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Liquidity Induced Asset Bubbles via Flows of ELMMs*

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Abstract. We consider a constructive model for asset price bubbles, where the market price W is endogenously determined by the trading activity on the market and the fundamental price W^F is exogenously given, as in [R. Jarrow, P. Protter, and A. Roch, *Quant. Finance*, 12 (2012), pp. 1339–1349]. To justify W^F from a fundamental point of view, we embed this constructive approach in the martingale theory of bubbles (see [R. Jarrow, P. Protter, and K. Shimbo, *Math. Finance*, 20 (2010), pp. 145–185] and [F. Biagini, H. Föllmer, and S. Nedelcu, *Finance Stoch.*, 18 (2014), pp. 297–326]) by showing the existence of a flow of equivalent martingale measures for W, under which W^F equals the expectation of the discounted future cash flow. As an application, we study bubble formation and evolution in a financial network.

Key words. bubbles, equivalent martingale measures, financial networks, liquidity-based model

AMS subject classifications. 91G80, 60H30, 60G99

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1. Introduction.

The formation of asset price bubbles has been thoroughly investigated from an economical point of view in many contributions; see Tirole [54], Allen and Gale [3], Choi and Douady [16], [15], Harrison and Kreps [26], Kaizoji [33], Earl, Peng, and Potts [21], DeLong et al. [20], Scheinkman [50], Scheinkman and Xiong [51], Xiong [59], Abreu and Brunnermeier [1], Föllmer, Horst, and Kirman [23], Miller [39], and Zhuk [60]. Different causes have been indicated as triggering factors for bubble birth, such as heterogeneous beliefs between interacting agents (as in [23], [26], [50], [51], [59], [60]), a breakdown of the dynamic stability of the financial system [16], [15], the diffusion of new investment decision rules from a few expert investors to a larger population of amateurs (see [21]), the tendency of traders to choose the same behavior as the other traders' behavior as thoroughly as possible (see [33]), and the presence of short-selling constraints (see [39]).

From the mathematical point of view, one of the main approaches is given by the martingale theory of bubbles as introduced by Cox and Hobson [18] and Loewenstein and Willard [34] and mainly developed by Jarrow and Protter and colleagues [28], [29], [31], [32], [27]. See Protter [48] for an overview. In this setting a Q-bubble is defined as the difference between the market price of a given financial asset and its fundamental value, given by the expectation of the future cash flows under an equivalent local martingale measure Q.

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Defined in this way, the bubble is a nonnegative local martingale under Q, and it is strictly positive if and only if the market wealth W is a strict Q-local martingale (for a complete analysis, see, for example, [10], [18], [31], [32], [34], [48]).

In a complete market (see [31]), where only one equivalent local martingale measure (ELMM) exists, only two possibilities are given: either no bubble appears at all, or a bubble is already present at the beginning. In [32] and [10] incomplete markets have been taken into consideration: the birth and the evolution of a bubble are then determined by a flow of different ELMMs that gives rise to a corresponding shifting perception of the fundamental value of the asset. In [32] the underlying pricing measures may change only at certain stopping times; in [10] a continuous flow in the space of martingale measures is considered.

On the other hand, an alternative model is given by Jarrow, Protter, and Roch in [30], where the fundamental value is exogenously given, whereas the market value is endogenously determined by the trading activity of investors and studied through the analysis of the liquidity supply curve. For another constructive model, see also [11].

In this setting a bubble is still defined as the difference between the market value W and the fundamental value W^F; however, it does not always coincide with the Q-bubble under a given equivalent martingale measure Q.

A natural question then is whether it is possible to embed a constructive model, where the fundamental price is exogenous and the market price endogenous, in the martingale theory of bubbles by determining a suitable flow of ELMMs for W under which W^F is justified from a fundamental point of view.

More precisely, given a liquidation time T for the financial asset, we look for a flow $(Q^t)_{t\in[0,T)}$ of ELMMs for the market wealth W such that the fundamental value of the asset is given as the expectation of the future cash flow as in (3.3). Note, however, that we do not obtain that W^F is also a (local) martingale under each measure of the flow, as thoroughly discussed in Remark 3.1.

Then our main result is that we can explicitly determine the form of such a flow of ELMMs in a liquidity-driven model under very general assumptions; see Theorem 3.16. This requires a consistent technical effort, mostly devoted to guaranteeing the martingale property of the chosen flows of (eventual) probability densities. In this way we are able to directly connect the impact of the underlying macroeconomic factors to the shift of the resulting pricing measure, which may change over time.

As an application of our method, we consider the evolution of a bubble in a financial network and compute the generating flow of ELMMs. However, this example is also of independent interest, as it studies how the interaction of market participants in a financial network can affect asset price formation and the consequent birth of a bubble. Different studies show how contagion between investors and herding behavior may play an essential role when a bubble grows up: euphoria and exuberance can propagate among market participants, due to exchanges of ideas (see Lux [35]) or to the fact that investors may be attracted by the short period earnings of acquaintances investing in the bubbly asset, as observed by Bayer, Mangum, and Roberts in [8], where, analyzing data from the housing bubble in Los Angeles in the 2000s, the authors noticed a strong contagion between neighbors.

Several contributions in recent years have focused on how some properties of the network, such as mean degree or degree heterogeneity, can influence the contagion of failures and

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losses between banks during a financial crisis (see, for example, Acemoglu et al. [2], Allen and Gale [4], Amini, Cont, and Minca [5], Cont, Moussa, and Santos [17], Gai and Kapadia [24], Newman, Strogatz, and Watts [41], Watts [56], and Watts and Strogatz [57]). Some investigation has been proposed about how bubbles are generated at the microeconomic level by the interaction of market participants (see, among others, Lux [35], Scheinkman [50], Scheinkman and Xiong [51], Tirole [54], and Zhuk [60]). However, only a few studies have been devoted to understanding how the structure of a given financial network can influence the spread of contagion between investors that generates a bubble. In [35], for example, the author models the bubble as caused by a self-organizing process of infection between traders, expressed by a system of PDEs, leading to equilibrium prices that deviate from the fundamental value. However, they consider a world in which everybody is connected with everybody, so that the network structure does not enter into play.

In our special case we focus on a model for the signed volume of market orders of X dependent on some characteristics of the underlying networks of investors, such as the degree distribution. In particular we use a modeling approach derived from the literature on infectious processes in a population by following the so-called Susceptible-Infected-Susceptible (SIS) model (see Pastor-Satorras and Vespignani [45], [46]). We provide numerical simulations to investigate how different networks generate different contagion mechanisms and then to bubbles with different evolutions. In particular, it turns out that in more heterogeneous networks (i.e., networks with a more right skewed degree distribution) contagion spreads faster at the beginning, so that the bubble builds up faster and bursts sooner: the nodes with high degree, which on average get infected faster, contribute with a higher weight in the more right skewed distributions.

The paper is therefore organized as follows: in section 2 we describe the setting of the liquidity model, define the fundamental value of the asset, and specify how the trading activity of investors influences the market price of the asset. In section 3 we determine a possible flow $(Q^t)_{t\in[0,T)}$ of ELMMs satisfying (3.3) and show that the density process $(Z_{t,s})_{s\in[0,T)}$ with $Z_{t,s} = \frac{dQ^t}{dP}|_{\mathcal{F}_s}$ is a true martingale with respect to s. In section 4 we give an example showing how contagion between investors can develop the bubble in a network and compute the generating flow of ELMMs.

2. The setting.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T > 0$ a random time on it, representing the maturity or liquidation time of the underlying risky asset as in the setting of [32]. We assume that $(\Omega, \mathcal{F}, P)$ is endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual assumptions of completeness and right continuity.

On $(\Omega, \mathcal{F}, \mathbb{F}, P)$ we have (B^1, B^2, B^3, B^4, N), where $B^i = (B^i_t)_{t\in[0,T]}$, $i = 1, 2, 3, 4$, are standard $\mathbb{F}$-Brownian motions and $N_t = \mathbb{1}_{\{\tau\le t\}}$ is a jump process with τ totally inaccessible stopping time with intensity process $\pi = (\pi_t)_{t\in[0,T]}$. We assume that (B^1, B^2, B^3, B^4, N) are independent processes.

We follow the approach of [30] and study how trading activity may impact prices and generate the formation and bursting of speculative asset price bubbles. We consider a continuous time setting where a stock is traded through a limit order book. The *fundamental wealth or value* of the stock is given as a primitive of the model and represents the price process if market orders have no quantity impact on the price. The *market wealth* equals the fun-

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damental value until market orders are executed. Market orders, which deplete or fill in the limit order book, produce a variation in the price over a small interval of time. If new market orders quickly enter before the price has time to decay again to the fundamental value, these short-term price variations may accumulate and result in a deviation from the fundamental wealth with a consequent bubble birth.

More specifically, we consider a financial asset whose fundamental wealth $W^F = (W_t^F)_{t\in[0,T]}$ (associated to the cumulative dividend process $(D_t)_{t\in[0,T]}$ and to the liquidation value F of the asset at time T) is given by

$$dW_t^F = W_t^F(adt + bdB_t^1), \quad 0 \le t \le T, \tag{2.1}$$

with $W_0^F > 0$, $a \ge 0$, and $b > 0$.

We interpret τ as the time of birth of a bubble for this financial asset. The bubble follows the dynamics

$$d\beta_t = M_t\Lambda_t(-k\beta_t dt + 2dX_t + 2xW_t^F dN_t), \quad 0 \le t < T, \tag{2.2}$$

where X is the signed volume of market orders (buy market orders minus sell market orders), x is the signed volume of market orders at τ, and $M = (M_t)_{t\in[0,T]}$, $\Lambda = (\Lambda_t)_{t\in[0,T]}$ are, respectively, a measure of illiquidity and the so-called *resiliency*. We put $\beta_\tau = 2x\Lambda_\tau M_\tau W_\tau^F$ for a given $x > 0$.

Remark 2.1. As in [30], we assume that the supply curve for the stock is linear, i.e., that the variation induced by a market order of size y is proportional to y via the stochastic coefficients M and Λ. Here M quantifies the illiquidity level, while the resiliency coefficient Λ represents the degree to which the limit order book recovers from small trades. In this way $\frac{1}{2M_t\Lambda_t}$ gives the depth of the order book at time t, i.e., the size of the order required to change the price of an asset by one unit. This linearity assumption is better justified in the case of frequently traded and large volume stocks; see [12]. For less liquid companies, statistical analysis (see, for example, Cetin et al. [14]) shows that the supply is at best piecewise linear. For more details about the economical motivation of this setting, we refer the reader to [30].

We consider that X satisfies the following dynamics:

$$\begin{aligned} X_t &= 0 && \text{for } 0 \le t < \tau, \\ dX_t &= \mu_t dt + \sigma_t dB_t^2 && \text{for } \tau \le t < T, \end{aligned} \tag{2.3}$$

where $\mu = (\mu_t)_{t\in[0,T]}$ and $\sigma = (\sigma_t)_{t\in[0,T]}$ are progressively measurable processes that a priori can also depend on X itself or on the bubble β.

In [30] the signed volume of market orders is modeled as in (2.3) with $\mu \equiv 0$ and $\sigma_t = \alpha\beta_t$. Here we introduce the drift μ in order to see the influence of the network on the size of the bubble, as we specify in section 4.

Here the fundamental wealth process W^F is exogenously given, while the market wealth process $W = (W_t)_{t\in[0,T]}$ is endogenously determined as

$$W_t = W_t^F + \beta_t, \quad 0 \le t < T.$$

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At liquidation time T we have $W_T = W_T^F$: the asset is liquidated at time T at the estimated firm's value, i.e., at the fundamental value. In particular we require in what follows that there exists an equivalent local martingale measure for W only on the open interval $[0, T)$, since around time T the liquidation procedure is not subjected to market equilibrium mechanisms.

Assumption 2.2.
(i) $\int_\tau^T \mu_s^2 ds < \infty$ *a.s.*
(ii) $\int_\tau^T \sigma_s^2 ds < \infty$ *a.s., and* $\int_\tau^T \frac{1}{\sigma_s^4} ds < \infty$ *a.s.*
(iii) μ *and* σ *are such that there exists a unique solution of* (2.3) *(see, e.g., Theorem* 7 *in Chapter* V.3 *in* [47]*).*
(iv) $M = (M_t)_{t\in[0,T]}$ *is an adapted process that satisfies the dynamics*

$$dM_t = \tilde{\mu}(M_t)dt + \tilde{\sigma}(M_t)dB_t^3, \quad 0 \le t \le T,$$

where $\tilde{\mu}$ *and* $\tilde{\sigma}$ *are such that there exists a unique solution of* (2.4) *according to Theorem* 7 *in Chapter* V.3 *in* [47]*. Moreover,* $\int_a^b \tilde{\sigma}^{-4}(x)dx < \infty$ *for every* a, b *such that* $0 < a < b < \infty$.
(v) $\Lambda = (\Lambda_t)_{t\in[0,T]}$ *satisfies the dynamics*

$$d\Lambda_t = \mu'(\Lambda_t)dt + \sigma'(\Lambda_t)dB_t^4, \quad 0 \le t \le T,$$

$\Lambda_0 \in (\lambda, 1)$*, with* μ'*,* σ' *that satisfy the conditions in Theorem* 7 *in Chapter* V.3 *in* [47]*. Furthermore,* $\mu'(\lambda) > 0$*,* $\mu'(1) < 0$*,* $\sigma'(1) = 0$*,* $\sigma'(\lambda) = 0$ *a.s., so that we obtain* $\lambda \le \Lambda_t \le 1$ *a.s. for all* $t \in [0, T]$.
(vi) $\pi = (\pi_t)_{t\in[0,T]}$ *is bounded, i.e.,* $|\pi_t| \le \Pi < \infty$ *a.s. for all* $t \in [0, T]$.
(vii) T *is a bounded a.s. (possibly by a very large constant)* $\mathbb{F}$*-stopping time independent of* (B^1, B^2, N) *such that* $\tau < T$ *a.s.*

Notice that we assume $\tau < T$ and T bounded a.s. for the sake of simplicity. The following results still hold without these conditions by imposing some integrability conditions on T. For example, it would be sufficient to have $T < \infty$ a.s., $\mathbb{E}_P[e^T|\mathcal{F}_t] < \infty$, and $\mathbb{E}_P[T - \tau|\mathcal{F}_t] > 0$ a.s. for $t \in [0, T]$.

