# Liquidity Based Modeling of Asset Price Bubbles via Random Matching[*]

Francesca Biagini[†], Andrea Mazzon[‡], Thilo Meyer-Brandis[†], and Katharina Oberpriller[†]

**Abstract.** In this paper we study the evolution of asset price bubbles driven by contagion effects spreading among investors via a random matching mechanism in a discrete-time version of the liquidity based model of [R. A. Jarrow, P. Protter, and A. F. Roch, *Quant. Finance*, 12 (2012), pp. 1339–1349]. To this scope, we extend the Markov conditionally independent dynamic directed random matching of [D. Duffie, L. Qiao, and Y. Sun, *J. Econ. Theory*, 174 (2018), pp. 124–183] to a stochastic setting to include stochastic exogenous factors in the model. We derive conditions guaranteeing that the financial market model is arbitrage-free and present some numerical simulation illustrating our approach.

**Key words.** asset price bubbles, dynamic directed random matching with stochasic intensities, contagion, liquidity

**MSC codes.** 60G07, 91G15, 91G30

**DOI.** 10.1137/22M1531580

**1. Introduction.** The formation of asset price bubbles has been the object of many investigations in the economic and mathematical literature. Different causes have been indicated as triggering factors for bubble birth and evolution, such as a risk shifting problem in [2], the joint effect of the individual incentive to *time the market* and the inability of arbitrageurs to coordinate their selling strategies in [1], heterogeneous beliefs between interacting traders as in [19], [21], [37], [38], [46], [47], a disruption of the dynamic stability of the financial system in [11], [10], the diffusion of new investment decision rules from a few expert traders to a larger population of amateurs in [17], the tendency of investors to adopt the behavior of other agents in [28], and the presence of short-selling constraints in [35] and of noise traders with erroneous stochastic beliefs in [13].

However, mathematical models for microfinancial interactions leading to the formation of asset price bubbles are still missing. This paper aims at filling this gap by studying a random matching mechanism among investors which impacts the trading volume of an asset and then its price via illiquidity effects. To this purpose, we first introduce a discrete time version of the liquidity based model of [27], where the fundamental price of the asset is exogenously given, while the market price is influenced by the trading activities of investors via an erosion of the

[†]Workgroup Financial Mathematics, Department of Mathematics, Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 Munich, Germany (biagini@math.lmu.de, oberpriller@math.lmu.de, meyerbr@math.lmu.de).
[‡]Department of Economics, University of Verona, Via Cantarane 24, 37129 Verona, Italy (andrea.mazzon@univr.it).

limit order book. The birth of a bubble is then caused by a deviation of the market value from the fundamental one.

Here, we model the signed volume of market orders by assuming that the investment attitudes of the traders on the market are influenced via a random matching mechanism. To this scope, we suppose that agents on the market can be of three types, i.e., optimistic, neutral, and pessimistic regarding the future returns of the asset, and that they trade according to their type. This means that an optimistic agent places a buy market order while a pessimistic one places a selling market order. Neutral agents neither buy nor sell the asset. The evolution of the signed volume of market orders is thus determined by the fraction of optimistic, neutral, and pessimistic agents, respectively.

In order to model the evolution of these quantities, we extend the Markov conditionally independent dynamic directed random matching of [15] to a stochastic setting. More precisely, the model in [15] describes a mechanism for how a continuum of agents search in a directed way for a suitable counterparty. The word "directed" refers to the fact that the search is not purely random, but the agents are motivated to meet another agent that provides them with some benefit. In particular, every agent is described by its type which may change at any time step, and can randomly mutate to another type and randomly match with another agent. This meeting may induce a further type change. Furthermore, agents can also enter some potentially enduring partnerships with random break-up times.

These models have a broad application, for example, in the fields of financial markets, monetary theory, and labor economics. The first mathematical basis for this approach in a discrete time setting is provided in [15] and strongly relies on techniques of nonstandard analysis, as a continuum of agents is considered. Given some deterministic functions describing the probabilities associated to the random matching and random changes introduced above, they prove existence of a dynamical system with independent agents types and deterministic cross-sectional distribution of types. We now extend this model by allowing the probabilities driving the system to also depend on an additional state of the world to allow the random matching mechanism to be driven by some stochastic exogenous factors. Hereby, the technical difficulty is to find a suitable setting to extend the results in [15] in a consistent way. For this purpose we construct a probability space $\Omega$ as the product of the space $\hat{\Omega}$ of the random matching and the space $\tilde{\Omega}$ of the factors which may influence the transition probabilities, and introduce a Markov kernel on $\Omega$. After proving the existence of such a dynamical system with input processes, we study conditional type distributions.

We then apply these results to model investment attitudes leading to bubble formation in the discretized version of [27]. More precisely, we assume that the signed volume of market orders is described by a random matching mechanism, where the agents can be of positive, negative, or neutral type, as explained above. The stochasticity of the transition probabilities is crucial here as it reflects the impact of heterogeneous factors such as socioeconomic indicators, external events, and public news. We are able to show that the market model is arbitrage-free by proving the existence of an equivalent martingale measure under suitable assumptions. Furthermore, we provide some examples for the input processes of the random matching mechanism in an arbitrage-free market model. We illustrate these results with numerical simulations.

