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On the Curvature Properties of "Long" Social Welfare Functions

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Abstract: This study characterizes the concavity properties of the Jorgenson and Slesnick's social welfare function that is likely the most empirically relevant function among the family of "long" welfare functions. We bridge this knowledge gap using the definition of generalized concavity to show the conditions necessary for the long social welfare function of interest to be decreasing and quasi-convex with respect to prices. Thanks to this result, "long" social welfare functions with regular curvature can be suitable for applied social welfare analysis and policy evaluations.

Keywords: generalized convexity; social welfare function; inequality aversion

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1. Introduction

This research uses generalized concavity results [1–3]to characterize the curvature properties of the Jorgenson and Slesnick Social Welfare Function (JS SWF) [4–8], which is the most relevant "long" SWF from an empirical point of view. The characterization of the conditions ensuring that the JS SWF is decreasing and quasi-convex with respect to prices is important to guarantee the regularity of social preferences required to maximize social welfare.

In contrast to abbreviated, or reduced form, social welfare functions that are based on vectors of incomes, we define long, or structural form, social welfare functions that are instead based on individual indirect welfare functions. Both social welfare functions encompass a concern for efficiency and equity. While empirical applications of abbreviated social welfare functions are widespread, the long functions, though much richer objects, are relegated to a niche probably because of high estimation costs. We hope that the formalization of the "missing" curvature properties may uplift the empirical attractiveness of long SWFs.

The next section discusses the empirical relevance of long social welfare functions. Section 3 formally describes the long JS SWF. Section 4 characterizes the missing curvature property. The conclusive section summarizes the main result and discusses future developments that completely characterized long SWFs may disclose.

2. Applied Relevance of Long Social Welfare Functions

The social welfare function was initially formalized by [9] and further developed by [10] to form the so-called Bergson–Samuelson individualistic social welfare function. If each individual has the same concave utility function, then the utilitarian social welfare function, which is increasing in each individuals' utility, is maximized if income is equally distributed. Any deviation from equality results in a welfare loss. Refs. [11–13] add to the efficiency value judgement implicit in utilitarian welfare functions, the value judgement for a more equitable distribution by scaling the efficiency term μ, the mean of a vector of incomes, with an index of equality expressed as the difference between 1 and an inequality index I: $SWF = \mu(1 - I)$. The literature refers to this class of SWFs, including a wide class of inequality indices and ordinal transformations incorporating society's preferences toward inequality, as abbreviated, or reduced form, SWFs [14–17] because the statistical distribution of income is abbreviated in terms of its mean income and dispersion measure. Ref. [18] establish a general relationship between the standard form of the individualistic social welfare function and the reduced form version. Abbreviated SWFs can easily be made operational to provide social rankings of policy reforms and/or to express judgements about the equity and efficiency tradeoff using the information about incomes present in household surveys without necessarily estimating an expenditure function or formally accounting for inter-household comparability.

As it is described in detail in the next section, Ref. [19] proposes a social welfare functional analogous to the abbreviated object (in sum rather than product form) where judgments can also be affected by both the size and distribution of welfares as captured by an index of deviations from the mean. It is more general because it has direct utilities as arguments and incorporates information concerned with the cardinal and comparable nature of welfares [19–22]. The empirical counterpart of the Roberts canonical form is the SWF proposed by [4–6] where direct utilities are substituted by indirect utilities representing household welfare. It maintains the same cardinal and fully comparable informational basis of the Roberts' form because the welfare weights are exact: that is, independent of the base level of income, or utility, at which comparisons are implemented [23–25]. The JS SWF is in the "welfarist" tradition where individual well-being is derived from the observed demand behavior of consumption goods, leisure and services. The econometric estimates of complete demand systems must be theoretically plausible; that is, the Slutsky matrix of substitution effects must be symmetric and negative semidefinite in order to recover by integration individual welfare functions that can be consistently aggregated into a social welfare function. The concavity of the individual welfare functions is a necessary condition for the concavity of the social welfare function. While the abbreviated SWFs are relatively simple, the JS SWF is a convoluted and highly parameterized object. In contrast with the term "abbreviated" or "reduced form", we refer to the empirical representations of Roberts' canonical form, such as the JS SWF, as "long" or "structural form." The long form reduces to the abbreviated form by inverting the indirect welfare function to obtain the associated income levels and mapping the equivalence scale functions at reference prices and demographic characteristics into scalar values specific to each household.

To the best of our knowledge, the applications of the long SWF amount to a handful of publications. The applications are either based on Translog preferences to model consumer choices [4–8,26–32] or on AIDS preferences [33,34]. These applications span from the measurement of social cost of living [35], poverty and inequality, and the implementation of sophisticated policy analysis based on equity and efficiency evaluations. Within a general equilibrium framework consistent with national accounts, Ref. [31] evaluate energy and environmental policies in terms of their impacts on individual and social welfare.

The long SWF has also been used for learning about important features of society [34] devise an objective method to elicit a society's degree of aversion to inequality. This requires the maximization of the JS SWF to determine the set of prices coherent with a given choice of degree of aversion to inequality. Maximization of a SWF is also naturally invoked in an optimal taxation or trade tariff context. Ref. [36] explains how to formalize optimal commodity and income taxation by maximizing a Bergson-Samuelson SWF, though empirical implementations are rare [37–39]. It would seem natural, though, to extend the social maximization context to the realm of long SWFs that recognize the relevance of distributional concerns.

The empirical implementation of long SWFs is a demanding programming task because it requires the econometric estimation of complete demand systems that obey the theory requirements for sound welfare analysis. The integrability of demands ensures that individual welfare functions can be recovered so that a theoretically consistent SWF can be constructed. Another implementation barrier is data collection and preparation. Expenditure surveys normally record household consumption without reporting price information or quantities that may allow the derivation of unit values. A recent procedure developed by [40] circumvents this lack of information by constructing pseudo-unit values. These recent developments make the estimation of a long SWF a manageable task.