Remark 2.3. Here we exclude that σ can depend on β. However, the following results also hold for the case $\sigma_t = \alpha\beta_t$, $t \in [\tau, T]$, $\alpha \in \mathbb{R}$, considered in [30] to model the evolution of the bubble given by illiquidity effects. We refer the reader to [36] for more details in this case.

Proposition 2.4. *From the hypothesis on* M *it follows that* $\int_0^T M_s^\alpha ds < \infty$ *a.s. for all* $\alpha \in \mathbb{R}$.

Proof. Following the same argument as in [37], we have that

$$\int_0^T M_s^\alpha ds = \int_0^T \frac{M_s^\alpha}{\tilde{\sigma}^2(M_s)} d[M, M]_s = \int_0^\infty \frac{x^\alpha}{\tilde{\sigma}^2(x)} L_T^x dx, \tag{2.4}$$

where L_T^x is the local time at T and the last equality follows by the occupation time formula (see, for example, Corollary 1 in Chapter IV of [47]).

Then the integral is finite since, by the fact that $0 < M_s < \infty$ a.s. for each $s \in [0, T]$, we have that the occupation time L_T^a has compact support in $(0, \infty)$. ∎

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From Remark 2.3 we have that β satisfies the SDE

$$d\beta_t = 2\Lambda_t M_t \left[(-k\beta_t + \mu_t)dt + \sigma_t dB_t^2 + xW_t^F dN_t\right], \quad \tau \le t < T.$$

The bubble therefore takes the following explicit expression:

$$\beta_t = \beta_\tau e^{-k\int_\tau^t \Lambda_s M_s ds} + \int_\tau^t \mu_s \Lambda_s M_s e^{-k\int_s^t \Lambda_u M_u du} ds$$
$$+ \int_\tau^t \sigma_s \Lambda_s M_s e^{-k\int_s^t \Lambda_u M_u du} dB_s^2, \quad \tau \le t < T. \tag{2.5}$$

3. Flow of equivalent local martingale measures.

Let $\mathcal{M}_{loc}(W)$ be the space of equivalent local martingale measures for $W = (W_t)_{t\in[0,T)}$. Given $Q \in \mathcal{M}_{loc}(W)$, a Q-bubble β^Q is defined as

$$\beta_t^Q = W_t - \mathbb{E}_Q[W_T|\mathcal{F}_t] \tag{3.1}$$

in the approach of [31] and [32]. In particular, we have that the bubble introduced in (2.2) coincides with a Q-bubble if and only if

$$W_t^F = \mathbb{E}_Q[W_T|\mathcal{F}_t], \quad t \in [0,T), \tag{3.2}$$

for some $Q \in \mathcal{M}_{loc}(W)$.

This is of course not possible in our setting. However, we can find a flow $(Q^t)_{t\in[0,T)} \subseteq \mathcal{M}_{loc}(W)$ such that

$$W_t^F = \mathbb{E}_{Q^t}[W_T|\mathcal{F}_t] = \mathbb{E}_{Q^t}[W_T^F|\mathcal{F}_t]. \tag{3.3}$$

In this way the bubble described in (2.2) is the result of the shift in the pricing measure induced by the change in the macroeconomic and financial conditions in the market.

Remark 3.1. We wish to point out the relation between our constructive approach and the martingale theory of bubbles described in [31], [32], and [10]. In our setting as well as in the approach of [30], the bubble β is defined as

$$\beta_t = W_t - W_t^F, \tag{3.4}$$

where W^F is a primitive of the model. According to the martingale theory of bubbles as illustrated in [31] and [32], the market wealth W is given a priori and for a given $\mathbb{Q} \in \mathcal{M}_{loc}(W)$ the Q-bubble process β^Q is defined as in (3.1), which also implies that β^Q is nonnegative. The two definitions coincide if the fundamental wealth process W^F in (3.4) is also a (local) $\mathbb{Q}$-martingale for $\mathbb{Q} \in \mathcal{M}_{loc}(W)$, i.e., if (3.2) holds; otherwise they differ.

In our setting as well as in [30, sect. 5], we have that $\mathcal{M}_{loc}(W) \cap \mathcal{M}_{loc}(W^F) = \emptyset$, so the bubble process cannot be a local martingale under any equivalent local martingale measure $\mathbb{Q} \in \mathcal{M}_{loc}(W)$ for the wealth process W and may also assume negative values. Hence the appearance of negative bubbles is not in contrast with arbitrage theory in our approach.

However, while in the martingale approach the model is automatically arbitrage-free because $\mathcal{M}_{loc}(W) \neq \emptyset$ is assumed a priori, in our "constructive" model for bubbles we need

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to explicitly exclude arbitrage possibilities. Since in Theorem 3.16 we show the existence of a flow $(Q^t)_{t\in[0,T)} \subseteq \mathcal{M}_{loc}(W)$, i.e., that $\mathcal{M}_{loc}(W) \neq \emptyset$, we obtain that our market model is arbitrage-free; see also Remark 3.17.

It is then a challenging question whether our constructive model can be included in the more fundamental view of the martingale theory of bubbles of [31] and [32] by following [10]. For this purpose we investigate the existence of a flow $(Q^t)_{t\in[0,T)} \subseteq \mathcal{M}_{loc}(W)$ which can "fundamentally explain" the a priori given fundamental wealth, i.e., such that (3.3) holds. This is not in contrast with our comments above since now the measure Q^t is not fixed over the entire interval $[0,T)$ but may change in time. In fact, (3.3) does not imply that W^F is a martingale under Q^t over the interval $[0,T)$ because (3.3) holds t-wise, and in general it is not true that

$$W_s^F = \mathbb{E}_{Q^t}[W_T|\mathcal{F}_s]$$

for $s \neq t$, $s,t \in [0,T)$.

We now explicitly compute a flow $(Q^t)_{t\in[0,T)} \in \mathcal{M}_{loc}(W)$ justifying the existence of the bubble in (2.2) from a fundamental point of view.

Let $Q \in \mathcal{M}_{loc}(W)$. Then the density process $Z = (Z_t)_{t\in[0,T)}$ of Q with respect to P is given by

$$Z_t = \frac{dQ}{dP}|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^\cdot \alpha_s^1 dB_s^1 + \int_0^\cdot \alpha_s^2 dB_s^2 + \int_0^\cdot \alpha_s^3 d\tilde{N}_s + \int_0^\cdot \alpha_s^4 dB_s^3 + \int_0^\cdot \alpha_s^5 dB_s^4 + L_t\right)_t,$$

$0 \le t < T$, where $\tilde{N}_t = N_t - \int_0^{t\wedge\tau} n_s ds$, $t \in [0,T)$, L is a martingale strongly orthogonal to (B^1,B^2,B^3,B^4,N), and the processes α^i, $i = 1,\dots,5$, are such that for $0 \le s < T$ the following equality holds:

$$W_s^F(a + b\alpha_s^1) + 2\Lambda_s M_s\left(\mu_s + \sigma_s\alpha_s^2 - k\beta_s\right)\mathbb{1}_{\{s\ge\tau\}} + 2\pi_s x W_s^F \Lambda_s M_s(\alpha_s^3+1)\mathbb{1}_{\{s<\tau\}} = 0. \tag{3.5}$$

Since (3.5) does not involve α^4, α^5, or L, we put $\alpha^4 \equiv \alpha^5 \equiv L \equiv 0$.

We can split (3.5) as

$$b\alpha_s^1 = -a - 2\pi_s x \Lambda_s M_s(\alpha_s^3+1) \quad \text{for } s < \tau \tag{3.6}$$

and

$$b\alpha_s^1 = -a + \frac{2\Lambda_s M_s}{W_s^F}\left(k\beta_s - \mu_s - \sigma_s\alpha_s^2\right) \quad \text{for } s \ge \tau. \tag{3.7}$$

We look for a flow of the form

$$Z_{t,s} = \frac{dQ^t}{dP}|_{\mathcal{F}_s} = \mathcal{E}\left(\int_0^\cdot \alpha_u^{t,1} dB_u^1 + \int_0^\cdot \alpha_u^{t,2} dB_u^2 + \int_0^\cdot \alpha_u^{t,3} d\tilde{N}_u\right)_s, \quad s \in [0,T), \tag{3.8}$$

since (3.5) does not involve conditions on $\alpha^{t,4}$, $\alpha^{t,5}$, and $\alpha^{t,6}$. In particular, we note that the laws of M, Λ, and T are invariant under this change of measure.

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If $\alpha^{t,1}$, $\alpha^{t,2}$, and $\alpha^{t,3}$ satisfy (3.6) and (3.7), the fundamental process under Q^t is given by

$$\frac{dW_s^F}{W_s^F} = \tilde{\mu}_s^t ds + b d\tilde{B}_s^t, \quad 0 \le s \le T, \tag{3.9}$$

where $\tilde{B}^t$ denotes the Q^t-standard Brownian motion given by

$$\tilde{B}_s^t = B_s^1 - \int_0^s \alpha_u^{t,1} du, \quad 0 \le s \le T,$$

and

$$\tilde{\mu}_s^t = \begin{cases} -2\pi_s x \Lambda_s M_s(\alpha_s^{t,3} + 1) & \text{for } s < \tau, \\ \frac{2\Lambda_s M_s}{W_s^F}\left(k\beta_s - \mu_s - \sigma_s \alpha_s^{t,2}\right) & \text{for } s \ge \tau. \end{cases} \tag{3.10}$$

If the condition

$$\mathbb{E}_{Q^t}\left[\int_t^T (W_s^F)^2 ds\right] < \infty \tag{3.11}$$

is satisfied, we have that (3.3) is equivalent to

$$\mathbb{E}_{Q^t}\left[\int_t^T W_s^F \tilde{\mu}_s^t ds \Big| \mathcal{F}_t\right] = 0;$$

that is,

$$0 = \mathbb{E}_{Q^t}\left[\int_t^\tau W_s^F \pi_s x \Lambda_s M_s(\alpha_s^{t,3} + 1) ds + \int_\tau^T \Lambda_s M_s \left(k\beta_s - \mu_s - \sigma_s \alpha_s^{t,2}\right) ds \Big| \mathcal{F}_t\right] \tag{3.12}$$

for $t < \tau$, and

$$\mathbb{E}_{Q^t}\left[\int_t^T \Lambda_s M_s \left(k\beta_s - \mu_s - \sigma_s \alpha_s^{t,2}\right) ds \Big| \mathcal{F}_t\right] = 0 \tag{3.13}$$

for $t \ge \tau$.

To show the existence of the flow $(Q^t)_{t\in[0,T)} \subseteq \mathcal{M}_{loc}(W)$, we choose $\alpha^{t,2}$ and $\alpha^{t,3}$ so that the integrals inside the conditional expectation in (3.12) and (3.13) are zero a.s. We show later that a posteriori this choice ensures as well that (3.11) holds.

For $t \ge \tau$, let

$$\alpha_s^{t,2} = \frac{1}{\Lambda_s M_s \sigma_s}\left(s - \frac{\mathbb{E}[T|\mathcal{F}_t] + t}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t] - \mathbb{E}[T^2|\mathcal{F}_t]}{2(\mathbb{E}[T|\mathcal{F}_t] - t)}\right) + \frac{k\beta_s}{\sigma_s} - \frac{\mu_s}{\sigma_s}, \quad t \le s < T.$$

Notice that such an $\alpha_s^{t,2}$ is well defined since from Assumption 2.2 it holds that $\Lambda_s > 0$, $M_s > 0$, $\sigma_s > 0$ a.s. for every $s \in [0, T]$.

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With this choice we have on $\{T > t\}$ that

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^t}\left[\int_t^T \Lambda_s M_s\left(k\beta_s - \mu_s - \sigma_s\alpha_s^{t,2}\right)ds\Big|\mathcal{F}_t\right]\\
=&\mathbb{E}_{\mathbb{Q}^t}\left[\int_t^T \left(s - \frac{\mathbb{E}[T|\mathcal{F}_t]+t}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t]-\mathbb{E}[T^2|\mathcal{F}_t]}{2(\mathbb{E}[T|\mathcal{F}_t]-t)}\right)ds\Big|\mathcal{F}_t\right]\\
=&\mathbb{E}_{\mathbb{Q}^t}\left[\left(\frac{T^2-t^2}{2} - (T-t)\frac{\mathbb{E}[T|\mathcal{F}_t]+t}{2} + (T-t)\frac{\mathbb{E}^2[T|\mathcal{F}_t]-\mathbb{E}[T^2|\mathcal{F}_t]}{2(\mathbb{E}[T|\mathcal{F}_t]-t)}\right)\Big|\mathcal{F}_t\right]\\
=&\frac{\mathbb{E}[T^2|\mathcal{F}_t]-t^2}{2} - \frac{(\mathbb{E}[T|\mathcal{F}_t]-t)(\mathbb{E}[T|\mathcal{F}_t]+t)}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t]-\mathbb{E}[T^2|\mathcal{F}_t]}{2} = 0, \qquad (3.14)
\end{aligned}
$$

since by Assumption 2.2 the law of T does not change under Q^t.

