(\exists V_0 \in \mathrm{Fin}(R))\, U, U_0 \vdash_{\mathcal{E}} V, V_0.$$

This $\vdash'$ is an entailment relation [19] the models of which are the prime ideals that contain $\mathfrak{a}$ but avoid R. Now, if R and $\mathfrak{a}$ are disjoint, then $\vdash'$ is consistent according to Lemma 6.7: that is, $\emptyset \not\vdash' \emptyset$. Hence $\vdash'$ has a model by (4), which is a consequence of PIT. □

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