In analogy with an estimated indirect utility (welfare) function that is theoretically plausible when it possesses the properties of continuity, the homogeneity of degree zero in prices and income, strictly increasing in income, decreasing in prices, and quasi-convex in prices and income, an estimated social welfare function should have analogous properties to be "well-behaved." However, the curvature properties of long SWFs have not been characterized yet. If the curvature properties cannot be verified, then there is no guarantee that a unique maximum is achieved. That regular properties of social indifference contours are crucial to lay the foundations for the "economics of a good society," as Samuelson closes his 1956 article, was also clear to [41,42]. Ref. [34] acknowledge that they empirically test the curvature properties of the estimated SWF by finding reasonable global solutions. Ref. [33] (p. 244) reports a surface plot and associated level curves of a long SWF showing that it is convex and may achieve a unique maximum. In general, welfare maximization is well-defined only if the social welfare function is at least strictly quasi-concave with respect to the individual indirect utilities for any given level of optimal prices. Because no previous study formally characterized the curvature properties of long SWFs, we intend to fill this knowledge gap. The next section introduces the long JS SWF.

3. The Long Jorgenson and Slesnick Social Welfare Function

A social welfare functional assigns a social ordering defined on the set of social states X to each possible profile of individual utility functions in its domain. In our context, a social state X is described by the vector of quantities x consumed by K individuals. A social ordering R is a reflexive, complete and transitive binary relation that orders social states. The set $\mathcal{R}$ represents the set of all orderings defined on X. We define person k's utility function on the set of social states X as $u_k : X \to \mathbb{R}$, continuous and differentiable. The individual utility function describes the level of welfare for a given individual in each state. We also define the profile U formed by the vector of all real-valued individual utility functions as $U = (u_1, \ldots, u_K) \in \mathcal{U}$, where $\mathcal{U}$ is the set of all possible profiles. For any $x \in X$, $U(x)$ denotes the vector $U(x) = (u_1(x), \ldots, u_K(x)) \in \mathbb{R}^K$. To obtain a social preference ordering based on the individual utility functions, Ref. [43] defines a social welfare functional $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{R}$ where $\mathcal{D} \subseteq \mathcal{U}$ is the set of admissible profiles defining the domain of $\mathcal{F}$. The social welfare functional $\mathcal{F}$ maps the set of admissible utility profiles $\mathcal{D}$ to the set of all possible social orderings $\mathcal{R}$. The social preference ordering that is obtained by applying $\mathcal{F}$ is denoted by $R_U = \mathcal{F}(U)$. The strict preference and the indifference relations corresponding to R_U are denoted by P_U and I_U, respectively.

Ref. [19] (Theorem 4) demonstrates that if W satisfies Unrestricted Domain (UD), the Independence of Irrelevant Alternatives (IR), the Weak Pareto Principle (WP) [19,21,44,45] and Cardinal Full Comparability (CFC), The degree of comparability can be described in terms of the invariance class Φ being the set of invariance transformations ϕ such that $\forall U^1, U^2 \in \mathcal{D}$, if $\forall x \in X$, $U^2(x) = \phi(U^1(x))$, then $R_{U^1} = R_{U^2}$. CFC holds if ϕ is a list of identical, strictly positive affine transformations, i.e., $\exists \alpha \in \mathbb{R}$ and $\exists \beta > 0$ such that $\phi_k(u_k) = \alpha + \beta u_k$, $k = 1, \ldots, K$. Then, there exists a function g, homogeneous of degree one, computed on the deviations of the levels of individual welfare from the mean level of welfare that defines the social welfare function in the following canonical form

$$W(U(x)) = \overline{W}(x) + g\big[U(x) - \overline{W}(x)\big], \text{ for } \overline{W}(x) = \sum_k a_k u_k(x) \text{ and } a \in \mathbb{R}^K_+. \tag{1}$$

The weights $a \in \mathbb{R}^K_+$ in [19] are all equal to $1/K$ because of the anonymity condition A requiring that the names of the individuals are irrelevant and all individuals in society are given the same weight. Such a class of admissible social judgments incorporates both an efficiency component given by the average individual welfare and an equity component in the form of an index of deviations from the mean measuring the inequality in the distribution of welfare. If dispersion increases, then social welfare decreases implying that g is a decreasing function. Then, for the SWF to be concave with respect to U, the function g must be concave in U. The Roberts specification encompasses the utilitarian class of SWF when the distributional concern is not important.

To incorporate non-welfare characteristics of social states, we need to replace WP with Positive Association (PA), ensuring that an increase in all levels of individual welfare must increase social welfare, and Non-Imposition (NI) granting that welfare characteristics are always deemed relevant. Maintaining UD, IR and CFC, Ref. [19] showed the existence of a social welfare function

$$W(U(x)) = F\left[\overline{W}(x) + g\left(U(x) - \overline{W}(x)\right), x\right] \tag{2}$$

with $F : \mathcal{U} \times X \to \mathbb{R}$ and $\overline{W}(x) = \sum_k a_k(x)u_k(x)$. It incorporates non-welfare characteristics of social states through the weights $a_k(x)$, $g(x)$ and through F that depends directly on the social state x.

As noted by [46], the SWF approach does not distinguish changes in individual well-beings from changes due to different measurement scales. They propose a scale-dependent approach that pairs each utility profile with a profile of measurement scales. Ref. [47] use the concept of reference set welfarism, which is based on the aggregation of reference money metric utilities used to represent the social order. This is still exposed to the criticism of [48] about the use of the sum of money metric utilities as an SWF, because it could be the result of the sum of non-concave functions (see also [49]). This class of welfare functionals is exempted from the criticism of welfarism, but it requires an informational basis often too large to make welfare judgments operational. What is crucial, as for individual utility profiles, is to contract the informational basis while maintaining the identifiability of social preference orderings. The informational constraint imposed on F being independent of x is similar to the one imposed at the individual level to permit inter-household comparability. In order to obtain an operational social welfare function, it is necessary to specify a functional form for g and for the individual welfare functions u_k compatible with the CFC requirements, as we now illustrate.