For $t < \tau$ define

$$C_{t,\tau} := \int_t^\tau W_s^F \pi_s x \Lambda_s M_s(\alpha_s^{t,3}+1)ds$$

and choose $\alpha_s^{t,2}$ to be such that

$$\mathbb{E}_{Q^t}\left[\int_\tau^T \Lambda_s M_s\left(k\beta_s - \mu_s - \sigma_s\alpha_s^{t,2}\right)ds\Big|\mathcal{F}_t\right] = -\mathbb{E}_{Q^t}\left[C_{t,\tau}|\mathcal{F}_t\right],$$

i.e.,

$$
\begin{aligned}
\alpha_s^{t,2} =&\frac{1}{\Lambda_s M_s \sigma_s}\left(s - \frac{\mathbb{E}_{Q^t}[C_{t,\tau}|\mathcal{F}_t]}{\mathbb{E}[T-\tau|\mathcal{F}_t]} - \frac{\mathbb{E}[T+\tau|\mathcal{F}_t]}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t]-\mathbb{E}[T^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau|\mathcal{F}_t]} - \frac{\mathbb{E}^2[\tau|\mathcal{F}_t]-\mathbb{E}[\tau^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau|\mathcal{F}_t]}\right)\\
&+\frac{k\beta_s}{\sigma_s} - \frac{\mu_s}{\sigma_s}, \quad t \le s < T,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^t}\left[\int_\tau^T \Lambda_s M_s\left(k\beta_s - \mu_s - \sigma_s\alpha_s^{t,2}\right)ds\Big|\mathcal{F}_t\right]\\
=&\mathbb{E}_{\mathbb{Q}^t}\left[\int_\tau^T\left(s - \frac{\mathbb{E}_{Q^t}[C_{t,\tau}|\mathcal{F}_t]}{\mathbb{E}[T-\tau|\mathcal{F}_t]} - \frac{\mathbb{E}[T+\tau|\mathcal{F}_t]}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t]-\mathbb{E}[T^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau|\mathcal{F}_t]} - \frac{\mathbb{E}^2[\tau|\mathcal{F}_t]-\mathbb{E}[\tau^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau|\mathcal{F}_t]}\right)ds\Big|\mathcal{F}_t\right]\\
=&\frac{\mathbb{E}[T^2-\tau^2|\mathcal{F}_t]}{2} - \mathbb{E}_{Q^t}[C_{t,\tau}|\mathcal{F}_t] - \frac{\mathbb{E}[T-\tau|\mathcal{F}_t]\mathbb{E}[T+\tau|\mathcal{F}_t]}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t]-\mathbb{E}[T^2|\mathcal{F}_t]}{2}\\
&- \frac{\mathbb{E}^2[\tau|\mathcal{F}_t]-\mathbb{E}[\tau^2|\mathcal{F}_t]}{2}\\
=&-\mathbb{E}_{Q^t}[C_{t,\tau}|\mathcal{F}_t],
\end{aligned}
$$

and then (3.12) holds.

For $s < t \vee \tau$ we set $\alpha_s^{t,2} = 0$.

Summarizing,

$$\alpha_s^{t,2} = \begin{cases} 0 & \text{for } s < \tau \vee t, \\ \frac{1}{\Lambda_s M_s \sigma_s}\left(s - \eta_{t,\tau}\right) + \frac{k\beta_s}{\sigma_s} - \frac{\mu_s}{\sigma_s} & \text{for } s \ge \tau \vee t, \end{cases} \qquad (3.15)$$

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where

$$\eta_{t,\tau} = \frac{\mathbb{E}_{Q^t}[\int_{t\wedge\tau}^{\tau} W_s^F \pi_s x \Lambda_s M_s(\alpha_s^{t,3}+1)ds|\mathcal{F}_t]}{2\mathbb{E}[T-\tau\vee t|\mathcal{F}_t]} - \frac{\mathbb{E}[T+\tau\vee t|\mathcal{F}_t]}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t]-\mathbb{E}[T^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau\vee t|\mathcal{F}_t]}$$
$$- \frac{\mathbb{E}^2[\tau\vee t|\mathcal{F}_t]-\mathbb{E}[(\tau\vee t)^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau\vee t|\mathcal{F}_t]}. \tag{3.16}$$

Remark 3.2. Notice that from Assumption 2.2 and from the fact that the integral in (3.16) is bounded, we have that $\eta_{t,\tau}$ is finite and $\mathcal{F}_t$-measurable and that, moreover, $\mathbb{E}[\eta_{t,\tau}^{\alpha}] < \infty$ for all $\alpha \in \mathbb{R}$.

Choosing

$$\alpha_s^{t,3} = \begin{cases} 0 & \text{for } s < t \text{ or } s \geq \tau, \\ \frac{1}{(M_s+1)(W_s^F+1)} - 1 & \text{for } t \leq s < \tau \end{cases} \tag{3.17}$$

and

$$\alpha_s^{t,1} = \begin{cases} 0 & \text{for } s < t, \\ -\frac{a}{b} - \frac{2}{b}\pi_s\Lambda_s\frac{M_s}{M_s+1}\frac{1}{W_s^F+1} & \text{for } t \leq s < \tau, \\ -\frac{a}{b} - \frac{2}{bW_s^F}(s-\eta_{t,\tau}) & \text{for } s \geq \tau \vee t, \end{cases} \tag{3.18}$$

we have that (3.12) and (3.13) hold.

Now we give the following.

Proposition 3.3. *Let $\alpha^{t,1}$, $\alpha^{t,2}$, and $\alpha^{t,3}$ be as defined in* (3.15)–(3.18). *Then*

$$\mathbb{E}_{Q^t}\left[\int_t^T (W_s^F)^2 ds\right] < \infty, \quad t \in [0,T].$$

Proof. From (3.10) and from the expressions of $\alpha^{t,1}$, $\alpha^{t,2}$, and $\alpha^{t,3}$ in (3.15)–(3.18) we have that

$$\tilde{\mu}_s^t = \begin{cases} -2\pi_s x\Lambda_s\frac{M_s}{M_s+1}\frac{1}{W_s^F+1} & \text{for } s < \tau, \\ \frac{1}{W_s^F}(\eta_{t,\tau}-s) & \text{for } s \geq \tau, \end{cases}$$

where $\eta_{t,\tau}$ is given in (3.16). Then from (3.9) it holds that under Q^t

$$dW_s^F = \psi_s ds + bW_s^F d\tilde{B}_s^t \qquad \text{for } s < \tau,$$
$$dW_s^F = (\eta_{t,\tau}-s)ds + bW_s^F d\tilde{B}_s^t \qquad \text{for } s \geq \tau,$$

where $\psi_s = -2\pi_s x\Lambda_s\frac{M_s}{M_s+1}\frac{1}{W_s^F+1}$.

Thus we have

$$W_s^F = \begin{cases} e^{b\tilde{B}_s^t - \frac{b^2}{2}s}\int_0^s \psi_u e^{-b\tilde{B}_u^t+\frac{b^2}{2}u}du & \text{for } s < \tau, \\ e^{b\tilde{B}_s^t - \frac{b^2}{2}s}\int_0^s (\eta_{t,\tau}-u) e^{-b\tilde{B}_u^t+\frac{b^2}{2}u}du & \text{for } s \geq \tau. \end{cases}$$

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Then

$$
\begin{aligned}
&\mathbb{E}_{Q^t}\left[\int_t^T (W_s^F)^2 ds\right]\\
&= \mathbb{E}_{Q^t}\left[\int_{t\wedge\tau}^{\tau}\left(\int_0^s \psi_u e^{b(\tilde B_s^t-\tilde B_u^t)-\frac{b^2}{2}(s-u)}du\right)^2 ds + \int_\tau^T\left(\int_0^s (\eta_{t,\tau}-u)e^{b(\tilde B_s^t-\tilde B_u^t)-\frac{b^2}{2}(s-u)}du\right)^2 ds\right]\\
&\le \mathbb{E}_{Q^t}\Bigg[4\Pi^2x^2\int_{t\wedge\tau}^{\tau}\left(\int_0^s e^{b(\tilde B_s^t-\tilde B_u^t)-\frac{b^2}{2}(s-u)}du\right)^2 ds + (|\eta_{t,\tau}|+T)^2\\
&\qquad\cdot\int_\tau^T\left(\int_0^s e^{b(\tilde B_s^t-\tilde B_u^t)-\frac{b^2}{2}(s-u)}du\right)^2 ds\Bigg]\\
&\le \left(4\Pi^2x^2+\mathbb{E}\left[(|\eta_{t,\tau}|+T)^4\right]^{\frac12}\right)\mathbb{E}_{Q^t}\left[\int_{t\wedge\tau}^T\left(\int_0^s e^{b(\tilde B_s^t-\tilde B_u^t)-\frac{b^2}{2}(s-u)}du\right)^4 ds\right]^{\frac12}.
\end{aligned}
$$

Since T is bounded and the first term is finite by Remark 3.2, it remains to prove

$$
\mathbb{E}_{Q^t}\left[\int_{t\wedge\tau}^T\left(\int_0^s e^{b(\tilde B_s^t-\tilde B_u^t)-\frac{b^2}{2}(s-u)}du\right)^4 ds\right]<\infty. \tag{3.19}
$$

We have that

$$
\begin{aligned}
&\mathbb{E}_{Q^t}\left[\int_{t\wedge\tau}^T\left(\int_0^s e^{b(\tilde B_s^t-\tilde B_u^t)-\frac{b^2}{2}(s-u)}du\right)^4 ds\right] = \mathbb{E}_{Q^t}\left[\int_{t\wedge\tau}^T\left(\int_0^s e^{b\tilde B_{s-u}^t-\frac{b^2}{2}(s-u)}du\right)^4 ds\right]\\
&= \mathbb{E}_{Q^t}\left[\int_{t\wedge\tau}^T\left(\int_0^s e^{b\tilde B_r^t-\frac{b^2}{2}r}dr\right)^4 ds\right] \le \mathbb{E}_{\mathbb{Q}^t}\left[(T-t\wedge\tau)^2\right]^{\frac12}\mathbb{E}_{Q^t}\left[\left(\int_0^T e^{b\tilde B_r^t-\frac{b^2}{2}r}dr\right)^8\right]^{\frac12}.
\end{aligned}
$$

The first term is finite by Assumption 2.2 on T and τ, whereas

$$
\mathbb{E}_{Q^t}\left[\left(\int_0^T e^{b\tilde B_r^t-\frac{b^2}{2}r}dr\right)^8\right] \le \mathbb{E}_{Q^t}\left[\int_0^T e^{8b\tilde B_r^t-4b^2r}dr\right] = \int_0^T \mathbb{E}_{Q^t}\left[e^{8b\tilde B_r^t-4b^2r}\right]dr<\infty.
$$

Then (3.19) holds and we have the result. ∎

We have therefore proved that if we take $\alpha^{t,1}$, $\alpha^{t,2}$, and $\alpha^{t,3}$ as defined in (3.15)–(3.18), then (3.6), (3.7), and (3.3) are satisfied.

From now on we denote $Z_{t,s} := \frac{dQ^t}{dP}|_{\mathcal{F}_s}$ for all $s\ge t$, and $Z_{t,s}=1$ for $s<t$.

Note that we have not yet used the hypothesis on μ and σ of Assumption 2.2 to derive (3.8). From now on we will need them to prove that $(Z_{t,s})_{s\in[t,T)}$ is a true martingale for each $t\in[0,T)$, i.e., that each Q^t, $t\in[0,T)$, in (3.8) belongs to $\in\mathcal{M}_{loc}(W)$.

Remark 3.4. By Assumption 2.2, as proved in Proposition 2.4, we exclude that the integral $\int_0^\cdot M_s^2ds$ can explode in finite time. This is a difference with respect to [30], where the bubble bursts (i.e., $\beta_t=0$) at $\inf\{s|\int_0^s M_u^2du=+\infty\}$.

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In our model, however, the bubble can be zero, and also negative, even if the liquidity is not zero: by (2.5) it can be seen that this can happen when the drift μ of the signed volume of market orders becomes negative. In this approach, therefore, whether or not the bubble is positive depends more on the attitude of the investors than on the liquidity. In section 4 we propose an example to show how contagion between traders in financial networks can determine the value of μ.

From now on, we fix $t \in [0,T)$. We begin the analysis by noticing that since $[B^1,N] \equiv [B^2,N] \equiv 0$,

$$\begin{aligned} Z_{t,s} &= \mathcal{E}\left(\int_0^s \alpha_u^{t,1} dB_u^1 + \int_0^s \alpha_u^{t,2} dB_u^2 + \int_0^s \alpha_u^{t,3} d\tilde{N}_u\right) \\ &= \mathcal{E}\left(\int_0^s \alpha_u^{t,1} dB_u^1 + \int_0^s \alpha_u^{t,2} dB_u^2\right) \mathcal{E}\left(\int_0^s \alpha_u^{t,3} d\tilde{N}_s\right) \end{aligned}$$

for $s \in [0,T)$.

Moreover,

$$\begin{aligned} \mathcal{E}\left(\int_0^s \alpha_u^{t,3} d\tilde{N}_u\right) &\le \exp\left\{\int_0^s \left[\alpha_u^{t,3} - \frac{1}{2}(\alpha_u^{t,3})^2\right] dN_u - \int_0^s \alpha_u^{t,3}\pi_u du\right\} \\ &\quad \cdot \prod_{0\le u\le s} (1+\Delta(\alpha_u^{t,3}N_u)) \exp\left\{\Delta(\alpha_u^{t,3}N_u) + \frac{1}{2}\Delta(\alpha_u^{t,3}N_u)^2\right\} \\ &\le 2\exp\left\{\frac{3}{2} + \int_0^s \left[|\alpha_u^{t,3}| + \frac{1}{2}|\alpha_u^{t,3}|^2\right] dN_u + \int_0^s |\alpha_u^{t,3}|\pi_u du\right\} \\ &\le 2e^{3+T\Pi}, \end{aligned}$$

since by (3.17) it holds that $|\alpha_s^{t,3}| \le 1$.

Then, taking $(\bar{Z}_{t,s})_{s\in[0,T)}$ with

$$\bar{Z}_{t,s} = \mathcal{E}\left(\int_0^s \alpha_u^{t,1} dB_u^1 + \int_0^s \alpha_u^{t,2} dB_u^2\right),$$

we have

$$Z_{t,s} \le 2e^{3+T\Pi}\bar{Z}_{t,s}. \tag{3.20}$$

We give the following.