The paper is organized as follows. In section 2 we introduce a discrete time version of the liquidity based model of asset prices in [27]. In section 3 we extend the directed random matching mechanism in [15] to a stochastic setting. We combine these two constructions in section 4, where we propose a model of the signed volume of market orders influenced by a random matching mechanism. In this setting we derive some conditions guaranteeing that the financial market model is arbitrage-free and we conclude with some numerical simulations.

**2. The formation of asset price bubbles.** We consider a word-of-mouth mechanism spreading among investors who meet by random matching, giving rise to the formation of asset price bubbles. To this scope, we introduce a discretized version of the liquidity based model in [27].

**2.1. A liquidity based model for asset price bubbles in discrete time.** We here present a discrete time version of the continuous time model of [27], which explains the birth of bubbles as the deviation of the market price $S$ from the fundamental price $F$ caused by the impact of trading volume and illiquidity.

Let $T > 0$ be a given trading horizon and consider a time discretization $0 =: t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ of the interval $[0, T]$. Also introduce a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}^i)_{i=0,\dots,N}, P)$, where we set $\mathcal{F}^i := \mathcal{F}_{t_i}$ for $i = 1, \dots, N$. In section 3 we further specify a possible construction of this space in the context of random matching. The market model consists of the money market account $B \equiv 1$ and one liquid financial asset (stock), which is traded through limit and market orders.

*Remark* 2.1. In order to be consistent with the notation of the random matching mechanism (see section 3), we indicate the dependence on time $t_i$ with a superscript $i$ for filtrations or processes.

The fundamental price of the asset is given by the stochastic process $F = (F^i)_{i=0,\dots,N}$, where $F^i$ represents the value at time $t_i$ for $i = 0, \dots, N$. Such a process is exogenously given. On the other hand, the market price of the asset is generated by the trading activity of the investors as we describe in the following. In particular, agents place buy and sell orders through a limit order book, which acts as market maker.

Coherently with the construction of [27], we assume that the average price to pay per share for a transaction of size $x$ via a market order at time $t_i$ is given by

$$(2.1) \qquad S^i(x) = S^i + M^i x, \quad x \in \mathbb{R}^+, \ i = 0, \dots, N,$$

where $S = (S^i)_{i=0,\dots,N}$ and $M = (M^i)_{i=0,\dots,N}$ are nonnegative, adapted processes on the space $(\Omega, \mathcal{F}, (\mathcal{F}^i)_{i=0,\dots,N}, P)$, representing the quoted price and a measure of illiquidity, respectively. Fix a time $t_i$ for $i = 1, \dots, N$. The limit order book at $t_i$ is described by the density function $\rho^i(\cdot)$, where $\rho^i(z)$ is the number of shares offered at price $z$ at time $t_i$. As in [27], the total amount paid by a trader who wants to buy $x$ shares at time $t_i$ is given by

$$(2.2) \qquad \int_{S^i}^{z_x} z \rho^i(z) dz,$$

where $z_x$ is the solution of

$$\int_{S^i}^{z_x} \rho^i(z) dz = x.$$

Due to the linear structure in (2.1) it follows that $\rho^i(z) = 1/2M^i$ and $z_x = S^i + 2M^i x$; see [27] for further details.

Let $X = (X^i)_{i=0,\ldots,N}$ be an adapted stochastic process representing the signed volume of aggregate market orders (buy minus sell orders). Next, we introduce a process $R = (R^i)_{i=0,\ldots,N}$ with values in $[0,1]$ to describe the short-term resiliency of the limit order book. In particular, if $\Delta X$ buy market orders are executed at time $t_i$, $R^i$ represents the proportion of new sell limit orders placed from $t_i$ to $t_{i+1}$, having therefore the effect to partly fill the temporary gap $[S^i, S^i + \Delta X]$ in the limit order book. If the gap caused by the new buy market orders is not fully filled before other market orders are executed, the market price of the asset deviates from the fundamental value, thus creating a bubble. However, it is observed that such a deviation decays in the long run; see [27] for details. Such an effect is quantified by the speed of decay process $\kappa = (\kappa^i)_{i=0,\ldots N}$.

The evolution of the market price process $S = (S^i)_{i=1,\ldots,N}$ is then given by

$$(2.3) \qquad S^i = S^{i-1} + F^i - F^{i-1} - \kappa^i(S^{i-1} - F^{i-1})\Delta t_i + 2\Lambda^i M^i \Delta X^i, \quad i = 1,\ldots,N,$$

where $\Lambda^i := 1 - R^i$, $i = 0,\ldots,N$. Moreover, $\Delta t_i := t_i - t_{i-1}$, $\Delta X^i := X^i - X^{i-1}$ for $i = 1,\ldots,N$. At initial time we have $X^0 = 0$ and $S^0 = F^0$. In particular, (2.3) is a discretized version of the SDE considered in [27].

Following [27], we now provide the definition of an asset price bubble in this setting.