Extending the seminal work of [4–8,19] define a social welfare function on the vector V of the logarithms of individual indirect utility functions V_k, that is $V = (\ln V_1, \ldots, \ln V_K)$, belonging to the indirect utility possibility set $\mathcal{V}$. Jorgenson and Slesnick use $\ln \overline{V}(V, p) = \sum_k m_o(p, d_k) \ln V_k / \sum_k m_o(p, d_k)$ as a mean level of welfare with weighting functions $m_o(p, d_k)$ and $g(V, p|\rho) = -\gamma(p)M(V, p)$ as a weighted deviation of the individual welfares from the mean, where

$$\gamma(p) = \left\{ \frac{\sum_{k \neq j} m_o(p, d_k)}{\sum_k m_o(p, d_k)} \left[1 + \left(\frac{\sum_{k \neq j} m_o(p, d_k)}{m_o(p, d_j)} \right)^{-(\rho+1)} \right] \right\}^{\frac{1}{\rho}} \text{ and} \tag{3}$$

$$M(V, p) = \left[\frac{\sum_k m_o(p, d_k) |\ln V_k - \ln \overline{V}(V, p)|^{-\rho}}{\sum_k m_o(p, d_k)} \right]^{-\frac{1}{\rho}}. \tag{4}$$

The JS SWF takes the form analogous to (2)

$$W(V, p|\rho) = \ln \overline{V}(V, p) + g(V, p|\rho). \tag{5}$$

The notation in Equation (5) makes the dependence on prices p explicit. The first term is a weighted average of individual welfare levels. The second term is a mean value function of degree ρ of the deviations of household welfare from the average. Using the definition of mean value function [50], the function g can be generalized as $g(V) = \phi^{-1}\{\sum_k \phi_k(f(V_k))\}$ where $\phi(f(V_k))$ is a continuous and strictly monotonic function of the form $f(V_k)^\rho$. The function $-g$ is ρ-concave if and only if it satisfies Definition A5 in Appendix A.1. The constant ρ determines the curvature of the SWF and measures the degree of aversion to inequality in the distribution of welfare levels. The function g is homogenous of degree one being a mean value function of order ρ. It is a negative function that reaches the value of zero in the perfectly equal case where $\ln V_k = \ln \overline{V}(V, p), \forall k = 1, \ldots, K$.

Note that the weighting function $m_o(p, d_k)$ is an indicator of the size of consuming units depending on prices p and on the vector of attributes d_k used to construct equivalent total expenditure $y = y_k / m_o$, where y_k is the total expenditure of household k. The scale for the reference household is $m_o(p, d_j) = \min_k m_o(p, d_k)$. Ref. [4] incorporated a notion of horizontal equity that treats different individuals differently by introducing a weak form of anonymity requiring that all individuals with same characteristics receive the same weight. The SWF therefore maintains symmetry in V for identical individuals. Both the weights and the measure of inequality given by the function g must be the same for individuals with identical characteristics.

The SWF is equity regarding in the sense that it obeys Dalton's principle of transfers requiring that a transfer from a richer to a poorer individual, that does not reverse their relative positions, must increase the level of social welfare. As a consequence, the weights associated to the individual welfare function must be $a_k(p) = m_o(p, d_k) / \sum_k m_o(p, d_k)$, with $\sum_k a_k(p) = 1,\ 0 < a_k(p) < 1$. Because we focus on changes in p, for analytical convenience, we assume that prices and demographic characteristics do not interact, so that we can drop the notation associated with demographic characteristics without loss of generality.

The SWF reaches a maximum when $\gamma(p)$ is equal to zero and it is positive only if $\gamma(p) < \ln \overline{V} / M(V, p)$. In order to simplify the notation, in Equation (3), we substitute $a_j(p) = \min_k a_k(p)$ and

$$\frac{\sum_{k \neq j} m_o(p, d_k)}{m_o(p, d_j)} = \frac{1 - a_j(p)}{a_j(p)}, \tag{6}$$

so that $\gamma(p)$ can be written as

$$\gamma(p) = \left\{ (1 - a_j(p)) \left[1 + (1/a_j(p) - 1)^{-(\rho+1)} \right] \right\}^{1/\rho}. \tag{7}$$

Note that when $a_j(p) \to 1$, as if there was only one individual in the society, then $\gamma(p) \to +\infty$. While if $a_j(p) \to 0$, as usually happens when there is a high number of observations (For instance, if there is a sample of 15,000 households of single persons and in the reference household, the equivalent adult is 1, then the value of $a_j(p) = m_o(p, d_j) / \sum_k m_o(p, d_k)$ is equal to 1/15,000), then $\gamma(p) \to 0$, except for the case $\rho = -1$ for which $\gamma(p) \to 1/2$). With $a_j(p) = 1/2$, then $\gamma(p) = 1$, independently from the value of ρ. Notice that with $0 < a_j(p) < 1/2$, then $0 < \gamma(p) < 1$. In particular, $a_j(p) \to 0$, and hence $\gamma(p) \to 0$ implies that the last individual is given a very small weight. While with $a_j(p) = 1/2$, there is no need for further ethic considerations except for the dispersion among the individuals.

The values of $\gamma(p)$ also depend on the choice of ρ. The parameter ρ measures the society's constant degree of aversion to inequality. Within the admissible interval $(-\infty, -1]$, it affects the curvature of the social welfare function in the individual welfare space. Recall that $\gamma(p) \in (0, +\infty)$ and $\rho \in (-\infty, -1]$. The function $\gamma(p)$ is increasing with respect to ρ. This implies that the weight given to dispersion depends also on ρ. To illustrate the range of the function $\gamma(p)$, suppose that the household sample consists of 20,000 units so that $a_j(p) = 0.00005$. When $-2 < \rho < -1$, which is the empirically interesting case [34], then we have $0.007 < \gamma(p) < 0.5$, while for $-10 < \rho < -2$, then $0.00001 < \gamma(p) < 0.007$. Therefore, $\gamma(p)$ becomes increasingly more relevant as ρ approaches -1.