Lemma 3.5. *Let X, Y be two positive stochastic processes such that $Y_t \le X_t$ a.s. for all $t \ge 0$, and let X be of class DL.*[1] *Then Y is of class DL as well.*

Proof. By Theorem 11 of Chapter I of [47] we have that a family of random variables $(U_\alpha)_{\alpha\in A}$ is uniformly integrable if and only if there exists a function G defined on $[0,\infty)$, positive, increasing, and convex, such that $\lim_{x\to\infty} \frac{G(x)}{x} = +\infty$ and $\sup_\alpha \mathbb{E}[G \circ |U_\alpha|] < \infty$.

[1] A stochastic process X is of class DL if, for each $t \ge 0$, $\{X_\tau : \tau \le t \text{ stopping time}\}$ is uniformly integrable.

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Now fix $t \geq 0$, and call $J_t = \{\tau : \tau \leq t \text{ stopping time}\}$, $U_X^t = \{X_\tau : \tau \in J_t\}$, and $U_Y^t = \{Y_\tau : \tau \in J_t\}$.

Since by hypothesis U_X^t is uniformly integrable, there exists a function G that satisfies the properties stated before. We have that

$$G(Y_\tau) \leq G(X_\tau) \quad \text{a.s. for } \tau \in J_t$$

and then that

$$\mathbb{E}[G(Y_\tau)] \leq \mathbb{E}[G(X_\tau)], \quad \tau \in J_t.$$

Thus

$$\sup_{\tau \in J_t} \mathbb{E}[G(Y_\tau)] \leq \sup_{\tau \in J_t} \mathbb{E}[G(X_\tau)] < \infty.$$

Therefore U_Y^t is uniformly integrable and Y is of class DL. ∎

We then have the following.

Proposition 3.6. *$(Z_{t,s})_{s\in[0,T)}$ in (3.8) is a martingale if $(\bar{Z}_{t,s})_{s\in[0,T)}$ is a martingale.*

Proof. Since a local martingale is a true martingale if and only if it is of class DL (see Proposition 1.7 of Chapter IV of [49]), we have that if $\bar{Z}$ is a true martingale, then $2e^{3+T\Pi}\bar{Z}$, being a martingale as well, is of class DL. Thus, by Lemma 3.5 and by (3.20), Z is of class DL, and therefore by Proposition 1.7 of Chapter IV of [49] it is a true martingale. ∎

To prove that $\bar{Z}$ is a martingale we rely on some results provided by Mijatovic and Urusov [38] and by Wong and Heyde [58]. We first need some preliminaries.

Consider the state space $J = (l, r)$, $-\infty \leq l < r \leq \infty$, and a J-valued diffusion $Y = (Y_s)_{s\in[0,T)}$ on some filtered probability space, governed by the SDE

$$dY_s = \mu_Y(Y_s)ds + \sigma_Y(Y_s)dB_s, \quad 0 \leq s < T, \tag{3.21}$$

with $Y_0 = x_0 \in J$, W Brownian motion, and deterministic functions $\mu_Y(\cdot)$ and $\sigma_Y(\cdot)$, which from now on we will simply denote by μ_Y and σ_Y, such that

$$\sigma_Y(x) \neq 0 \quad \forall x \in J \tag{3.22}$$

and

$$\frac{1}{\sigma_Y^2}, \ \frac{\mu_Y}{\sigma_Y^2} \in L^1_{loc}(J), \tag{3.23}$$

where $L^1_{loc}(J)$ denotes the class of locally integrable functions on J, i.e., the measurable functions $(J, \mathcal{B}(J)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that are integrable on compact subsets of J.

Consider the stochastic exponential

$$\mathcal{E}\left(\int_0^s f(Y_u)dB_u\right), \quad 0 \leq s < T, \tag{3.24}$$

with $f(\cdot)$ such that

$$\frac{f^2}{\sigma_Y^2} \in L^1_{loc}(J), \tag{3.25}$$

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and the auxiliary J-valued diffusion $\tilde{Y}$ governed by the SDE

$$d\tilde{Y}_s = \Big(\mu_Y(\tilde{Y}_s) + f(\tilde{Y}_s)\sigma_Y(\tilde{Y}_s)\Big)\,ds + \sigma_Y(\tilde{Y}_s)d\tilde{B}_s, \quad 0 \le s < T, \tag{3.26}$$

where $\tilde{B}$ is a Brownian motion on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

Put $\bar{J} = [l, r]$ and, fixing an arbitrary $c \in J$, define

$$\rho(x) := \exp\left\{-\int_c^x \frac{2\mu_Y}{\sigma_Y^2}(y)dy\right\}, \quad x \in J, \tag{3.27}$$

$$\tilde{\rho}(x) := \rho(x)\exp\left\{-\int_c^x \frac{2f}{\sigma_Y}(y)dy\right\}, \quad x \in J, \tag{3.28}$$

$$s(x) := \int_c^x \rho(y)dy, \quad x \in \bar{J}, \tag{3.29}$$

$$\tilde{s}(x) := \int_c^x \tilde{\rho}(y)dy, \quad x \in \bar{J}. \tag{3.30}$$

Denote $\rho = \rho(\cdot)$, $s = s(\cdot)$, $s(r) = \lim_{x\to r^-} s(x)$, $s(l) = \lim_{x\to l^+} s(x)$, and analogously for $\tilde{s}(\cdot)$ and $\tilde{\rho}(\cdot)$.

Recall that by Feller's test for explosions $\tilde{Y}$ exits its state space with positive probability at the boundary point r if and only if

$$\tilde{s}(r) < \infty \quad \text{and} \quad \frac{\tilde{s}(r) - \tilde{s}}{\tilde{\rho}\sigma_Y^2} \in L^1_{loc}(r-), \tag{3.31}$$

where $L^1_{loc}(r-) := \{g | g : (J, \mathcal{B}(J)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\int_x^r g(y)dy < \infty$ for some $x \in J\}$. Similarly, $\tilde{Y}$ exits its state space with positive probability at the boundary point l if and only if

$$\tilde{s}(l) > -\infty \quad \text{and} \quad \frac{\tilde{s} - \tilde{s}(l)}{\tilde{\rho}\sigma_Y^2} \in L^1_{loc}(l+), \tag{3.32}$$

where $L^1_{loc}(l+) := \{g | g : (J, \mathcal{B}(J)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\int_l^x g(y)dy < \infty$ for some $x \in J\}$. Moreover, the endpoint r of J is said to be *good* if

$$s(r) < \infty \quad \text{and} \quad \frac{(s(r) - s)f^2}{\rho\sigma_Y^2} \in L^1_{loc}(r-), \tag{3.33}$$

or, equivalently (see [38]), if

$$\tilde{s}(r) < \infty \quad \text{and} \quad \frac{(\tilde{s}(r) - \tilde{s})f^2}{\tilde{\rho}\sigma_Y^2} \in L^1_{loc}(r-). \tag{3.34}$$

Similarly, the endpoint l of J is said to be *good* if

$$s(l) > -\infty \quad \text{and} \quad \frac{(s - s(l))f^2}{\rho\sigma_Y^2} \in L^1_{loc}(l+), \tag{3.35}$$

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or, equivalently, if

$$\tilde{s}(l) > -\infty \quad \text{and} \quad \frac{(\tilde{s}-\tilde{s}(l))f^2}{\tilde{\rho}\sigma_Y^2} \in L^1_{loc}(l+). \tag{3.36}$$

We recall here Theorem 2.1 in [38].

Theorem 3.7. *Let the functions μ_Y, σ_Y, and f satisfy conditions (3.22), (3.23), and (3.25), and let Y be a solution of the SDE (3.21).*

Then the Doléans exponential given by (3.24) is a martingale for any $T < \infty$ if and only if both of the following requirements are satisfied:

(a) *Condition (3.31) does not hold or conditions (3.33)–(3.34) hold.*
(b) *Condition (3.32) does not hold or conditions (3.35)–(3.36) hold.*

We now obtain the following.

Proposition 3.8. *Let $S = (S_s)_{s\in[0,T)}$ be a geometric Brownian motion*

$$dS_s = \mu_0 S_s ds + \sigma_0 S_s dB_s, \quad 0 \le s < T, \tag{3.37}$$

where B is a Brownian motion, $\mu_0 \in \mathbb{R}$, and $\sigma_0 > 0$.

Then the process

$$\mathcal{E}\left(\int_0^s (S_u)^{-1} dB_u\right), \quad 0 \le s < T,$$

is a martingale.

Proof. We show that the requirements of Theorem 3.7 hold for $Y = S$, with $\mu_Y(x) = \mu_0 x$, $\sigma_Y(x) = \sigma_0 x$, and $f(x) = x^{-1}$. Notice that μ_Y, σ_Y, and f satisfy conditions (3.22), (3.23), and (3.25) with $J = (0,\infty)$. Then, taking $c = 1$ for the functions (3.27)–(3.30) and first assuming $\frac{2\mu_0}{\sigma_0^2} \neq 1$, we have

$$\rho(x) = \exp\left\{-\int_1^x \frac{2\mu_Y}{\sigma_Y^2}(y)dy\right\} = x^{\frac{-2\mu_0}{\sigma_0^2}}, \tag{3.38}$$

$$\tilde{\rho}(x) = \rho(x)\exp\left\{-\int_1^x \frac{2f}{\sigma_Y}(y)dy\right\} = x^{\frac{-2\mu_0}{\sigma_0^2}} \exp\left(\frac{2}{\sigma_0}\left(\frac{1}{x}-1\right)\right), \tag{3.39}$$

$$s(x) = \int_1^x \rho(y)dy = \frac{\sigma_0^2}{2\mu_0 - \sigma_0^2}(1 - x^{-\gamma_0}), \tag{3.40}$$

$$\tilde{s}(x) = \int_1^x \tilde{\rho}(y)dy = e^{-\frac{2}{\sigma_0}}\left(-\frac{2}{\sigma_0}\right)^{-\gamma_0}\left[\bar{\Gamma}\left(\gamma_0, -\frac{2}{x\sigma_0}\right) - \bar{\Gamma}\left(\gamma_0, -\frac{2}{\sigma_0}\right)\right], \tag{3.41}$$

with $\gamma_0 = \frac{2\mu_0}{\sigma_0^2} - 1$ and where $\bar{\Gamma}(a,z) = \int_z^\infty e^{-t}t^{a-1}dt$, $a \in \mathbb{R}^+$, $z \in \mathbb{R}$, is the incomplete Gamma function extended to all $\mathbb{R}$.

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Notice that in (3.41) we have that

$$\begin{aligned}\tilde{s}(x) &= e^{-\frac{2}{\sigma_0}}\left(-\frac{2}{\sigma_0}\right)^{-\gamma_0}\left[\bar{\Gamma}\left(\gamma_0, -\frac{2}{x\sigma_0}\right) - \bar{\Gamma}\left(\gamma_0, -\frac{2}{\sigma_0}\right)\right] \\ &= e^{-\frac{2}{\sigma_0}}\left(\frac{2}{\sigma_0}\right)^{-\gamma_0}(-1)^{-\gamma_0}\int_{-\frac{2}{x\sigma_0}}^{-\frac{2}{\sigma_0}} e^{-t}(-1)^{\gamma_0-1}|t|^{\gamma_0-1}dt \\ &= -e^{-\frac{2}{\sigma_0}}\left(\frac{2}{\sigma_0}\right)^{-\gamma_0}\int_{-\frac{2}{x\sigma_0}}^{-\frac{2}{\sigma_0}} e^{-t}|t|^{\gamma_0-1}dt \in \mathbb{R}.\end{aligned} \tag{3.42}$$

We obtain the following:

- In $l = 0$ we have

$$\tilde{s}(0) = -e^{-\frac{2}{\sigma_0}}\left(\frac{2}{\sigma_0}\right)^{-\gamma_0}\int_{-\infty}^{-\frac{2}{\sigma_0}} e^{-t}|t|^{\gamma_0-1}dt = -\infty;$$

thus condition (3.32) does not hold, and the first requirement of (b) in Theorem 3.7 is fulfilled.

- If $\gamma_0 < 0$, we have

$$\tilde{s}(\infty) = e^{-\frac{2}{\sigma_0}}\left(\frac{2}{\sigma_0}\right)^{-\gamma_0}\int_{-\frac{2}{\sigma_0}}^{0} e^{-t}|t|^{\gamma_0-1}dt = \infty;$$

then condition (3.31) does not hold, and the first requirement of (a) in Theorem 3.7 is fulfilled.

- If $\gamma_0 > 0$, then $s(\infty) = \frac{\sigma_0^2}{2\mu_0 - \sigma_0^2} = C < \infty$, and condition (3.33) holds since

$$\frac{s(r) - s}{\rho\sigma_0^2} = C\frac{x^{-\gamma_0}x^{\frac{2\mu_0}{\sigma_0^2}}}{x^4} = \frac{1}{x^3}.$$

Therefore the second requirement of (a) in Theorem 3.7 is fulfilled.

So we have that if $\gamma_0 \neq 0$, the requirements of Theorem 3.7 are satisfied, and thus Z is a martingale.

In the case $\gamma_0 = 0$, i.e., $\mu_0 = \frac{\sigma_0^2}{2}$, we have that the process $S = (S_u)_{u\in[0,T)}$ in (3.37) takes the form $S_u = e^{\sigma_0 B_u}$, $0 \le u < T$. We can thus apply the results of Theorem 3.7 taking $J = (-\infty, \infty)$, $\mu_Y \equiv 0$, $\sigma_Y \equiv 1$, $f(x) = e^{-\sigma_0 x}$, and $c = 0$ in (3.27)–(3.30). We have

$$\begin{aligned}\rho(x) &= \exp\left\{-\int_0^x \frac{2\mu_Y}{\sigma_Y^2}(y)dy\right\} = 1, \\ \tilde{\rho}(x) &= \rho(x)\exp\left\{-\int_0^x \frac{2f}{\sigma_Y}(y)dy\right\} = \exp\left(2(e^{-\sigma_0 x} - 1)/\sigma_0\right), \\ s(x) &= \int_0^x \rho(y)dy = x, \\ \tilde{s}(x) &= \int_0^x \tilde{\rho}(y)dy = \frac{1}{\sigma_0}e^{-\frac{2}{\sigma_0}}\left(Ei\left(2/\sigma_0\right) - Ei\left(2e^{-\sigma_0 x}/\sigma_0\right)\right),\end{aligned} \tag{3.43}$$

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where $Ei(z) = -\int_{-z}^{\infty} \frac{e^{-u}}{u} du$ is the exponential integral function that satisfies $\lim_{z\to\infty} Ei(z) = \infty$ and $\lim_{z\to 0} Ei(z) = -\infty$. Therefore $\tilde{s}(\infty) = \infty$ and $\tilde{s}(-\infty) = -\infty$; then the first requirements of (a) and (b) of Theorem 3.7 are both satisfied, and Z is a martingale. ∎

Then we have immediately the following.