**Definition 2.2.** *An* asset price bubble $\beta = (\beta^i)_{i=0,\ldots,N}$ *is defined as*

$$\beta^i := S^i - F^i, \quad i = 0,\ldots,N.$$

*By* (2.3) *we obtain that*

$$(2.4) \qquad \beta^i = \beta^{i-1} - \kappa^i \beta^{i-1}\Delta t_i + 2\Lambda^i M^i \Delta X^i, \quad i = 1,\ldots,N,$$
$$\beta_0 = 0.$$

*Remark* 2.3. In this construction, the birth of the bubble occurs when $M$, $\Lambda$, and $\Delta X$ are strictly positive. Afterward the bubble can revert to zero due to the declining impact of $M\Lambda\Delta X$ or to the growing influence of the mean reverting term $-\kappa\beta\Delta t$, when the size of $\beta$ increases.

Equation (2.4) shows that the main force driving the bubble evolution is the signed volume of market orders $X$. Market orders, indeed, deplete or fill in the limit order book, producing a variation in the price over a small interval of time. If new market orders quickly enter before the price has time to decay again to the fundamental value, these short-term price variations may accumulate and result in a deviation from the fundamental wealth.

We now focus on modeling $X$ by assuming that the investment attitudes of the traders on the market are influenced via a random matching mechanism. To this scope, we suppose that agents on the market can be of three types, i.e., optimistic, neutral, and pessimistic regarding the future returns of the asset, and that they trade according to their type. This means that an optimistic agent places a buy market order while a pessimistic one places a selling market order. Neutral agents neither buy nor sell the asset. Based on this characterization, from

now on we refer to optimistic and pessimistic agents also as *buyers* and *sellers,* respectively. We admit that agents may influence each other if they meet and that they may change their type at each time $t_i$, $i = 1, \ldots, N$, via a random matching mechanism as we explain in section 3. The evolution of $X$ is determined by the processes $\hat{p}_i = (\hat{p}_i^j)_{j=0,\ldots,N}$, $i = 1, 2, 3$, standing for the fraction of optimistic, neutral, and pessimistic agents, respectively. In particular, the value of $X$ at time $t_i$ is given by

$$(2.5) \qquad X^i = \Theta^i(\hat{p}_1^i - \hat{p}_3^i), \quad i = 0, \ldots, N,$$

where $\Theta = (\Theta^i)_{i=0,\ldots,N}$ is an adapted stochastic process modeling the average size of buy market orders as in [8]. We assume that at time $t_0 = 0$ it holds that $\hat{p}_0^1 = \hat{p}_3^i$, i.e., that the fraction of optimistic agents is equal to that of pessimistic ones, so that $X^0 = 0$. We now model the evolution of the fractions $\hat{p}_1$, $\hat{p}_2$, and $\hat{p}_3$ by using a special case of the *Markov conditionally independent dynamic directed random matching* which we introduce in the next section.

**3. Markov conditionally independent dynamic directed random matching.** We here extend the setting in [15] by using stochastic kernels. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be a probability space and $(\hat{\Omega}, \hat{\mathcal{F}})$ another measurable space. We consider the product space

$$(3.1) \qquad (\Omega, \mathcal{F}) := (\tilde{\Omega} \times \hat{\Omega}, \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}})$$

and denote the elements of $\Omega$ by $\omega := (\tilde{\omega}, \hat{\omega})$. Let $\hat{P}$ be a Markov kernel[1] (or stochastic kernel) from $\tilde{\Omega}$ to $\hat{\Omega}$. Given $\tilde{\omega} \in \tilde{\Omega}$, we set $\hat{P}^{\tilde{\omega}} := \hat{P}(\tilde{\omega})$ with a slight notational abuse. We then introduce a probability measure $P$ on $(\Omega, \mathcal{F})$ as the semidirect product of $\tilde{P}$ and $\hat{P}$, that is,

$$(3.2) \qquad P(\tilde{A} \times \hat{A}) := (\tilde{P} \ltimes \hat{P})(\tilde{A} \times \hat{A}) = \int_{\tilde{A}} \hat{P}^{\tilde{\omega}}(\hat{A}) d\tilde{P}(\tilde{\omega})$$

for $\tilde{A} \in \tilde{\mathcal{F}}$, $\hat{A} \in \hat{\mathcal{F}}$ and consider the probability space $(\Omega, \mathcal{F}, P)$ representing all possible states of the world. We assume that the probability space $(\Omega, \mathcal{F}, P)$ is endowed with a filtration $\mathbb{F} = (\mathcal{F}_n)_{n \geq 1}$.

We then introduce the space of agents as given by an atomless probability space $(I, \mathcal{I}, \lambda)$. Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be a rich Fubini extension[2] of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. We classify all agents in $I$ according to their type belonging to the finite space $S = \{1, 2, \ldots, K\}$. We say that an agent has type $J$ if he is not matched. We denote $\hat{S} := S \times (S \cup \{J\})$ and call it the extended type space. If an agent has the extended type $(k, l)$, this means that she is of type $k \in S$ and is currently matched to another agent of type $l \in S$. If an agent of type $k$ is not matched at the moment, the agent's extended type is $(k, J)$. We consider probability distributions on $\hat{S}$. In particular, we introduce the space $\hat{\Delta}$ of extended type distributions,

---

[1]We recall the definition of a Markov kernel from Definition 8.25 in [31] for the reader's convenience.