Further inspection of Equation (7) reveals that when $\rho \to -\infty$, then we place the least possible weight upon equity as if all individuals had the same level of welfare and the social welfare function collapses to the weighted utilitarian case. If $\rho = -1$, then one recovers the egalitarian case giving maximum consideration to the inequality function $g(V, p|\rho)$.

When the weights $a_k(p)$ take the same value for all k, then the potentially available level of welfare is maximum. This is Jorgenson and Slesnick's measure of efficiency because the inequality function receives minimum consideration. Note that a greater inequality aversion corresponds to a lower value of ρ. If ρ increases, then $\gamma(p)$ and $M(V, p)$ increases. This implies that the social planner is more willing to give up some welfare from the utilitarian position and s/he is less averse to inequality.

Ref. [7] (p. 311) state that "although the magnitude of money metric social welfare depends on the degree of aversion to inequality, we find that the qualitative features of comparisons among alternative policies for different values of this parameter are almost identical." However, this may be true only when alternative policies are ranked in relation to one society. If the same alternatives were compared across societies, then different degrees of society's aversion to inequality may significantly affect social orderings. When the degree of aversion to inequality is estimated endogenously, ρ becomes a distinctive attribute of each society.

Ref. [34] propose a scheme in which a benevolent social planner, or ethical observer, first chooses economic policies by maximizing W, specified *à la* JS, with respect to $\ln V_k$, with $k = 1, \ldots, K$, for a set of prices at each given ρ. In the second part of the scheme, the households are asked to reveal the ρ that maximizes each household's welfare $\ln V_k$. Society is assumed to choose according to a majority rule [51]. The mechanism critically depends on the choice of the set of prices p^* that minimizes society's welfare at each given ρ. The existence of a solution to this problem requires that the social welfare functional W be quasi-convex in prices p. The composition mapping W is strictly increasing in each function $\ln V_k$ and homogeneous of degree one in levels of the individual welfare. The logarithm transformation of the indirect utility function $\ln V_k(p, y_k)$ preserves its properties of (a) homogeneity of degree 0 in (p, y_k), (b) continuity at all strictly positive p, (c) non-increasing in p and non-decreasing in y_k, and (d) quasi-convexity in p.

A benevolent social planner elicits society's preferences toward inequality by gaining knowledge on the set of relative prices that corresponds to the maximization of $W(V, x|\rho)$ with respect to V at each $\rho \in (-\infty, -1]$. Then, the set of relative prices that maximizes the level of welfare of each household in the society associated with each level of ρ can be recovered. When each household selects the level of ρ that maximizes its level of welfare, than the social planner selects the median level of society's degree of aversion ρ as the majority winner. Here, V is a K-dimensional vector of individual welfare functions $\ln V_k(p, y_k)$ for $k = 1, \ldots, K$. The welfare maximization is well defined only if $W(V(\bar{p}, y), \bar{p}|\rho)$ is at least strictly quasi-concave with respect to V for any fixed level of prices $\bar{p}$. A dual problem can be defined as shown in the next section. It leads to the minimization of an indirect welfare function $\mathcal{W}(V^*(p, \bar{y}), p|\rho)$ with respect to p, where the level of income $\bar{y}_k$ is now given in the indirect utilities $V_k^*(p, \bar{y}_k)$.

We now investigate the curvature properties of a long SWF using well-known notions of concavity and convexity that we report in Appendix A.1 for the readers' convenience.

4. The "Missing" Curvature Properties

To study the curvature properties of long SWFs, we first recover the indirect social welfare function to be maximized with respect to prices. Consider the maximization of social welfare with respect to the vector of indirect utility functions V with exogenous prices that are predetermined at level $\bar{p}$ and a fixed level of aversion to inequality ρ

$$\max\{W(V(\bar{p}, y),\ \bar{p}\,|\rho)\ :\ V \in \mathcal{V}\}, \tag{8}$$

where $\mathcal{V}$ is the indirect utility possibility set. The optimal value functions

$$V^*(p, \bar{y}) = (\ln V_1^*(p, \bar{y}_1), \ldots, \ln V_K^*(p, \bar{y}_K)),$$

solution of problem (8), depend on prices p and describe the maximum level of individual welfare attainable for a given level of equivalent total expenditure $\bar{y}$.

The indirect social welfare function $\mathcal{W}(V^*(p,\bar{y}),\ p\,|\rho)$ represents the maximum value of welfare for any $p \in \mathcal{P}$. [35] (p. 134) establishes the maximizing society's market demand functions and the associated aggregate indirect social welfare functions. Pollak also provides an alternative proof of Samuelson's [10] theorem stating that if expenditure is distributed among households so as to maximize a social welfare function, then the implied market demand functions can be rationalized by a utility function. Ref. [10] also assumes that individual and social indifference contours have the usual convexity properties. Problem (8) is equivalent to

$$\max\{W(V(\bar{p},y),\ \bar{p}\,|\rho) - \mathcal{W}(V^*(p,\bar{y}),\ p\,|\rho)\ :\ V \in \mathcal{V},\ p \in \mathcal{P}\}, \tag{9}$$

where $\mathcal{P} = \{p : 0 \le p \le 1, \text{ with } \sum_i p_i = 1\}$ is the set of feasible normalized prices. Problem (9) is called the primal-dual problem in [52]. It reaches a maximum at zero and solving it with respect to prices p is equivalent to solving the following problem

$$\min\{\mathcal{W}(V^*(p,\bar{y}),\ p\,|\rho)\ :\ p \in \mathcal{P}\}. \tag{10}$$

The indirect SWF $\mathcal{W}$ measures what the society is willing to give up to reach a given level of welfare. The decision variables are the prices, because they represent the direction along which to move to achieve the equilibrium. Notice that the properties of $\mathcal{W}$ are the same as those of an household indirect utility function: (a) homogeneity of degree 0 in (p,y), (b) continuity, (c) non-increasing in p and non-decreasing in y and (d) quasi-convexity in p. As noted in [47], when implementing social welfare comparisons, it is necessary to specify a reference price vector as commonly completed with money metrics of individual utilities. Incidentally, the possibility to maximize the SWF $\mathcal{W}$ with respect to prices may provide an admissible set of reference prices corresponding to the optimal solution of the maximization of the social welfare function.