Corollary 3.9. *Under Assumption 2.2, the process*

$$\mathcal{E}\left(\int_\tau^s \frac{1}{W_u^F} dB_u^1\right), \quad \tau \le s < T, \tag{3.44}$$

is a martingale for every fixed $T < \infty$.

To prove that Corollary 3.9 also implies that $\mathcal{E}(\int_\tau^s \alpha_u^{t,1} dB_u^1)$ is a martingale, we extend the results of Wong and Heyde in [58].

For this purpose we consider an $\mathbb{F}$-progressively measurable d-dimensional process $H = (H_s)_{s\in[0,T)}$ of the form

$$H_s = \xi(B(\cdot), s)\zeta_s + \eta_s, \tag{3.45}$$

where $\xi \in C_0(\mathbb{R}^{d+1}, \mathbb{R}^d)$, B is a d-dimensional progressively measurable Brownian motion, and ζ, η are d-dimensional stochastic processes independent of B. Here the product between ξ and ζ is intended componentwise.

Define

$$\tau_N^{M_H} = \inf\left(s \in [0,T) : M_H(t) := \int_0^t \|H_u\|^2 du \ge N\right),$$

with the convention that $\inf \emptyset = \infty$, and then

$$\tau^{M_H} = \lim_{N\to\infty} \tau_N^{M_H}. \tag{3.46}$$

Then we have the following.

Proposition 3.10. *Let H be as in (3.45) and defined up to the explosion time τ^{M_H} in (3.46). Then there also exists a d-dimensional $\mathbb{F}$-progressively measurable process, $Y = (Y_s)_{s\in[0,T)}$ with $Y_s = \xi(W(\cdot) + \int_0^\cdot Y_u du, s)\zeta_s + \eta_s$ defined up to the explosion time τ^{M_Y} with*

$$\tau^{M_Y} = \lim_{N\to\infty} \tau_N^{M_Y},$$

where

$$\tau_N^{M_Y} = \inf\left(s \ge 0 : M_Y(s) := \int_0^s \|Y_u\|^2 du \ge N\right) \wedge T,$$

such that the stochastic exponential $Z^H = (Z_s^H)_{s\in[0,T)}$ with $Z_s^H = \mathcal{E}\left(\int_0^s H_u dW_u\right)$ satisfies

$$P(\tau^{M_Y} > T) = \mathbb{E}[Z_T^H].$$

Hence Z^H is a (true) martingale if and only if $P(\tau^{M_Y} > T) = 1$.

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Proof. Since the proof is a long but easy extension of the result in [58], we omit it here and refer the reader to [36]. ∎

Proposition 3.11. *In the setting of section 2, the process*

$$\mathcal{E}\left(\int_0^s |\alpha_u^{t,1}| dB_u^1\right), \quad 0 \le s < T,$$

with $\alpha^{t,1}$ in (3.18) is a martingale for each $t \in [0,T)$.

Proof. For $s < \tau$ we have

$$|\alpha_s^{t,1}| = \frac{a}{b} + \frac{2}{b}\pi_s \Lambda_s \frac{M_s}{M_s+1}\frac{1}{W_s^F+1} \le \frac{a}{b} + \frac{2}{b}\Pi;$$

then $\mathcal{E}(\int_0^\cdot |\alpha_u^{t,1}| dB_u^1)$ is a martingale up to time τ since it satisfies the Novikov condition because

$$\mathbb{E}\left[\exp\left(\int_0^\tau (\alpha_s^{t,1})^2 ds\right)\right] \le \mathbb{E}\left[\exp(c^2\tau)\right]$$

with $c = \frac{a}{b} + \frac{2}{b}\Pi$.

Consider now $s \ge \tau$. We have that the process Y associated to $|\alpha_s^{t,1}|$ as in Proposition 3.10 satisfies

$$Y_s = \frac{2}{bW_s^F}(s + |\eta_{t,\tau}|)\exp\left(-b\int_{t\wedge\tau}^s Y_u du\right), \quad t\wedge\tau \le s < T,$$

with $\eta_{t,\tau}$ in (3.16). On the other hand, we have

$$\tilde{Y}_s = \frac{1}{W_s^F}\exp\left(-b\int_{t\wedge\tau}^s \tilde{Y}_u du\right), \quad t\wedge\tau \le s < T,$$

where $\tilde{Y}$ is the process associated to $\frac{1}{W^F}$. By Corollary 3.9 and Proposition 3.10 it holds that

$$\int_{t\wedge\tau}^T \tilde{Y}_s^2 < \infty. \tag{3.47}$$

We show that the integral of Y^2 does not explode as well.

We have that

$$\Delta_s = \frac{\tilde{Y}_s}{Y_s} = \frac{b}{s+|\eta_{t,\tau}|}\cdot\exp\left(b\int_{t\wedge\tau}^s (Y_u - \tilde{Y}_u)du\right), \quad t\wedge\tau \le s < T. \tag{3.48}$$

Define the stopping time

$$\tau_1 = \inf\{s \in [t\wedge\tau, T) : \Delta_s \le 1\} \wedge T,$$

and notice that, since Y and $\tilde{Y}$ are continuous, $\Delta_{\tau_1} = 1$.

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Define

$$\tau_2 = \inf\{s \geq \tau_1 : \Delta_s \geq 1\} \wedge T.$$

If $\tau_1 = T$, we are done. Otherwise consider $s \in (\tau_1, \tau_2)$.

Since for $\tau_1 < s < \tau_2$ we have

$$\Delta_s = \frac{b}{s + |\eta_{t,\tau}|} \cdot \exp\left(b \int_{t\wedge\tau}^{s} (Y_u - \tilde{Y}_u) du\right) \geq \frac{b}{T + |\eta_{t,\tau}|} \exp\left(b \int_{t\wedge\tau}^{\tau_1} (Y_u - \tilde{Y}_u) du\right),$$

it follows that

$$Y_s \leq \frac{\tilde{Y}_s\,(T + |\eta_{t,\tau}|)}{b} \exp\left(b \int_{t\wedge\tau}^{\tau_1} (\tilde{Y}_u - Y_u) du\right) \leq \frac{\tilde{Y}_s\,(T + |\eta_{t,\tau}|)}{b} \exp\left(b \int_{t\wedge\tau}^{\tau_1} \tilde{Y}_u du\right)$$

for $\tau_1 < s < \tau_2$, which implies, together with (3.47), that $M_Y(s) := \int_{t\wedge\tau}^{s} Y_s^2 ds$ does not explode before τ_2.

But after τ_2, up to $\tau_3 = \inf\{s \geq \tau_2 : \Delta_s \leq 1\} \wedge T$, Y is smaller than $\tilde{Y}$; hence $M_Y(s) \leq M_{\tilde{Y}_s}$ on $[\tau_2, \tau_3]$.

Repeating this argument up to T, we obtain that $\mathcal{E}(\int_0^s |\alpha_u^{t,1}| dB_u^1)$ is a martingale by Proposition 3.10. ■

We want now to prove that

$$\mathcal{E}\left(\int_0^s |\alpha_u^{t,2}| dB_u^2\right), \quad 0 \leq s < T, \tag{3.49}$$

with $\alpha^{t,2}$ in (3.15) is a martingale as well.

We start with the following.

Proposition 3.12. *Let β be the bubble as in (2.5). Under Assumption 2.2, the Doléans exponential*

$$\mathcal{E}\left(\int_0^s \beta_u dB_u^2\right), \quad 0 \leq s < T,$$

is a martingale.

Proof. If we rewrite β in the form (3.45), we obtain that

$$\xi(B^2(\cdot), s) = \int_\tau^s \sigma_u \Lambda_u M_u e^{-k \int_u^s k\Lambda_r M_r dr} dB_u^2, \quad \tau \leq s < T;$$

i.e., the process Y associated to β in Proposition 3.10 is given by

$$\begin{aligned} Y_s = {} & \beta_\tau e^{\int_\tau^s (-k+\sigma_u)\Lambda_u M_u ds} + \int_\tau^s \mu_u \Lambda_u M_u e^{\int_u^s (-k+\sigma_r)\Lambda_r M_r dr} du \\ & + \int_\tau^s \sigma_u \Lambda_u M_u e^{\int_u^s (-k+\sigma_r)\Lambda_r M_r dr} dB_u^2, \quad \tau \leq s < T. \end{aligned} \tag{3.50}$$

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We first prove that $Y_s < \infty$ for each $s \in [\tau, T)$. We have $\int_u^s (-k+\sigma_r)\Lambda_r M_r dr < \infty$ a.s. for each $s \in [\tau, T)$ by the hypothesis on σ and Λ in Assumption 2.2 and by Proposition 2.4.

Thus by Theorem 2.4 of [37] and by the fact that T is bounded, we obtain

$$\int_\tau^T e^{\alpha \int_u^s (-k+\sigma_r)\Lambda_r M_r dr} du < \infty \tag{3.51}$$

for all $\alpha \in \mathbb{R}$, and then by the hypothesis on μ in Assumption 2.2, and again by Proposition 2.4, we have

$$\int_\tau^s \mu_u \Lambda_u M_u e^{\int_u^s (-k+\sigma_r)\Lambda_r M_r dr} du < \infty, \quad \tau \le s < T.$$

By (3.51) and by Assumption 2.2 it follows that the stochastic integral in (3.50) does not explode before T, so we have that $Y_s < \infty$ for each $s \in [\tau, T)$.

We prove that this implies $\int_{t\wedge\tau}^T Y_s^2 ds < \infty$. By the expression of Y in (3.50) we have

$$\int_{t\wedge\tau}^T Y_s^2 ds = \int_{t\wedge\tau}^T Y_s^2 \frac{1}{M_s^2 \Lambda_s^2 \sigma_s^2} d[Y,Y]_s$$

(by the Kunita–Watanabe inequality)

$$\le \left(\int_{t\wedge\tau}^T Y_s^4 d[Y,Y]_s \right)^{1/2} \left(\int_{t\wedge\tau}^T \frac{1}{M_s^4 \Lambda_s^4 \sigma_s^4} d[Y,Y]_s \right)^{1/2}$$

(by the occupation time formula)

$$= \left(\int_{-\infty}^{\infty} a^4 L_T^a da \right)^{1/2} \left(\int_{t\wedge\tau}^T \frac{1}{M_s^2 \Lambda_s^2 \sigma_s^2} ds \right)^{1/2} < \infty, \tag{3.52}$$

where the first integral is finite because the local time L_T^a has bounded support in $(-\infty, \infty)$, since Y does not explode before T, and the second one is finite by Assumption 2.2 and Proposition 2.4. Then the result follows by Proposition 3.10. ■

Proposition 3.13. *Under Assumption 2.2 the process*

$$\mathcal{E}\left(\int_0^s |\alpha_u^{t,2}| dB_u^2 \right), \quad 0 \le s < T,$$

with $\alpha^{t,2}$ in (3.15) is a martingale for each $t \in [0, T)$.

Proof. We have that

$$|\alpha_s^{t,2}| \le \frac{1}{\sigma_s} \left(\frac{\eta_{t,\tau} + T}{\lambda M_s} + k\mu_s + k|\beta_s| \right), \quad \tau \wedge t \le s < T.$$

Let $\tilde{Y}$ be the process associated to $\frac{\eta_{t,\tau}+T}{\lambda M_s} + k\mu_s + k|\beta|$ in Proposition 3.10, and let $\bar{Y}$ be the one associated to $k|\beta|$.

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We have

$$\begin{aligned}
\tilde{Y}_s &= \frac{\eta_{t,\tau}+T}{\lambda M_s} + k\mu_s + k|\beta_s| + k\int_\tau^s \sigma_u \Lambda_u M_u \tilde{Y}_u e^{-k\int_u^s \Lambda_r M_r dr} du \\
&= \frac{\eta_{t,\tau}+T}{\lambda M_s} + k\mu_s + k|\beta_s| + k\int_\tau^s \sigma_u \Lambda_u M_u \bar{Y}_u e^{-k\int_u^s \Lambda_r M_r dr} du \\
&\quad + k\int_\tau^s \sigma_u \Lambda_u M_u (\tilde{Y}_u - \bar{Y}_u) e^{-k\int_u^s \Lambda_r M_r dr} du \\
&= \frac{\eta_{t,\tau}+T}{\lambda M_s} + k\mu_s + \bar{Y}_s + k\int_\tau^s \sigma_u \Lambda_u M_u (\tilde{Y}_u - \bar{Y}_u) e^{-k\int_u^s \Lambda_r M_r dr} du, \quad \tau \le s < T,
\end{aligned}$$

and consequently, for $\bar{D}_s := \tilde{Y}_s - \bar{Y}_s$,

$$d\bar{D}_s = d\left(\frac{\eta_{t,\tau}+T}{\lambda M_s} + k\mu_s\right) + k\Lambda_s M_s \left[(\sigma_s - 1)\bar{D}_s + \frac{\eta_{t,\tau}+T}{\lambda M_s} + k\mu_s\right] ds, \quad \tau \le s < T,$$

so that we can write

$$\bar{D}_s = \frac{\eta_{t,\tau}+T}{\lambda M_s} + k\mu_s + k\int_\tau^s \left(\frac{\eta_{t,\tau}+T}{\lambda} + k\mu_u M_u\right) \sigma_u \Lambda_u e^{k\int_u^s \Lambda_r M_r(\sigma_r - 1)dr} du, \quad \tau \le s < T.$$

By Assumption 2.2 and Proposition 2.4, with the same argument as in the proof of Proposition 3.12, we have that

$$\int_{t\wedge\tau}^T \bar{D}_s^2 ds = \int_{t\wedge\tau}^T |\tilde{Y}_s - \bar{Y}_s|^2 ds < \infty.$$

Then, since by Proposition 3.12 we have $\int_{t\wedge\tau}^T |\bar{Y}_s|^2 ds < \infty$, we obtain

$$\int_{t\wedge\tau}^T |\tilde{Y}_s|^2 ds < \infty. \tag{3.53}$$

Now call Y the process associated to $R^{t,2}$ in Proposition 3.10.