    *Definition 3.1. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}), (\hat{\Omega}, \hat{\mathcal{F}})$ be measurable spaces. A map $\kappa : \tilde{\Omega} \times \hat{\mathcal{F}} \to [0, \infty]$ is called a Markov kernel or stochastic kernel from $\tilde{\Omega}$ to $\hat{\Omega}$ if*

       1. *$\kappa(\cdot, \hat{A})$ is $\tilde{\mathcal{F}}$-measurable for any $\hat{A} \in \hat{\mathcal{F}}$;*

       2. *$\kappa(\tilde{\omega}, \cdot)$ is a probability measure on $(\hat{\Omega}, \hat{\mathcal{F}})$ for any $\tilde{\omega} \in \tilde{\Omega}$.*

[2]For the definition of such a Fubini extension, see Appendix A.

which is the set of probability distributions $p$ on $\hat{S}$ satisfying $p(k,l) = p(l,k)$ for any $k$ and $l$ in $S$. We endow $\hat{\Delta}$ with the topology $\mathcal{T}^{\Delta}$ induced by the topology on the space of matrices with $|S|$ rows and $|S| + 1$ columns. Moreover, let $\hat{p} = (\hat{p}^n)_{n \geq 1}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$ with values in $\hat{\Delta}$, representing the evolution of the underlying extended type distribution. We assume that $\hat{p}^0$ is deterministic.

We now introduce a dynamical system where agents match with each other and change type with probabilities depending on exogenous, random factors defined on the space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. As random matching mechanisms are used to model decentralized environments, e.g., OTC markets or labor markets, this has a wide area of applications. In the context of [30] and [34], where agents meet to trade with goods, stochastic processes on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ can model factors which influence the availability of the good on the market, e.g., weather when the good is food. In [34], where in addition two different currencies are introduced, a further stochastic process defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ can be used to describe the exchange rate. Moreover, in the labor market model of [3], the space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ can be used to represent the stock market which is an indicator of the general economic situation. A further application for asset price bubbles modeling is given in section 4.

The rest of the section is devoted to defining and constructing such a dynamical system together with a suitable Fubini extension. Our presentation is structured as follows. We first give the definition of random matching and of dynamical system; see Definitions 3.2 and 3.3, respectively. We then introduce stochastic processes on $(\Omega, \mathcal{F}, P)$ which identify the probabilities of matching and type change. These processes are the parameters of our model and depend in particular on the state of the world $\tilde{\omega} \in \tilde{\Omega}$ and on the current distribution of types $\hat{p}$. In Definition 3.5 we introduce a specific dynamical system characterized by these parameters. In Theorem 3.13 we then show that for the introduced input parameters there exists a rich Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which our dynamical system can be constructed. In order to do that, we fix $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and exploit (3.1)–(3.2). We finally show relevant properties of the dynamical system in Theorem 3.14.

We now describe a matching mechanism among the agents, by following Definition 2 in [16].

**Definition 3.2.**
1. *A full matching $\phi : I \to I$ is a one-to-one mapping such that for each $i \in I$, $\phi(i) \neq i$, and $\phi(\phi(i)) = i$.*
2. *A (partial) matching $\psi$ is a matching from $I$ to $I$ such that for some subset $B$ of $I$, the restriction of $\psi$ to $B$ is a full matching on $B$, and $\psi(i) = i$ on $I \setminus B$. This means that agent $i$ is matched with agent $\psi(i)$ for $i \in B$, whereas any agent $i$ not in $B$ is unmatched, represented by setting $\psi(i) = i$.*
3. *A random matching on $\Omega$ is a mapping $\pi : I \times \Omega \to I$ such that $\pi_{\omega} := \pi(\cdot, \omega)$ is a matching for each $\omega \in \Omega$.*

In what follows we use "matching" to denote a partial matching for the sake of simplicity. We now provide the general definition of dynamical systems.

*Definition* 3.3. *A dynamical system* $\mathbb{D}$ *defined on* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *is a triple* $(\alpha, \pi, g) = (\alpha^n, \pi^n, g^n)_{n \in \mathbb{N} \setminus \{0\}} =: (\Pi_n)_{n \in \mathbb{N} \setminus \{0\}}$ *such that for each integer period* $n \geq 1$ *we have the following:*

1. $\alpha^n : I \times \Omega \to S$ *is the* $\mathcal{I} \boxtimes \mathcal{F}$-*measurable agent-type function. The corresponding end-of-period type of agent* $i$ *under the realization* $\omega \in \Omega$ *is given by* $\alpha^n(i, \omega) \in S$.
2. *A random matching* $\pi^n : I \times \Omega \to I$, *describing the end-of-period agent* $\pi^n(i)$ *to whom agent* $i$ *is currently matched, if agent* $i$ *is currently matched. If agent* $i$ *is not matched, then* $\pi^n(i) = i$. *The associated* $\mathcal{I} \boxtimes \mathcal{F}$-*measurable partner-type function* $g^n : I \times \Omega \to S \cup \{J\}$ *is given by*

$$g^n(i, \omega) = \begin{cases} \alpha^n(\pi^n(i, \omega), \omega) & \text{if } \pi^n(i, \omega) \neq i, \\ J & \text{if } \pi^n(i, \omega) = i, \end{cases}$$