Problem (8) can be interpreted as a pre-transfer problem, where $\bar{p}$ is the vector of market prices, while Problem (10) can be interpreted as a post-transfer problem, where prices change. If, for instance, $p = \bar{p} + T$, with $T = (T_1, \ldots, T_n)$ being the vector of the amounts of transfers T, from condition $p \in \mathcal{P}$ we have that $\sum_i T_i = 0$. Note, however, that the pricing rule at the basis of the transfer principle can be more general [53] accounting for different transformations of prices.

We now specialize on the JS SWF and provide the conditions that make these properties hold in the next Section 4.1.

4.1. Curvature Properties of the Jorgenson and Slesnick Social Welfare Function

Reconsider the Jorgenson and Slesnick specification for an SWF:

$$W(V,p|\rho) = \ln \overline{V}(V,p) - \gamma(p)\left\{\sum_k a_k(p)\,|\ln V_k(p) - \ln \overline{V}(V,p)|^{-\rho}\right\}^{-\frac{1}{\rho}}, \tag{11}$$

where $\ln \overline{V}(V,p) = \sum_k m_o(p,d_k) ln V_k(p) / \sum_k m_o(p,d_k) = \sum_k a_k(p) \ln V_k(p)$ with $a_k(p) \in [0,1], \sum_k a_k(p) = 1$, $\gamma(p)$ is given in Equation (7), k is the number of households in the society, and $m_o(p,d_j) = \min_k m_o(p,d_k)$ is the scale for the reference household. In line with [8], the SWF embeds the following properties.

The JS SWF is equity regarding, because at a given level of average welfare, social welfare declines as the distribution of welfare levels becomes more dispersed. It is also efficiency regarding because it is strictly increasing if an individual utility function increases, all other things equal. Furthermore, the increase in the average level of individual welfare $\ln \overline{V}(V,p)$ must be larger than $g(V,p|\rho)$ representing the dispersion in individual welfare levels if the individual welfares are considered as "goods". The weight $\gamma(p)$ is therefore chosen as a function of the weight $a_j(p) = m_o(p,d_j)/\sum_k m_o(p,d_k)$ respecting this condition.

This monotonicity property is known as positive association (PA). It has been formalized by [54,55] in order to generalize the Pareto principle. PA requires that if the ordering of each individual was preferred before the appearance of an alternative state, then it is still preferred. Equity considerations represented by the function g are affected by the size of the population through γ which depends on $a_j(p)$.

The concavity with respect to V can be deduced studying the curvature of the components $\ln \overline{V}(V,p)$ and $g(V,p|\rho)$. The term $\ln \overline{V}(V,p) = \sum_k a_k(p) \ln V_k(p)$ is a weighted sum of concave functions. The term $g(V,p|\rho) = -\gamma(p)\left[\sum_k a_k(p)|\ln V_k - \ln \overline{V}(V,p)|^{-\rho}\right]^{-\frac{1}{\rho}}$ is concave, because $M(V,p)$ is convex, being a weighted $\ell^{-\rho}$ norm of non-negative elements.

Recalling the equivalence between Problems (8) and (10), we now focus on the properties of the SWF with respect to changes in prices. To learn about the curvature properties of the JS social function with respect to prices p, it is crucial to know the curvature properties of the different functions that compose it. The weighting function $a_k(p)$ depends on the structure of the equivalence scale $m_o(p,d_k)$. The function $m_o(p,d_k)$ assigns a weight to each household in proportion to its needs. This weight depends on prices and exogenous attributes d_k and represents the number of household equivalent members. Ref. [56] show that for the household income $y_k = y\, m_o(p,d_k)$ to be a plausible expenditure function (an expenditure function is theoretically plausible if it is (a) homogeneous of degree 1 in p, (b) positive, strictly increasing in u and non-decreasing in p, (c) concave in p, and d) continuous in p and u), the equivalence scale $m_o(p,d_k)$ must satisfy the conditions described below.

Properties of the Household Equivalence Scales (ES). The equivalence scale $m_o(p,d_k)$ is positive and non-decreasing in p_i, homogeneous of degree zero in p and quasi-concave.

Consider now the normalized scale $a_k(p) = m_o(p,d_k)/\sum_k m_o(p,d_k)$. It weights the household equivalence scales relative to the sum of all types in the sample. It ranges in the $[0,1]$ interval and can be therefore interpreted as a relative frequency of a certain type in the population. An increase in p results in an increase in the cost of the needs of an individual. The increase can be more or less than proportional, depending on the compensation effect of the economies of scale. Notice also that the scale $a_k(p)$ has the same properties of $m_o(p,d_k)$, because the sum of equivalent incomes $y_k = y\, m_o(p,d_k)$ gives

$$\sum_k y_k = \sum_k y\, m_o(p,d_k) \text{ and } \frac{\sum_k y_k}{\sum_k m_o(p,d_k)} = \frac{\sum_k y\, m_o(p,d_k)}{\sum_k m_o(p,d_k)}. \tag{12}$$

Then,we have

$$\frac{Y}{\sum_k m_o(p,d_k)} = \frac{\sum_k y\, m_o(p,d_k)}{\sum_k m_o(p,d_k)} = \sum_k a_k(p) y = y, \tag{13}$$

where Y represents the total income in the society and $a_k(p)y$ measures the total income per equivalent household. The weight $a_k(p)$ scales income y, and therefore, it satisfies the same properties of $m_o(p,d_k)$.

We now show that the SWF is decreasing and quasi-convex with respect to p, splitting the steps that lead to this statement. The proofs are in Appendix A.2. We recall that the properties of UD, IR, PA, NI and CFC hold as explained in Section 3. The first step states that the weights $a_k(p)$ are non-decreasing in prices but in a way that does not dominate the decrease of $\ln V_k(p)$, depending also on $\varepsilon_k = \frac{\partial \ln V_k(p)/\partial p_i}{\partial a_k(p)/\partial p_i} \frac{a_k(p)}{\ln V_k(p)}$, that is the elasticity of the function $\ln V_k(p)$ with respect to $a_k(p)$.