It holds that

$$\begin{aligned}
Y_s &= \frac{1}{\sigma_s}\left(\frac{\eta_{t,\tau}+T}{\lambda M_s} + k\mu_s + k|\beta_s| + k\int_\tau^s \Lambda_u M_u \tilde{Y}_u e^{-k\int_u^s \Lambda_r M_r dr} du\right) \\
&\quad + \frac{1}{\sigma_s} k\int_\tau^s \Lambda_u M_u (Y_u - \tilde{Y}_u) e^{-k\int_u^s \Lambda_r M_r dr} du \\
&= \frac{1}{\sigma_s}\left(\tilde{Y}_s + k\int_\tau^s \Lambda_u M_u (Y_u - \tilde{Y}_u) e^{-k\int_u^s \Lambda_r M_r dr} du\right), \quad \tau \le s < T.
\end{aligned}$$

Then we have

$$\sigma_s Y_s - \tilde{Y}_s = \Psi_s + k\int_\tau^s \Lambda_u M_u (\sigma_u Y_u - \tilde{Y}_u) e^{-k\int_u^s \Lambda_r M_r dr} du, \quad \tau \le s < T,$$

where $(\Psi_s)_{s\in[\tau,T)}$ is given by

$$\Psi_s = k\int_\tau^s \Lambda_u M_u (\tilde{Y}_u - \sigma_u \tilde{Y}_u) e^{-k\int_u^s \Lambda_r M_r dr} du, \quad \tau \le s < T. \tag{3.54}$$

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It follows that $D_s = \sigma_s Y_s - \tilde{Y}_s$ satisfies

$$dD_s = d\Psi_s + k\Lambda_s M_s \Psi_s ds, \quad \tau \le s < T,$$

and consequently that it takes the form

$$D_s = \Psi_s + k\int_\tau^s \Lambda_u M_u \Psi_u du, \quad \tau \le s < T.$$

Since by Assumption 2.2 the process Ψ in (3.54) does not explode before T, $D_s = \sigma_s Y_s - \tilde{Y}_s < \infty$ a.s. for each $s \in [0, T)$.

Thus, with the same argument as in the proof of Proposition 3.12 it can be proved that

$$\int_{t\wedge\tau}^T |\sigma_s Y_s - \tilde{Y}_s|^2 ds < \infty.$$

By (3.53) we then have

$$\int_{t\wedge\tau}^T |\sigma_s Y_s|^2 ds < \infty.$$

Then by the integrability hypothesis on $\frac{1}{\sigma^4}$ in (ii) of Assumption 2.2, it holds that

$$\int_{t\wedge\tau}^T |Y_s|^2 ds < \infty.$$

The result then follows by Proposition 3.10 and by the fact that if Y^α is the process associated to $|\alpha^{t,2}|$, it can easily be seen that $Y_s^\alpha \le Y_s$ a.s. for each $s \in [\tau, T)$. ■

Proposition 3.14. *Consider* $(Z^1_{t,s})_{s\in[0,T)}$ *and* $(Z^2_{t,s})_{s\in[0,T)}$, *with*

$$Z^1_{t,s} = \mathcal{E}\left(\int_0^s \alpha_u^{t,1} dB_u^1\right) \tag{3.55}$$

and

$$Z^2_{t,s} = \mathcal{E}\left(\int_0^s \alpha_u^{t,2} dB_u^2\right), \tag{3.56}$$

where $\alpha^{t,1}$ *and* $\alpha^{t,2}$ *are defined as in* (3.18) *and* (3.15), *and suppose that* Assumption 2.2 *holds. Then* $(Z^1_{t,s})_{s\in[0,T)}$ *and* $(Z^2_{t,s})_{s\in[0,T)}$ *are true martingales.*

The proof follows by Proposition 3.11, by Proposition 3.13, and by the following.

Lemma 3.15. *Consider* $H_s = \int_0^s Y_u dB_u$ *and* $\bar{H}_s = \int_0^s |Y_u| dB_u$, $s \ge 0$, *where* Y *is a stochastic process such that the stochastic integral is well defined. Then* $\mathcal{E}(H)$ *is a martingale if and only if* $\mathcal{E}(\bar{H})$ *is a martingale.*

Proof. Theorem 4.1 in [13] states that, for a general continuous local martingale H, $\mathcal{E}(H)$ is a martingale if and only if

$$\lim_{n\to\infty} Q_s(\{A_s < n\}) = 1 \quad \forall s \ge 0,$$

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where $A_s = [H, H]_s$, $dQ_s = \mathcal{E}(H_{T_s})dP$, and $T_s := \inf\{u \geq 0 : A_u > s\}$. Since $[H, H]_s = \int_0^s Y_u^2 du = \int_0^s |Y_u|^2 du = [\bar{H}, \bar{H}]_s$, this property holds for H if and only if it holds for $\bar{H}$. Hence we have the result. ∎

We are now ready to state the main result of the section.

Theorem 3.16. *Under Assumption 2.2, Q^t defined as in (3.8) belongs to $\mathcal{M}_{loc}(W)$ for each $t \in [0, T)$.*

Proof. The proof follows by the fact that if we take $\alpha^{t,1}$ and $\alpha^{t,2}$ as in (3.18) and (3.15), with μ_t, σ_t, M, Λ, and π satisfying Assumption 2.2, then $(\bar{Z}_{t,s})_{s\in[0,T)}$ with

$$\bar{Z}_{t,s} = \mathcal{E}\left(\int_0^s \alpha_u^{t,1} dB_u^1 + \int_0^s \alpha_u^{t,2} dB_u^2\right)$$

is a martingale with respect to time s.

This follows immediately from Proposition 3.14: $(Z_{t,s}^1)_{s\in[0,T)}$ in (3.55) and $(Z_{t,s}^2)_{s\in[0,T)}$ in (3.56) are martingales, so by Proposition 3.10 we know that $H^1 = \alpha^{t,1}$ and $H^2 = \alpha^{t,2}$ are such that the associated processes Y^1 and Y^2 defined in Proposition 3.10 do not explode before T. Taking now $H = (H^1, H^2)$, the associated process $Y = (Y^1, Y^2)$ does not explode before T as well, and this concludes the proof. ∎

Remark 3.17. Theorem 3.16 shows that our constructive model can be included in the more fundamental view of the martingale theory of bubbles of [31] and [32]. For this purpose we need to admit the possibility of shifting pricing views over time as suggested in [10]. However, we emphasize that our definition of bubbles and the models proposed in section 2 and further investigated in section 4 are independent of any choice of $\mathbb{Q} \in \mathcal{M}_{loc}(W)$. This can be seen as an advantage of this framework since the definition of the $\mathbb{Q}$-bubble could raise some criticisms (see [25]).

Note that Theorem 3.16 also implies that $\mathcal{M}_{loc}(W) \neq \emptyset$, and hence that our market model is arbitrage-free on $[0, T)$.

4. Liquidity induced bubbles in a network. As an illustration of the previous results, we focus on a particular example. We note, however, that the results of this section are of independent interest since we provide one of the few contributions on mathematical modeling of bubbles in a network. For further results on this topic, we also refer the reader to [7], where it is shown how bubbles can have an impact on the structure of a banking network, and to [19], where the authors describe the passage from a well-connected network with high global confidence to a poorly connected network with low global confidence, producing a boom and bust cycle. Our approach, however, is quite different: we consider an *information* network of N investors who may be influenced by the trading activity of their neighborhoods. In particular we assume that the number N of traders in the network is big enough to guarantee that our hypothesis on the linearity of the supply curve holds. Investors may place a buy market order on the bubbly asset because they imitate neighbors in the network who have successfully bought the asset as well, eventually leading to some self-exciting herding effect.

We refer the reader to some literature about information networks (see, among others, Ozsoylev and Walden [43], Ozsoylev et al. [44], and Walden [55]) where investors share information with neighbors so that, as in [44], two traders linked together buy or sell the same

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stock at a similar point in time.

We model the trading contagion mechanism between agents taking place from time τ via the evolution dynamics of the signed volume of market orders. Our analysis is based on some epidemiological studies, which describe how diseases spread in social networks, or how computer viruses spread from computer to computer. In particular, we focus here on the SIS model, studied, for example, by Pastor-Satorras and Vespignani (see [45] and [46]) to analyze virus diffusion in a population. We reinterpret virus diffusion as trading contagion and consider as a first step in our model building process the following stochastic version of the SIS model for the contagion evolution of the fraction $(\rho_t^k)_{\tau\le t\le T}$ of traders of degree $0 \le k \le N$ (i.e., traders with information channels to k other traders) who have bought the asset before or at time t:

$$d\rho_t^k = \left(-\delta\rho_t^k + \lambda k m_t(1-\rho_t^k)\right) dt + \bar\sigma_t^k(\rho_t^k)^\alpha(1-\rho_t^k)^\alpha dB_t^2, \quad \tau \le t \le T, \quad 0 < \rho_\tau^k < 1. \tag{4.1}$$

Here m_t is the probability that an individual at the end of an edge has bought the asset before or at time t, λ is the rate of buying contagion, δ is the rate of selling, $\bar\sigma^k = (\bar\sigma_t^k)_{\tau\le t\le T}$, $k = 1, \dots, N$, are progressively measurable processes, which we assume bounded from above and away from zero, and $\alpha \ge 1$. Then the evolution (4.1) guarantees that $0 \le \rho^k \le 1$, and we further assume that the parameters are chosen such that even $0 < \rho^k < 1$, which can be shown, for example, for $\alpha > 1$ and $\frac{1}{z}\lambda p_k k^2 - \delta > 0$ by using Proposition 2.4 of [37].

To determine the probability m_t, we observe that by Bayes' rule, and since for any given node v it holds that

$$P(\text{meet } v|deg(v) = k) = \frac{k}{\sum_j jq_j},$$

where q_j is the number of nodes with degree j, we have that

$$P(deg(v) = k|\text{meet } v) = \frac{P(\text{meet } v|deg(v)=k)P(deg(v)=k)}{P(\text{meet } v)} = \frac{k}{\frac{1}{N}\sum_j jq_j} p_k = \frac{kp_k}{z},$$

where $z := \frac{1}{N}\sum_j jq_j$ is the average degree and $p_k = P(deg(v) = k) = q_k/N$. Therefore we have

$$m_t = \sum_k P(deg(v) = k|\text{meet } v)\rho_t^k = \frac{1}{z}\sum_k kp_k\rho_t^k, \quad \tau \le t < T. \tag{4.2}$$

Given the contagion evolution of the fraction ρ^k, we model the average signed volume of market orders of an agent of degree k by $X_t^k = \theta_t^k\rho_t^k$, where the size of market orders $(\theta_t^k)_{\tau\le t\le T}$ of a trader of degree k who buys the asset is given by a positive continuous process with dynamics

$$d\theta_t^k = \mu_t^k dt + \sigma_t^k dB_t^2, \quad \tau \le t < T, \quad 0 < \theta_\tau^k, \tag{4.3}$$

where for all $k = 1, \dots, N$, $(\mu_t^k)_{\tau\le t\le T}$ is an adapted continuous process and $(\sigma_t^k)_{\tau\le t\le T}$ is a positive adapted continuous process.[2]

[2]Note that the following analysis still holds under different integrability and measurability conditions on $\bar\sigma$ and σ^k, μ^k.

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Since we have $d[\rho^k, \theta^k]_t = \bar{\sigma}_t^k \sigma_t^k (\rho_t^k)^\alpha (1-\rho_t^k)^\alpha dt$, by Itô's formula it holds that

$$
\begin{aligned}
dX_t^k &= \theta_t^k d\rho_t^k + \rho_t^k d\theta_t^k + d[\rho^k, \theta^k]_t \\
&= \left(-\delta X_t^k + \lambda k m_t (\theta_t^k - X_t^k) + \rho_t^k \mu_t^k + \bar{\sigma}_t^k \sigma_t^k (\rho_t^k)^\alpha (1-\rho_t^k)^\alpha \right) dt \\
&\quad + \left(\theta_t^k \bar{\sigma}_t^k (\rho_t^k)^\alpha (1-\rho_t^k)^\alpha + \rho_t^k \sigma_t^k \right) dB_t^2.
\end{aligned} \tag{4.4}
$$

Finally, we obtain that the signed volume of total market orders is given by $X_t = \sum_{k=0}^N q_k X_t^k$, where q_k is the number of investors of degree k. From (4.4) we thus obtain

$$
dX_t = (-\delta X_t + \lambda m_t(\theta_t - n_t) + \eta_t)\, dt + \bar{\Sigma}_t dB_t^2, \tag{4.5}
$$

with

$$
n_t = \sum_k k q_k X_t^k, \qquad \theta_t = \sum_k k q_k \theta_t^k, \qquad \eta_t = \sum_k k q_k \left(\rho_t^k \mu_t^k + \bar{\sigma}_t^k \sigma_t^k (\rho_t^k)^\alpha (1-\rho_t^k)^\alpha \right), \tag{4.6}
$$

and

$$
\bar{\Sigma}_t = \sum_k q_k \left(\bar{\sigma}_t^k \theta_t^k (\rho_t^k)^\alpha (1-\rho_t^k)^\alpha + \rho_t^k \sigma_t^k \right). \tag{4.7}
$$

We are thus in the framework[3] of section 2, with

$$
\mu_t = -\delta X_t + \lambda m_t(\theta_t - n_t) + \eta_t \tag{4.8}
$$

and $\sigma_t = \bar{\Sigma}_t$, leading to the following SDE for the bubble β:

$$
d\beta_t = \Lambda_t M_t \left[-k\beta_t + 2\left(-\delta X_t + \lambda m_t(\theta_t - n_t) + \eta_t \right) \right] dt + 2\Lambda_t M_t \bar{\Sigma}_t dB_t^2 \tag{4.9}
$$

for $\tau \le t < T$, with explicit solution

$$
\begin{aligned}
\beta_t =& \beta_\tau e^{-k\int_\tau^t \Lambda_s M_s ds} + \int_\tau^t \left(-\delta X_s + \lambda m_s(\theta_s - n_s) + \eta_s \right) \Lambda_s M_s e^{-k\int_s^t \Lambda_u M_u du} ds \\
&+ \int_\tau^t \bar{\Sigma}_s \Lambda_s M_s e^{-k\int_s^t \Lambda_u M_u du} dB_s^2, \quad \tau \le t < T.
\end{aligned} \tag{4.10}
$$

Remark 4.1. Setting $\mu^j \equiv \bar{\sigma}^j \equiv \sigma^j \equiv 0$ for all $0 \le j \le N$ in (4.1) and (4.3), respectively, we can identify the driving deterministic contagion evolution for the signed volume of market orders as implied by the SIS network model approach:

$$
dX_t = (-\delta X_t + \lambda m_t(\theta_t - n_t))\, dt. \tag{4.11}
$$

[3]The assumption that θ^k is driven by the same Brownian motion of ρ^k allows us to show the existence of the flow by using directly the results of section 2, but it can be easily relaxed, letting θ^k depend also on an additional Brownian motion B^θ independent of B^2, as we do in subsection 4.1.