*providing the type of the agent to whom agent* $i$ *is matched, if agent* $i$ *is matched, or* $J$ *if agent* $i$ *is not matched.*

A dynamical, directed random matching mechanism was studied for the first time in [15], where the probabilities describing the random matching only depend on the current probability distribution on the space of pairs consisting of an agent's type and the type of the agent with whom she is matched. We now generalize this approach by allowing the probabilities also to depend on the state of the world, which influences the random matching. Hereby, the technical difficulty is to find a suitable setting to extend the results in [15] in a consistent way, as the existence of the random matching system in [15] relies extensively on techniques of nonstandard analysis. This is necessary in order to construct a Fubini extension by working with Loeb spaces which satisfy a Fubini property.

In this setting we describe how agents may change their type by random matching with other agents. Consider time periods $(n)_{n \geq 1}$.

Each time period $n$ can be divided into three steps: *mutation, random matching,* and *match-induced type changing with break-up.* In the first step, agents might mutate their type, and in the second step, any currently unmatched agent can be matched. Finally, in the third step a currently matched pair of agents can either break up or stay in their relationship. They can also change type as a consequence of their matching or their break up. We assume that the probabilities of these mechanisms depend on input processes $(\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n)_{n \geq 1}$ on $(\Omega, \mathcal{F}, P)$ as we describe next. Here, $(\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n)$ are matrix valued processes with

$$(\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n) = (\eta^n_{kl}, \theta^n_{kl}, \xi^n_{kl}, \sigma^n_{kl}[r, s], \varsigma^n_{kl}[r])_{k, l, r, s \in S \times S \times S \times S}$$

for $n \geq 1$. In particular, we assume that the input processes are of the following form. For any $n \geq 1$, $k, l \in S$, and any $\iota = \eta, \theta, \xi, \sigma[r, s], \varsigma[r]$ we set

$$\iota^n_{kl} : (\Omega, \mathcal{F}, P) \to ([0, 1], \mathcal{B}([0, 1])),$$

(3.3)
$$\iota^n_{kl}(\omega) := \iota_{kl}(\tilde{\omega}, n, \hat{p}^n(\tilde{\omega}, \hat{\omega}))$$

with $\iota_{kl} : \tilde{\Omega} \times \mathbb{N} \times \hat{\Delta} \to [0, 1]$. Precisely, if $\hat{p}^n(\tilde{\omega}, \hat{\omega})$ is the underlying extended type distribution at time $n \in \mathbb{N}$ under the scenario $(\tilde{\omega}, \hat{\omega}) \in \Omega$, then

- $\eta_{kl}(\tilde{\omega}, n, \hat{p}^n(\tilde{\omega}, \hat{\omega}))$ is the probability that an agent of type $k$ becomes an agent of type $l$ at the first, mutation step time $n$ given $\tilde{\omega}$;
- $\theta_{kl}(\tilde{\omega}, n, \hat{p}^n(\tilde{\omega}, \hat{\omega}))$ is the probability that an unmatched agent of type $k$ is matched to an agent of type $l$ given the scenario $\tilde{\omega}$ at time $n$; we define $b_k^n(\omega) := 1 - \sum_{l \in S} \theta_{kl}^n(\omega)$;
- $\xi_{kl}(\tilde{\omega}, n, \hat{p}^n(\tilde{\omega}, \hat{\omega}))$ is the probability that a matched pair of types $k$ and $l$ breaks up under $\tilde{\omega}$ at time $n$;
- $\sigma_{kl}[r, s](\tilde{\omega}, n, \hat{p}^n(\tilde{\omega}, \hat{\omega}))$ is the probability that a matched pair of agents of respective types $k$ and $l$, which stays in their relationship, becomes a pair of agents of type $r$ and $s$, at the third time step at time $n$ given $\tilde{\omega}$;
- $\varsigma_{lk}^n[r](\tilde{\omega}, n, \hat{p}^n(\tilde{\omega}, \hat{\omega}))$ is the probability that given a pair of agents of type $k$ and $l$ that break up at time $n$, the agent of type $k$ becomes an agent of type $r$ given $\tilde{\omega}$.