Lemma 1. *Given PA, the change in the average of individual welfares $\ln \overline{V}(V,p)$, due to an increase in prices, is such that*

$$\frac{\partial \ln \overline{V}(V,p)}{\partial p_i} = \sum_k \left[\frac{\partial a_k(p)}{\partial p_i} \ln V_k(p)(1+\varepsilon_k)\right] < 0,\ i = 1,\ldots,n, \tag{14}$$

where $\varepsilon_k = \frac{\partial \ln V_k(p)/\partial p_i}{\partial a_k(p)/\partial p_i} \frac{a_k(p)}{\ln V_k(p)}$.

This implies that the monotonicity of $\ln \overline{V}(V,p)$ depends on the values of ε_k, and the choice of the weights a_k must be consistent with Equation (14). In fact, the sign of each partial derivative of $\ln \overline{V}$ depends on the elasticity ε_k between $\ln V_k$ and a_k, with $k = 1, \ldots, K$. Let ES hold, then $\varepsilon_k \leq 0$. However, if further $\varepsilon_k \leq -1$, $\forall k$, then Equation (14) is satisfied. Note also that under an anonymity condition such that $a_k = 1/K$, $\forall k$, then $\partial \ln \overline{V}(V,p)/\partial p_i = (1/K)\sum_k \partial \ln V_k(p)/\partial p_i$ is negative because each $\partial \ln V_k(p)/\partial p_i$ is negative. We now show the generalized convexity of $\ln \overline{V}$.

Lemma 2. *Given PA and ES, the function* $\ln \overline{V}$ *is quasi-convex with respect to* p*.*

To examine the properties of the function $g(V,p|\rho) = -\gamma(p)M(V,p)$, we introduce the following property of Monotonicity of the Deviations of the individual welfare functions from the mean with respect to p_i, $i = 1, \ldots, n$.

Property of Monotonicity of the Deviations (MD). Define $\delta_k(V,p)$ as the deviation function $\delta_k(V,p) = \ln V_k(p) - \ln \overline{V}(V,p)$ such that $\partial \ln \delta_k(V,p)/\partial p_i \geq 0$, $\forall k = 1, \ldots, K$ and $\forall i = 1, \ldots, n$.

Recall that $\partial \ln \delta_k(V,p)/\partial p_i = (\partial \delta_k(V,p)/\partial p_i)/\delta_k(V,p)$. Note that the property requires that if $\delta_k(V,p) > 0$, then $\partial \delta_k(V,p)/\partial p_i \geq 0$. Therefore, if the ranking of the welfare of household k is above the mean level $\ln \overline{V}(V,p)$, that is $\ln V_k(p) > \ln \overline{V}(V,p)$, then $\partial \ln V_k(p)/\partial p_i \geq \partial \ln \overline{V}(V,p)/\partial p_i$. This means that the difference between $\ln V_k(p)$ and $\ln \overline{V}(V,p)$ is not decreasing. On the other hand, if $\delta_k(V,p) < 0$, then $\partial \delta_k(V,p)/\partial p_i \leq 0$, and the difference between $\ln V_k(p)$ and $\ln \overline{V}(V,p)$ is not increasing, or equivalently, the difference between $\ln \overline{V}(V,p)$ and $\ln V_k(p)$ is not decreasing.

We can describe $\partial \delta_k(V,p)/\partial p_i$ as a variation of the difference in the welfare of household k from the mean. For a rich household, given a change in p_i, the change in household welfare $|\partial \ln V_k(p,y_k)/\partial p_i|$ is lower than the change in mean welfare $|\partial \ln \overline{V}(V,p)/\partial p_i|$. The opposite holds for a poor household. Furthermore, MD describes the impact of a change in prices for the different households while maintaining the ranking of each household k after a price change. In other words, the household that was relatively richer before the price change maintains the relative ordering after the change. Therefore, any measure that redistributes income through price (or tax) policies does not have an effect on the ordering of the individuals [57].

Lemma 3. *Given ES and MD, the function*

$$\begin{aligned} g(V,p|\rho) &= -\gamma(p)M(V,p) = \\ &-\left\{(1-a_j(p))\left[1+\left(\frac{1-a_j(p)}{a_j(p)}\right)^{-(\rho+1)}\right]\right\}^{\frac{1}{\rho}}\left[\sum_k a_k(p)|\ln V_k(p) - \ln \overline{V}(V,p)|^{-\rho}\right]^{-\frac{1}{\rho}}, \end{aligned} \tag{15}$$

is non-increasing and quasi-convex with respect to p*.*

We now use these results to formalize the curvature property of the function $W(V,p|\rho)$.

Proposition 1. *Given PA, ES and MD, the function* $W(V,p|\rho)$ *is decreasing and quasi-convex with respect to* p*.*

Therefore, we can state that if a JS SWF satisfies UD, IR, PA, NI and CFC, that is, it maintains the measurement/comparability requirements of Roberts' canonical form, the equivalence scale functions m_o or a_k conform to the properties described in Property ES, and the deviations δ_k of each V_k from the mean are monotone as described in Property MD; then, Proposition 1 follows. Therefore, if a long SWF is quasi-convex with respect to p, then it can be used to solve optimization problems in the dual space and to maximize social welfare in general.

5. Conclusions

The main contribution of this study is the definition of the curvature properties of each object composing the long JS SWF and of the social functional in its aggregate. Only a regular SWF is suitable both for welfare maximization, robust microsimulations of policy impacts on individual and social costs of living, and the elicitation of society's aversion to inequality for different societies or groups of individuals. An SWF is well-behaved when the regularity properties described in the study are empirically respected.

By completing the set of requirements with the "missing" characterization of the curvature properties, the long JS SWF is ready for a more general use. The concavity results obtained here can be extended to other functional forms by using analogous lines of proof. Furthermore, the knowledge of the effects of price variations on the SWF allows analyzing the conditions necessary for optimal welfare-improving price subsidies, tariffs and taxation. We hope that our mathematical effort may help uplift the long SWFs from being down and draw the attention of economists interested in applied social welfare analysis and in the microsimulations of policy impacts.