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Remark 4.2. In subsection 4.1 we consider two different types of network topologies in order to see how the characteristics of the network influence the dynamics of the expected fraction of buyers through n_t. In the first type we have a connectivity distribution which is very peaked at the average value z and decaying exponentially fast for $k \gg z$ and $k \ll z$. Examples of this kind of network are random graph models [22] and the small-world model of Watts and Strogatz [57]. In the second type the degree distribution is more right skewed, following, for example, a power law, as in Barabási and Albert's preferential attachment model [6]. From (4.11) and (4.6) we can see that the expected contagion between buyers will spread faster in the second kind of network, since the distribution puts more weight on the nodes with higher degree, resulting in a bigger value of n_t in (4.6).

We conclude the introduction of the model by showing a sufficient condition under which the above bubble specification can be represented by a flow of local martingale measures as analyzed in the general framework of the previous sections, i.e., that there exists a flow $Q^t \in \mathcal{M}_{loc}(W)$ with Radon–Nykodim derivative process

$$Z_{t,s} = \frac{dQ^t}{dP}|_{\mathcal{F}_s} = \mathcal{E}\left(\int_0^\cdot \alpha_u^{t,1} dB_u^1 + \int_0^\cdot \alpha_u^{t,2} dB_u^2 + \int_0^\cdot \alpha_u^{t,3} d\tilde{N}_u\right)_s, \quad s \in [0,T), \tag{4.12}$$

such that

$$W_t^F = \mathbb{E}_{Q^t}[W_T^F|\mathcal{F}_t], \quad 0 \le t \le T.$$

Taking $\alpha^{t,1}$, $\alpha^{t,2}$, and $\alpha^{t,3}$ in (3.18), (3.15), and (3.17), respectively, we need only show that Z in (4.12) is, in fact, a martingale.

Proposition 4.3. *Assume that there exists a $\bar{k} \in 1, \dots, N$ such that $\theta_t^{\bar{k}} > \epsilon$ a.s. for all $t \in [\tau, T]$, where $\epsilon > 0$. Then for each $t \in [0,T]$, $(Z_{t,u})_{u\in[0,T)}$ is a $(P, \mathcal{F})$-martingale.*

Proof. We show that μ and Σ in (4.8) and (4.7) satisfy Assumption 2.2.

We have $\int_\tau^T \mu_s^2 ds < \infty$ since $m, \bar{\sigma}^j$ are bounded and $\sigma^j, \mu^j, X, \theta, n$ are continuous processes for $j \in 1, \dots, N$. Analogously one can show $\int_\tau^T \bar{\Sigma}_s^2 ds < \infty$.

Finally by using that $\sigma^{\bar{k}}, \rho^{\bar{k}} \ge 0$ and that $\theta^{\bar{k}}, \bar{\sigma}^{\bar{k}}$ are bounded away from zero, it is easy to see that

$$\int_\tau^T \frac{1}{\bar{\Sigma}_s^4} ds \le \frac{C}{q_{\bar{k}}^4} \int_\tau^T \frac{1}{(\rho_s^{\bar{k}})^{4\alpha}(1-\rho_s^{\bar{k}})^{4\alpha}} ds \tag{4.13}$$

for some constant C. We can show that the integral on the right-hand side of (4.13) is finite by applying Theorem 2.6 of [37]. ∎

4.1. Analysis of the model. We now comment on our model and specify how the evolution of the bubble described in (4.9) depends on the involved parameters as well as on the structure of the network.

The evolution of the bubble is characterized by two different phases: in the first the bubble builds up, since the quick increase of the signed volume of market orders X dominates in (2.2). However, after a while the processes ρ^k in (4.1) tend towards an equilibrium in which

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the drift of ρ^k vanishes. When this drift's component (and also the contribution of η in (4.5)) is sufficiently small, the mean reverting term of (2.2) starts to dominate, leading to the burst of the bubble and to the second phase, i.e., the decrease of the bubble towards zero.

In the ascending phase, assuming first for illustration purposes the process $(\theta_t)_{t\geq\tau}$ to be constantly equal to $\theta > 0$ and $\bar{\sigma}^j = 0$ for all $0 \leq j \leq N$, the essential force of the bubble is given by the deterministic contagion mechanism (4.11) driving the signed volume of market orders X in (4.5). The contagion accelerates to a maximum and then slows down. In this way X evolves along an "S" shape as shown in Figure 1, growing towards an equilibrium/maximum that is increasing in the volume term θ and the contagion rate λ and decreasing in the recovery rate δ.

Further, the speed at which X grows towards the maximum is increasing in λ and decreasing in δ. However, since both the length and the maximum of observed speculation bubbles are highly uncertain, we randomize this mechanism by letting θ be a stochastic process. The impact of a random volume term θ will be to modify the "S" pattern by allowing the bubble to slow down or pick up in a random way until it reaches a random maximum. In the bursting phase, the dynamics of the bubble will be dominated by the mean reverting factor k, which drives the bubble down.

We now focus on the impact of the choice of the underlying network on the dynamics of the bubble. We compare two different cases, an Erdős–Rényi network with Poisson degree distribution

$$p_j = \frac{e^{-\tilde{\lambda}}\tilde{\lambda}^j}{j!}, \quad j \in \mathbb{N}, \quad \tilde{\lambda} \in \mathbb{R},$$

and a scale-free network with a power law distribution

$$p_j \sim j^{-\gamma}, \quad 2 < \gamma < 3, \quad j \in \mathbb{N}. \tag{4.14}$$

The Erdős–Rényi network has a degree distribution which is very peaked around the mean degree z, whereas the scale-free one, which is well known to better represent real-world information networks (see [44]), has a much larger right tail, which allows for a more heterogeneous degree distribution with some nodes being very connected and others less so (core-periphery structure).

For simplicity, we consider the following deterministic specifications: we set $\bar{\sigma}^j = 0$ for all $0 \leq j \leq N$ and assume the processes $(M_t)_{t\geq\tau}$, $(\Lambda_t)_{t\geq\tau}$, and $(\theta_t)_{t\geq\tau}$ to be constantly equal to $M = \Lambda = \theta = 1$. Further, we choose $\tau = 0$.

We take two different values of γ in (4.14), i.e., $\gamma_1 = 2.2$ and $\gamma_2 = 2.5$, obtaining therefore a more connected network (with $z = z_1 \sim 3.2$) and a less connected one (with $z = z_2 \sim 1.9$). We consider as well two Erdős–Rényi networks with $z = z_1 \sim 3.2$ and $z = z_2 \sim 1.9$, respectively. We take the distribution p_j up to a maximum degree that corresponds to a network with 5000 nodes; see section 3.3.2 of [40].

In Figure 1 we illustrate the trajectories of X for the four different networks, taking $\lambda = 1$, $\delta = 1$. One can notice that both the mean degree and the degree heterogeneity play a key role in the evolution of X: in particular, both of them are positively correlated with the speed of the increase. It can also be seen that in the Erdős–Rényi network, i.e., in the less right skewed one, as well as in the less connected networks, the fraction reaches its equilibrium later in time.

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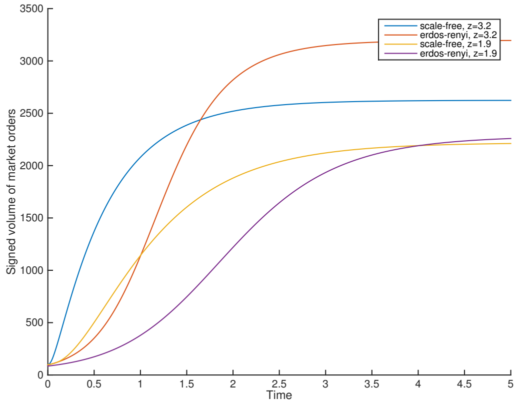

Figure 1. *Deterministic fraction of buyers for different networks, with* $\lambda = 1$, $\delta = 1$.

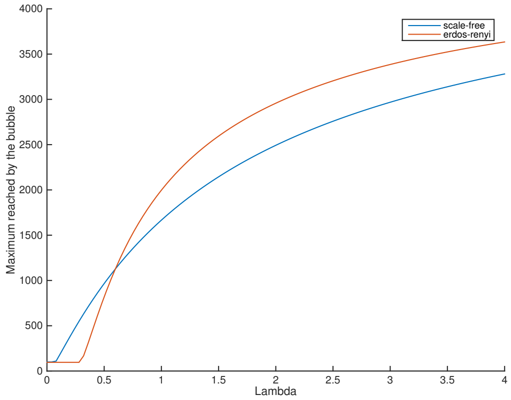

Figure 2. *Maximum value of the bubble as a function of* λ *with* $\delta = 1$, $k = 0.4$.

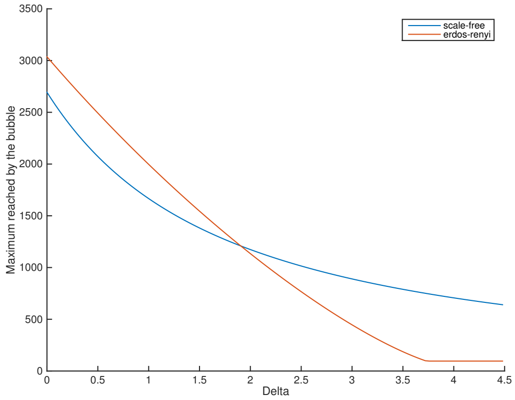

Figure 3. *Maximum value of the bubble as a function of* δ *with* $\lambda = 1$, $k = 0.4$.

We then focus on the behavior of the bubble and consider a mean reversion level $k = 0.4$ in (4.10). In Figures 2 and 3 we show the maximum reached by the bubble as a function of λ and δ, respectively, whereas in Figures 4 and 5 we plot the time needed to reach the maximum, again as a function of λ and δ, respectively.

Figure 5 shows that the time to the maximum is decreasing in δ in the scale-free network and increasing in δ in the Erdős–Rényi one; i.e., the two networks give rise to different behaviors. It can be seen that for small λ and big δ the maximum is higher in the case of the scale-free network, whereas the opposite holds for big λ and small δ. On the other hand, the time needed by the bubble to attain the maximum is always higher in the case of the Erdős–Rényi network.

In our analysis up to this point, we have taken the process $\theta = (\theta_t)_{t \geq \tau}$ to be constant. We now show the influence of the process θ on the dynamics of the bubble assuming that it

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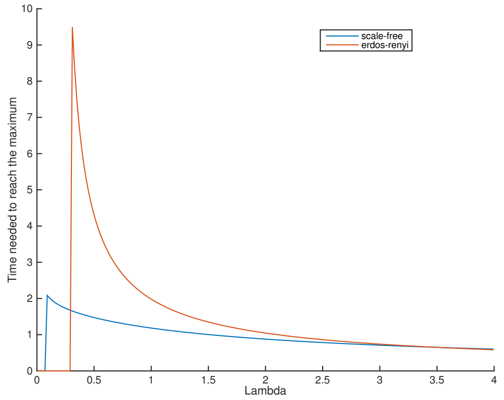

Figure 4. *Time to the maximum as a function of* λ *with* $\delta = 1$, $k = 0.4$.

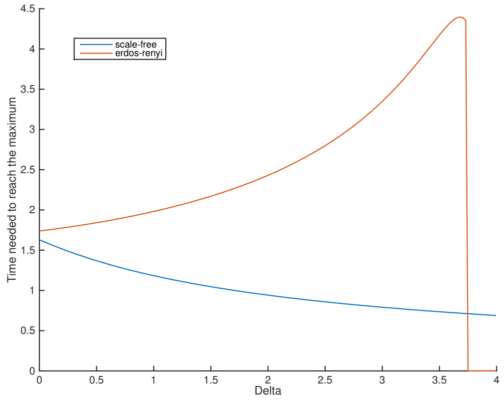

Figure 5. *Time to the maximum as a function of* δ *with* $\lambda = 1$, $k = 0.4$.