We assume the following:

- for any $n \geq 1$, $k, l \in S$ and every $\iota = \eta, \theta, \xi, \sigma[r, s], \varsigma[r]$, the function $\iota_{kl}^n$ is $(\mathcal{F}, \mathcal{B}([0, 1]))$-measurable.
- $(\eta^0, \theta^0, \xi^0, \sigma^0, \varsigma^0)$ are deterministic, i.e.,

$$\eta(0, \cdot), \theta(0, \cdot), \xi(0, \cdot), \sigma(0, \cdot), \varsigma(0, \cdot) : \hat{\Delta} \to [0, 1].$$

- For every $n \geq 1$ and each $k \in S$ and $\omega \in \Omega$ it holds that

$$(3.4) \qquad \sum_{l \in S} \eta_{kl}^n(\omega) = 1.$$

- For all $k, l \in S, \tilde{\omega} \in \tilde{\Omega}$, and $n \in \mathbb{N}$ the function $\hat{p}_{kJ}\theta_{kl}(\tilde{\omega}, n, \hat{p})$ is continuous in $\hat{p} \in \hat{\Delta}$ with respect to the topology $\mathcal{T}^\Delta$. Moreover, for any $k, l \in S, \tilde{\omega} \in \tilde{\Omega}, n \in \mathbb{N}$, and $\hat{p} \in \hat{\Delta}$ we suppose the following to hold:

$$(3.5) \qquad \hat{p}_{kJ}\theta_{kl}(\tilde{\omega}, n, \hat{p}) = \hat{p}_{lJ}\theta_{lk}(\tilde{\omega}, n, \hat{p}) \quad \text{and} \quad \sum_{r \in S} \theta_{kr}(\tilde{\omega}, n, \hat{p}) \leq 1.$$

- For every $n \geq 1$, $k, l \in S$, it holds that $\xi_{kl}^n = \xi_{lk}^n$.
- For every $k, l, r, s \in S$ and each $\omega \in \Omega$ it holds that

$$(3.6) \qquad \sum_{r, s \in S} \sigma_{kl}^n[r, s](\omega) = 1 \quad \text{and} \quad \sigma_{kl}^n[r, s](\omega) = \sigma_{lk}^n[s, r](\omega).$$

- For every $n \geq 1$, $k, l, r \in S$ and each $\omega \in \Omega$ it holds that

$$(3.7) \qquad \sum_{r \in S} \varsigma_{kl}^n[r](\omega) = 1.$$

*Remark* 3.4. Note that for fixed $\tilde{\omega} \in \tilde{\Omega}$, the processes $(\eta, \theta, \xi, \sigma, \varsigma)$ are deterministic and depend only on the current time $n$ and the extended type distribution $\hat{p}^n$ at time $n$. This means that for fixed $\tilde{\omega} \in \tilde{\Omega}$ the setting boils down to the framework of [15]. While in [15] only the input parameter $\theta$, i.e., the parameter determining the random matching step, depends on $\hat{p}$ and the other parameters $\eta, \xi, \sigma, \varsigma$ are independent of $\hat{p}$, we consider a dependence on $\hat{p}$ in every parameter. This allows us to take into account self-reinforcement mechanisms fueling

the shift of agents toward a given type; see also our application in section 4. Moreover, $\theta^n$ must depend on $\hat{p}^n$ in order for (3.5) to hold.

Next, we use the processes $(\eta, \theta, \xi, \sigma, \varsigma)$ introduced above to construct a specific dynamical system $\mathbb{D}$ as in Definition 3.3 with random probabilities, as an extension of a dynamical system in [15].

**Definition 3.5.** *Let an initial condition $\Pi^0 = (\alpha^0, \pi^0, g^0)$ of $\mathbb{D}$ be given, i.e., functions $\alpha^0 : I \to S$, $\pi^0 : I \to I$ and $g^0 : I \to S \cup \{J\}$. A dynamical system $\mathbb{D}$ defined on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ with parameters $(\eta, \theta, \xi, \sigma, \varsigma) = (\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n)_{n \geq 1}$ is defined as follows.*

*Assume that $\Pi^{n-1} = (\alpha^{n-1}, \pi^{n-1}, g^{n-1})$ is given for some $n \geq 1$, and define $\Pi^n = (\alpha^n, \pi^n, g^n)$ by characterizing the three substeps of random change of types of agents, random matchings, break-ups, and possible type changes after matchings and break-ups as follows.*

**Mutation.** *For $n \geq 1$ consider an $\mathcal{I} \boxtimes \mathcal{F}$-measurable postmutation function*

$$\bar{\alpha}^n : I \times \Omega \to S.$$

*In particular, $\bar{\alpha}_i^n(\omega) := \bar{\alpha}^n(i, \omega)$ is the type of agent $i$ after the random mutation under the scenario $\omega \in \Omega$. The type of the agent to whom an agent is matched is identified by an $\mathcal{I} \boxtimes \mathcal{F}$-measurable function*

$$\bar{g}^n : I \times \Omega \to S \cup \{J\},$$

*given by*

$$\bar{g}^n(i, \omega) = \bar{\alpha}^n(\pi^{n-1}(i, \omega), \omega)$$

*for any $\omega \in \Omega$. In particular, $\bar{g}_i^n(\omega) := \bar{g}^n(i, \omega)$ is the type of the agent to whom agent $i$ is matched under the scenario $\omega \in \Omega$. Given $\hat{p}^{n-1}$ and $\tilde{\omega} \in \tilde{\Omega}$, for any $k_1, k_2, l_1,$ and $l_2$ in $S$, for any $r \in S \cup \{J\}$, for $\lambda$-almost every agent $i$, by the definition of $(\eta_{k,l})_{k,l \in S}$ we have*