Regarding future developments that an uplift of long SWFs may disclose, the efficiency and equity considerations involved in the long JS SWF should also be extended to the exact aggregation process summing up the individual welfare of each family member to a household welfare function [58] and then to a long SWF based on the recent acquisitions of collective theory [59–61] that, in a Pareto household economy, would allow the recovering of the welfare function of each individual in the household. Ref. [10] (p. 8) asks, "Who after all is the consumer in the theory of consumer's (not consumers') behavior? Is he a bachelor? A spinster? Or is he a "spending unit" ...?" such as the family unit that consists, Samuelson notes, of a single individual in but a fraction of the total cases. Along the same lines, Ref. [62] (p. 2144) asks, "Whose welfare functions should serve as arguments of the social welfare functions?" This research endeavor toward a collective SWF is the next item on our research agenda.

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Appendix A

Appendix A.1. Generalized Concavity: Basic Definitions

The analysis of generalized concavity requires the use of basic notions of quasi-concavity and quasi-monotonicity.

Definition A1. *Quasi-concavity. A function $f : X \to \mathbb{R}$ defined on a convex subset $X \subseteq \mathbb{R}^n$ of a real vector space is quasi-concave if for all $x^0, x^1 \in X$ and $\alpha \in [0,1]$ we have*

$$f\left(\alpha x^0 + (1-\alpha)x^1\right) \geq \min\left\{f(x^0), f(x^1)\right\}. \tag{A1}$$

A function $f : X \to \mathbb{R}$ is said to be quasi-convex if $-f$ is quasi-concave.
An equivalent condition for functions that are differentiable at least once is the following.

Definition A2. *Differentiable Quasi-concavity. Let f be differentiable on the open convex set $X \subseteq \mathbb{R}^n$. Then, f is quasi-concave if for every $x^0, x^1 \in X$, the following inequality is verified*

$$f(x^0) \leq f(x^1) \Longrightarrow (x^1 - x^0)\nabla f(x^0) \geq 0 \tag{A2}$$

or vice versa

$$(x^1 - x^0)\nabla f(x^0) < 0 \Longrightarrow f(x^0) > f(x^1). \tag{A3}$$

If a function is both quasi-concave and quasi-convex, then it is quasi-monotone. Monotonicity and quasi-monotonicity are equivalent for univariate functions. In general, a function is both concave and convex if and only if it is affine. Quasi-monotone functions are generalizations of affine functions in the case the concave and convex functions are replaced by quasi-concave or quasi-convex functions.

Consider the vector of functions $v = (v_1, .., v_K) \in \mathbb{R}^K_+$ and the vector of weights $\sum \Lambda \in \Lambda$ with $\Lambda = \left\{\lambda | 0 \leq \lambda_k \leq 1,\ \forall k = 1, .., K; \sum_{k=1}^K \lambda_k = 1\right\}$. The mean value function $M_h(\lambda; v)$ on a K-dimensional space is defined as follows.

Definition A3. *Mean value function in vector space. Let h be a continuous and strictly monotone real valued function,*

$$M_h(\lambda; v) = h^{-1}\left(\sum_{k=1}^{K} \lambda_k h(v_k)\right), \tag{A4}$$

where $\lambda \in \Lambda$ and h^{-1} is the inverse function of h .

Note that if h is the identity function $h(v) = I(v) = v$ then $M_h(.)$ is the arithmetic mean, while if $h(v) = \ln v$, then $M_h(.)$ is the geometric mean and if $h(v) = v^\rho$ then $M_h(.)$ is a power mean of order ρ. The same notation can be extended to the case of vector functions.

Definition A4. *(h, G)-concavity ([2]). A real valued function f on $\mathbb{R}^K_+$ is (h, G)-concave if and only if $\forall \lambda \in \Lambda$ and $\forall v = (v_1, .., v_K) \in \mathbb{R}^K_+$:*

$$f(M_h(\lambda; v) \geq M_G(\lambda; f(v_1), .., f(v_K)), \tag{A5}$$

$$\text{that is, } f\left(h^{-1}\left(\sum_{k=1}^{K} \lambda_k h(v_k)\right)\right) \geq G^{-1}\left(\sum_{k=1}^{K} \lambda_k G(f(v_k))\right). \tag{A6}$$

The choice of $h(.)$ and $G(.)$ as identity functions $h(.) = G(.) = I(.)$ defines the family of concave functions. When only $h(.)$ is chosen as the identity function $h(.) = I(.)$, then the G-concave family of functions is generated. G-concave functions are concave functions transformable by a continuous increasing function over a range.

Definition A5. *ρ-concavity ([1,3,50]). Consider $\rho > 0$, a non-negative function f, with convex support is called ρ-concave if and only if*

$$f\left(\sum_{k=1}^{K} \lambda_k v_k\right) \geq \left[\sum_{k=1}^{K} \lambda_k f(v_k)^\rho\right]^{\frac{1}{\rho}}, \forall \lambda \in \Lambda. \tag{A7}$$

The definition refers to f^ρ being concave for positive ρ. For negative ρ, $-f^\rho$ is concave as it is in the case considered here. The parameter ρ is a measure of the degree of concavity of the function. The definition of ρ-concavity is also obtained as a special case of (A6) by letting $h(.)$ be the identity function and $G(.) = [f(v)]^\rho$. Note that the standard definition of concavity is obtained when $\rho = 1$ and the mean value function takes the form of an arithmetic mean. The case of $\rho = 0$ corresponds to log-concavity to which a geometric mean is associated.

Furthermore, recall that the sum of concave functions is concave, but this property does not hold in general for quasi-concave functions. Consider the sum of f_1, a strictly increasing convex function, and f_2, a strictly decreasing convex function. This function is convex, but it is not in general quasi-concave, even if both f_1 and f_2 are quasi-concave. For differentiable functions, quasi-concavity is related to a property of monotonicity of the gradient that is not guaranteed by the sum of an increasing and decreasing function. Therefore, it is useful to define the following class of quasi-concave functions.