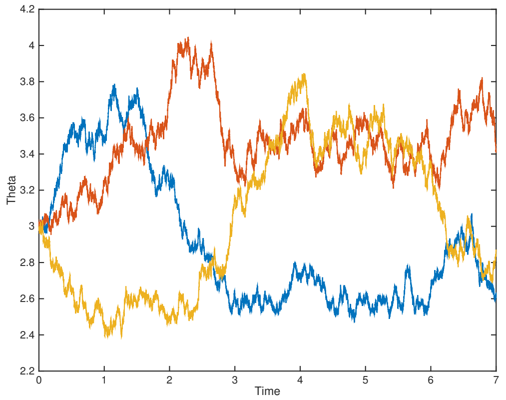

Figure 6. *Example of trajectories of* θ *with* $\mu^\theta = 0$, $\sigma^\theta = 0.2$.

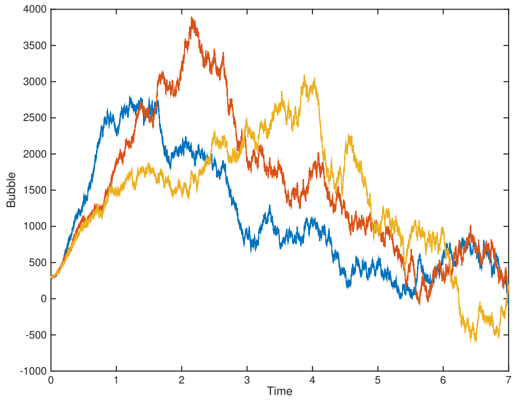

Figure 7. *Corresponding trajectories of the bubble in a scale-free network with mean degree* $z = 3.2$.

satisfies the SDE

$$d\theta_t = \sigma^\theta \theta_t dB_t^3, \quad \tau \leq t < T,$$

where $\sigma^\theta = 0.4$, taking $\delta = 0.2$, $\lambda = 0.4$, $\Lambda = 0.5$, $k = 1$, $\bar{\sigma}^j = 0.1$ for all $0 \leq j \leq N$, $\tau = 0$, $T = 7$, $M = 1$, $\theta_0 = 3$. See, for example, Figures 6 and 7 for the case of a scale-free network with mean degree $z = 3.2$.

The influence of the process θ on the bubble is apparent. If θ has an increase from its initial value, the bubble bursts relatively late (see the yellow dynamics): in this sense, the growth of θ can postpone the burst of the bubble. The other trajectories evolve similarly to each other up to the point where the corresponding processes θ differ. In the blue case, θ decreases and the bubble soon bursts. For the red dynamics, θ increases, making the bubble grow more.

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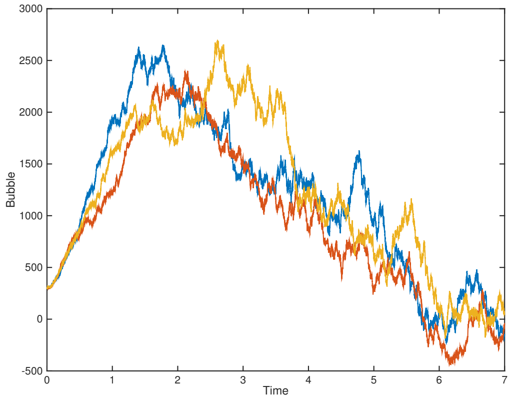

Figure 8. *Example of trajectories for a bubble in a scale-free network.*

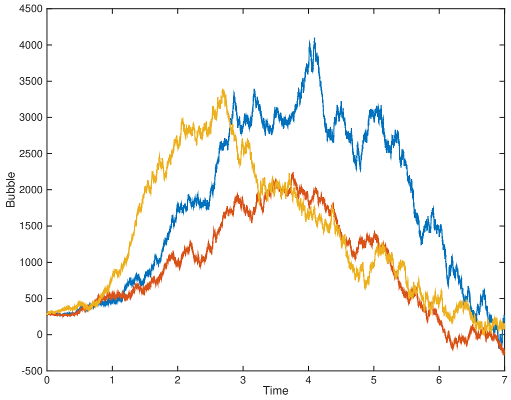

Figure 9. *Example of trajectories for a bubble in an Erdős–Rényi network.*

We conclude the section illustrating the impact of the structure of the network by showing three trajectories of the bubble in Figure 8 for the the scale-free case, and in Figure 9 for the Erdős–Rényi case. We choose $\delta = 0.2$, $\lambda = 0.3$, $\Lambda = 0.5$, $k = 1$, $\bar{\sigma}^j = 0.2$ for all $0 \leq j \leq N$, $\tau = 0$, $T = 3$, $M = 1$, $\theta_0 = 3$, and $\sigma^\theta = 0.2$. We can see that the bubble builds up faster in the scale-free network, but at the same time the trajectories have a steeper decrease, and therefore the effect of the burst of the bubble is more dramatic. On the other hand, Figure 7 shows that a quick decrease of θ can also lead to a quick burst and then to a hard landing.

4.2. Model testing on real data. In this subsection we test some features of our model on real data. Since we were not able to find tick by tick data for the signed volume of market orders of well-known bubbles of the past such as the *dot com bubble*, we consider the asset prices Alphabet Inc. (NASDAQ:GOOG) and Amazon.com Inc. (NASDAQ:AMZN). For these stocks we could obtain tick by tick data for the signed volume of their market orders starting from June 2016. These companies, as can also be seen by the prices reported in Figures 10 and 11, have experienced in the last years a boom, which has brought many financial analysts to propose the presence of a new tech bubble, similar to the *dot com mania* of the late 1990s (see, for example, [9], [42], [52], [53]).

We consider the realized signed volume of market orders since 2016. As shown in Figures 12 and 13, the signed volume tends to increase over time for both Alphabet and Amazon. This behavior indicates the tendency of traders to invest in these companies, contributing to the increase of the price in line with our model.

Our aim is to investigate whether typical trading behavior in a bubble environment is captured in our model. In particular, since we deal with a relatively small time window of a potential bubble, we calibrate the coefficients of the deterministic component $\bar{X}$ for X in (4.11), driving the signed volume of market orders on the observed data for Amazon and Alphabet by employing a quadratic regression. In doing so, we further assume that $\bar{\sigma}^j = 0$ for $j \in 1, \ldots, N$, that the process $(\theta_t)_{t \geq \tau}$ is constant, and that all the nodes of the network have the same degree $d = 3$, i.e., that the degree distribution of our network is a Dirac delta

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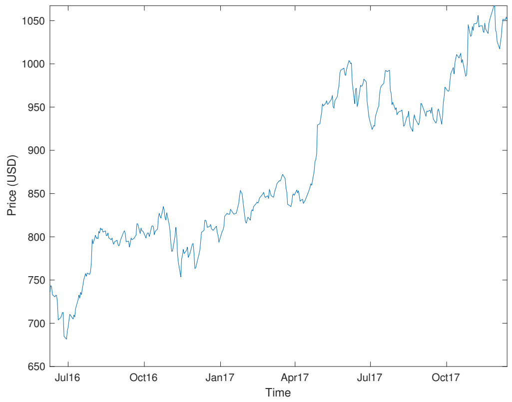

Figure 10. *Price in USD of Alphabet Inc., June* 2016*–October* 2017*.*

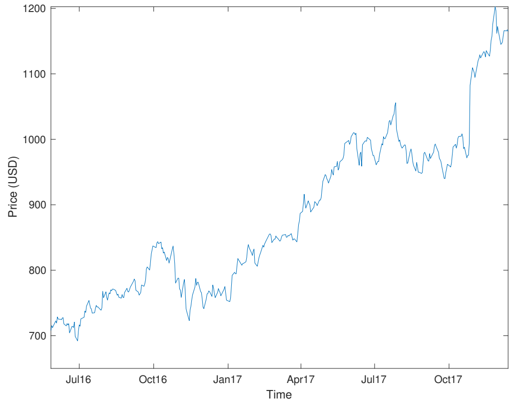

Figure 11. *Price in USD of Amazon.com Inc., July* 2016*–October* 2017*.*

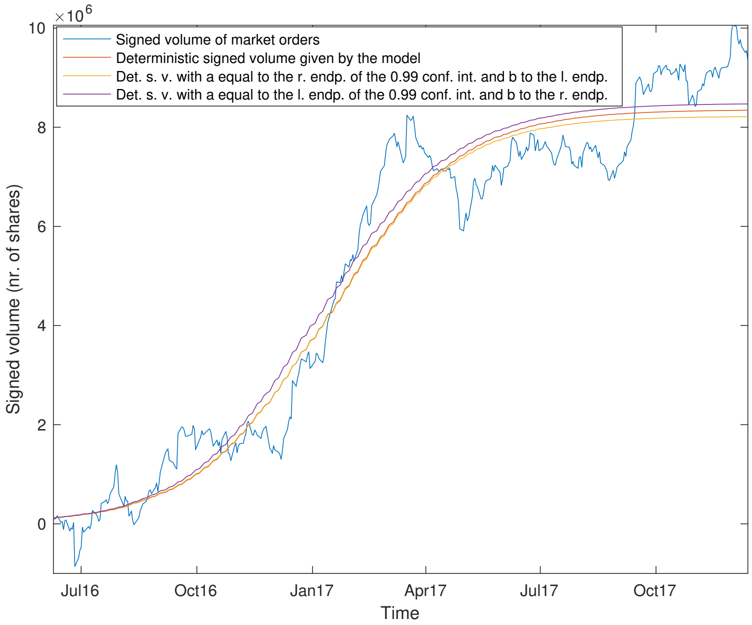

Figure 12. *Realized signed volume of market orders and deterministic signed volume given by* (4.11)*, Alphabet Inc., June* 2016*–October* 2017*.*

centered in $d = 3$. In this way, $\bar{X}$ has dynamics

$$d\bar{X}_t = (-a\bar{X}_t^2 + b\bar{X}_t)dt, \quad t \geq 0, \tag{4.15}$$

where $a = \frac{3\lambda}{\theta}$ and $b = 3\lambda - \delta$.

Further, for every asset we compute the mean relative squared error of the prediction, where the relative squared error at point t_i is defined as $(\bar{X}_{t_i} - X_{t_i}^{obs})^2/(X_{t_i}^{obs})^2$. Here $\bar{X}$ is the

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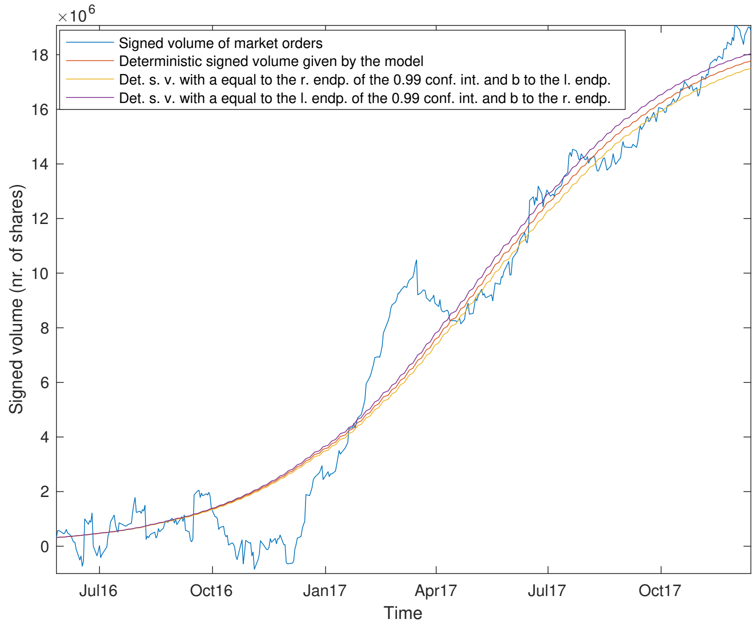

Figure 13. *Realized signed volume of market orders and deterministic signed volume given by* (4.11)*, Amazon.com Inc., June* 2016*–October* 2017*.*

signed volume predicted by our model, and X^{obs} is the observed signed volume. Moreover, we also compute the 99% and 95% confidence intervals for the estimates of the two parameters a and b in (4.15).

The mean relative squared errors are shown in Table 1.

Table 1

Mean relative squared error of the predicted signed volume of market orders versus the observed data for Alphabet Inc. and Amazon.com Inc.

Alphabet Inc.	Amazon.com Inc.
0.0511	0.0291

The confidence intervals of the parameters, together with the selected values, are shown in Tables 2 and 3 for Alphabet Inc. and Amazon.com Inc., respectively.

Table 2

Confidence intervals for parameters a and b in (4.15) *for Alphabet Inc.*

	99% c.i., l. endp.	95% c.i., l. endp.	Parameter	95% c.i., r. endp.	99% c.i., r. endp.
a	1.3013	1.3015	1.3016	1.3017	1.3019
b	10.8271	10.8381	10.8727	10.9074	10.9184

In Figures 12 and 13, we illustrate how our model changes when the parameters a and b are equal to the endpoints of the 99% confidence intervals. In particular, we show the trajectories of the predicted signed volume of market orders when a is equal to the right endpoint of

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Table 3

Confidence intervals for parameters a and b in (4.15) for Amazon.com Inc.

	99% c.i., l. endp.	95% c.i., l. endp.	Parameter	95% c.i., r. endp.	99% c.i., r. endp.
a	0.3557	0.3559	0.3560	0.3561	0.3563
b	6.6935	6.7113	6.7676	6.8239	6.8417

the confidence intervals and b to the left one and vice versa, i.e., when a is equal to the left endpoint and b to the right one.

In Figures 12 and 13 we can observe the "S" behavior discussed in subsection 4.1. We remark that since we perform a local analysis by considering a specific short time window with constant θ, this behavior cannot be directly interpreted as an indication of a decreasing phase of the bubble. In the next time window the signed volume may start to grow steeply again, due to the impact of a stochastic θ. In this case the curve describing the evolution of the signed volume would also grow for a longer time, distorting the "S" shape as illustrated in Figures 6 and 7.

We can conclude that the analysis shows the flexibility of our model and its capacity to

1. describe both the increasing and the descending phase of the bubble;
2. capture the impact of signed volume market orders on bubbles' formation and burst;
3. take into account the underlying network structure in the contagion process of a bubble's evolution;
4. describe typical features of a bubble's behavior such as steep increase and hard landing.

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