$$(3.8) \quad \begin{aligned} &\hat{P}^{\tilde{\omega}} \left( \bar{\alpha}_i^n(\tilde{\omega}, \cdot) = k_2, \ \bar{g}_i^n(\tilde{\omega}, \cdot) = l_2 | \alpha_i^{n-1}(\tilde{\omega}, \cdot) = k_1, \ g_i^{n-1}(\tilde{\omega}, \cdot) = l_1, \ \hat{p}^{n-1}(\tilde{\omega}, \cdot) \right) (\hat{\omega}) \\ &\quad = \eta_{k_1, k_2} \left( \tilde{\omega}, n, \hat{p}^{n-1}(\tilde{\omega}, \hat{\omega}) \right) \eta_{l_1, l_2} \left( \tilde{\omega}, n, \hat{p}^{n-1}(\tilde{\omega}, \hat{\omega}) \right), \end{aligned}$$

$$(3.9) \quad \begin{aligned} &\hat{P}^{\tilde{\omega}} \left( \bar{\alpha}_i^n(\tilde{\omega}, \cdot) = k_2, \ \bar{g}_i^n(\tilde{\omega}, \cdot) = r | \alpha_i^{n-1}(\tilde{\omega}, \cdot) = k_1, \ g_i^{n-1}(\tilde{\omega}, \cdot) = J, \ \hat{p}^{n-1}(\tilde{\omega}, \cdot) \right) (\hat{\omega}) \\ &\quad = \eta_{k_1, k_2} \left( \tilde{\omega}, n, \hat{p}^{n-1}(\tilde{\omega}, \hat{\omega}) \right) \delta_J(r). \end{aligned}$$

*We then define*

$$\bar{\beta}^n(\omega) = (\bar{\alpha}^n(\omega), \bar{g}^n(\omega)), \quad n \geq 1.$$

*The postmutation extended type distribution realized in the state of the world $\omega \in \Omega$ is denoted by $\check{p}(\omega) = (\check{p}^n(\omega)[k, l])_{k \in S, l \in S \cup \{J\}}$, where*

$$(3.10) \qquad \check{p}^n(\omega)[k, l] := \lambda(\{i \in I : \bar{\alpha}^n(i, \omega) = k, \ \bar{g}^n(i, \omega) = l\}).$$

**Matching.** *We introduce a random matching $\bar{\pi}^n : I \times \Omega \to I$ and the associated post-matching partner-type function $\bar{\bar{g}}^n$ given by*

$$\bar{\bar{g}}^n(i, \omega) = \begin{cases} \bar{\alpha}^n(\bar{\pi}^n(i, \omega), \omega) & \text{if } \bar{\pi}^n(i, \omega) \neq i, \\ J & \text{if } \bar{\pi}^n(i, \omega) = i, \end{cases}$$

*satisfying the following properties:*

1. $\bar{\bar{g}}^n$ is $\mathbb{I} \boxtimes \mathcal{F}$-measurable.
2. For any $\tilde{\omega} \in \tilde{\Omega}$, any $k, l \in S$, and any $r \in S \cup \{J\}$, it holds that

$$\hat{P}^{\tilde{\omega}}(\bar{\bar{g}}^n(\tilde{\omega}, \cdot) = r | \bar{\alpha}_i^n(\tilde{\omega}, \cdot) = k, \ \bar{g}_i^n(\tilde{\omega}, \cdot) = l)(\hat{\omega}) = \delta_l(r).$$

This means that

$$\bar{\pi}_\omega^n(i) = \pi_\omega^{n-1}(i) \quad \text{for any } i \in \{i : \pi^{n-1}(i, \omega) \neq i\}.$$

3. Given $\tilde{\omega} \in \tilde{\Omega}$ and the postmutation extended type distribution $\check{p}^n$ in (3.10), an unmatched agent of type $k$ is matched to an unmatched agent of type $l$ with conditional probability $\theta_{kl}(\tilde{\omega}, n, \check{p}^n)$, that is, for $\lambda$-almost every agent $i$ and $\hat{P}^{\tilde{\omega}}$-almost every $\hat{\omega}$, by the definition of $(\theta_{k,l}^n)_{k,l \in S}$ we have

$$(3.11) \qquad \hat{P}^{\tilde{\omega}}(\bar{\bar{g}}^n(\tilde{\omega}, \cdot) = l | \bar{\alpha}_i^n(\tilde{\omega}, \cdot) = k, \ \bar{g}_i^n(\tilde{\omega}, \cdot) = J, \ \check{p}^n(\tilde{\omega}, \cdot))(\hat{\omega}) = \theta_{kl}^n(\tilde{\omega}, \check{p}^n(\tilde{\omega}, \hat{\omega})).$$