Definition A6. *Uniform Quasiconcavity ([63]). Two functions f_1 and f_2 are said to be uniformly quasi-concave if and only if*

$$\min\{f_i(x_1), f_i(x_2)\} = f_i(x_1) \ \ \forall i = 1,2, \ \forall x_1, \ x_2 \in \mathbb{R}^n$$

or

$$\min\{f_i(x_1), f_i(x_2)\} = f_i(x_2) \ \ \forall i = 1,2, \ \forall x_1, \ x_2 \in \mathbb{R}^n.$$

Note that the sum of uniformly quasi-concave functions is also quasi-concave, and the same holds for the product.

Fact A1 ([63])**.** *Given two functions f_1 and f_2 uniformly quasi-concave and non-negative, then their product is quasi-concave.*

Appendix A.2. Proofs

In this section, we collect the Proofs of Lemma 1 to Lemma 3 and Proposition 1.

Proof of Lemma 1. The property of Positive Association (PA) ensures that $\ln \overline{V}(V,p)$ must be increasing in each $V_k(p)$ and decreasing in each p_i. Hence,

$$\frac{\partial \ln \overline{V}(V,p)}{\partial p_i} = \sum_k \left[\frac{\partial a_k(p)}{\partial p_i} \ln V_k(p) + \frac{\partial \ln V_k(p)}{\partial p_i} a_k(p) \right] < 0. \tag{A8}$$

Grouping terms in Equation (A8), we can see that the size of the change in the average of individual welfares depends on the relative change in $\ln V_k$ with respect to the change in the weight $a_k(p)$ through a change in price p_i

$$\sum_k \left[\frac{\partial a_k(p)}{\partial p_i} \ln V_k(p)(1+\varepsilon_k) \right] \leq 0, \text{ with } \varepsilon_k = \frac{\partial \ln V_k(p)/\partial p_i}{\partial a_k(p)/\partial p_i} \frac{a_k(p)}{\ln V_k(p)} \leq 0.$$

□

Proof of Lemma 2. From Positive Association and Lemma 1, each function $a_k(p) \ln V_k(p)$ is decreasing and quasi-convex by Definition A2. Then, $\ln \overline{V}(V,p)$ is the sum of uniformly quasi-convex functions (see Definition A6) and hence, it is quasi-convex. □

Proof of Lemma 3. Let $\partial g(V,p)/\partial p_i = -(M(V,p)\partial\gamma(p)/\partial p_i + \gamma(p)\partial M(V,p)/\partial p_i)$ and

$$\begin{aligned} \frac{\partial \gamma(p)}{\partial p_i} &= \frac{1}{\rho}\left\{ (1-a_j(p)) \left[1 + \left(\frac{1-a_j(p)}{a_j(p)} \right)^{-(\rho+1)} \right] \right\}^{\frac{1}{\rho}-1} \cdot \\ & \cdot \left(-\frac{\partial a_j(p)}{\partial p_i} \right) \left[1 + \left(\frac{1-a_j(p)}{a_j(p)} \right)^{-(\rho+1)} \left(1 - \frac{\rho+1}{a_j(p)} \right) \right] \geq 0 \end{aligned} \tag{A9}$$

because $a_j(p)$ is non-decreasing. Then, the sign of the partial derivative of $g(V,p|\rho)$ depends also on the partial derivative of $M(V,p)$, which is $\frac{\partial M(V,p)}{\partial p_i} = \mu_i^1(V,p)\mu_i^2(V,p)$, with $\mu_i^1(V,p) = -\frac{1}{\rho}\left[\sum_k a_k(p)|\ln V_k(p) - \ln \overline{V}(V,p)|^{-\rho}\right]^{-\frac{1}{\rho}-1} \geq 0$ and

$$\mu_i^2(V,p) = \sum_k \left[\frac{\partial a_k}{\partial p_i} |\ln V_k(p) - \ln \overline{V}(V,p)|^{-\rho}\right] \left[1 - \frac{\rho}{\frac{\partial \ln a_k}{\partial p_i}} \frac{\partial \ln\left(\ln V_k - \ln \overline{V}(V,p)\right)}{\partial p_i}\right]. \quad \text{(A10)}$$

Because $\partial a_k(p)/\partial p_i \geq 0$, then $\frac{\partial a_k(p)}{\partial p_i} |\ln V_k(p) - \ln \overline{V}(V,p)|^{-\rho} \geq 0$ and $-\rho \Big/ \frac{\partial \ln a_k(p)}{\partial p_i} \geq 0$. Finally, $\partial \ln\left(\ln V_k(p) - \ln \overline{V}(V,p)\right)/\partial p_i \geq 0$ because of the MD property. Hence, we have $\mu_i^2(V,p) \geq 0$ and $\partial g(V,p)/\partial p_i \geq 0$, $\forall i = 1,\ldots,n$. Then, γ is quasi-concave because it is an increasing transformation of the quasi-concave function $a_j(p)$. The function M is also quasi-concave in prices because it is an increasing transformation of the sum of uniformly quasi-concave functions. In fact, $M(V,p) = \left[\sum_k a_k(p) |\ln V_k - \ln \overline{V}(V,p)|^{-\rho}\right]^{-\frac{1}{\rho}} = \left[\sum_k f_k(V,p)\right]^{-\frac{1}{\rho}}$, where $f_k(V,p) = a_k(p) |\ln V_k - \ln \overline{V}(V,p)|^{-\rho}$ is non-decreasing and quasi-concave in prices because of the MD property. □

Proof of Proposition 1. Lemma 3 states that the product $g(V,p) = -\gamma(p) M(V,p)$ is non-increasing and quasi-convex. Lemmas 1 and 2 ensure that $\ln \overline{V}(V,p)$ is decreasing and quasi-convex. Consequently, $\ln \overline{V}(V,p)$ and $g(V,p)$ are uniformly quasi-convex, and their sum is decreasing and quasi-convex with respect to p. □

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