This also implies that

$$\begin{aligned} &\hat{P}^{\tilde{\omega}}\left(\bar{\bar{g}}^n(\tilde{\omega}, \cdot) = J | \bar{\alpha}_i^n(\tilde{\omega}, \cdot) = k, \ \bar{g}_i^n(\tilde{\omega}, \cdot) = J, \ \check{p}^n(\tilde{\omega}, \cdot)\right)(\hat{\omega}) \\ (3.12) \qquad &= 1 - \sum_{l \in S} \theta_{kl}^n(\tilde{\omega}, \check{p}^n(\tilde{\omega}, \hat{\omega})) = b^k(\tilde{\omega}, \check{p}^n(\tilde{\omega}, \hat{\omega})). \end{aligned}$$

The extended type of agent $i$ after the random matching step is

$$\bar{\bar{\beta}}_i^n(\omega) = (\bar{\alpha}_i^n(\omega), \bar{\bar{g}}_i^n(\omega)), \quad n \geq 1.$$

We denote the postmatching extended type distribution realized in $\omega \in \Omega$ by $\check{p}^n(\omega) = (\check{p}^n(\omega)[k, l])_{k \in S, l \in S \cup \{J\}}$, where

$$(3.13) \qquad \check{p}^n(\omega)[k, l] := \lambda(\{i \in I : \bar{\bar{\alpha}}^n(i, \omega) = k, \ \bar{\bar{g}}^n(i, \omega) = l\}).$$

**Type changes of matched agents with break-up.** We now define a random matching $\pi^n$ by

$$(3.14) \qquad \pi^n(i) = \begin{cases} \bar{\pi}^n(i) & \text{if } \bar{\pi}^n(i) \neq i, \\ i & \text{if } \bar{\pi}^n(i) = i. \end{cases}$$

We then introduce an $(\mathbb{I} \boxtimes \mathcal{F})$-measurable agent-type function $\alpha^n$ and an $(\mathbb{I} \boxtimes \mathcal{F})$-measurable partner-type function $g^n$ with

$$g^n(i, \omega) = \alpha^n(\pi^n(i, \omega), \omega), \quad n \geq 1,$$

for all $(i, \omega) \in I \times \Omega$. Given $\tilde{\omega} \in \tilde{\Omega}$, $\check{p}^n \in \hat{\Delta}$, for any $k_1, k_2, l_1, l_2 \in S$ and $r \in S \cup \{J\}$, for $\lambda$-almost every agent $i$, and for $\hat{P}^{\tilde{\omega}}$-almost every $\hat{\omega}$, we have

$$(3.15) \qquad \hat{P}^{\tilde{\omega}}\left(\alpha_i^n(\tilde{\omega}, \cdot) = l_1, \ g_i^n(\tilde{\omega}, \cdot) = r | \bar{\alpha}_i^n(\tilde{\omega}, \cdot) = k_1, \ \bar{\bar{g}}_i^n(\tilde{\omega}, \cdot) = J\right)(\hat{\omega}) = \delta_{k_1}(l_1)\delta_J(r),$$

$$\hat{P}^{\tilde{\omega}}\left(\alpha_i^n(\tilde{\omega},\cdot)=l_1,\ g_i^n(\tilde{\omega},\cdot)=l_2|\bar{\alpha}_i^n(\tilde{\omega},\cdot)=k_1,\ \bar{\bar{g}}_i^n(\tilde{\omega},\cdot)=k_2,\ \check{p}^n(\tilde{\omega},\cdot)\right)(\hat{\omega})$$

$$(3.16) \qquad = \left(1-\xi_{k_1k_2}(\tilde{\omega},n,\check{p}^n(\tilde{\omega},\hat{\omega}))\right)\sigma_{k_1k_2}[l_1,l_2](\tilde{\omega},n,\check{p}^n(\tilde{\omega},\hat{\omega})),$$

$$\hat{P}^{\tilde{\omega}}\left(\alpha_i^n(\tilde{\omega},\cdot)=l_1,\ g_i^n(\tilde{\omega},\cdot)=J|\bar{\alpha}_i^n(\tilde{\omega},\cdot)=k_1,\ \bar{\bar{g}}_i^n(\tilde{\omega},\cdot)=k_2,\ \check{p}^n(\tilde{\omega},\cdot)\right)(\hat{\omega})$$

$$(3.17) \qquad = \xi_{k_1k_2}(\tilde{\omega},n,\check{p}^n(\tilde{\omega},\hat{\omega}))\varsigma_{k_1k_2}^n[l_1](\tilde{\omega},n,\check{p}^n(\tilde{\omega},\hat{\omega})).$$

*The extended type function at the end of the period is*

$$\beta^n(\omega)=(\alpha^n(\omega),g^n(\omega)),\quad n\geq 1.$$

*We denote the extended type distribution at the end of period $n$ realized in $\omega\in\Omega$ by $\hat{p}^n(\omega)=(\hat{p}^n(\omega)[k,l])_{k\in S,l\in S\cup J}$, where*

$$(3.18) \qquad \hat{p}^n(\omega)[k,l]:=\lambda(\{i\in I:\alpha^n(i,\omega)=k,\ g^n(i,\omega)=l\}).$$

*Remark* 3.6. If we set

$$P(\hat{A}):=P(\tilde{\Omega}\times\hat{A})$$

for any $\hat{A}\in\hat{\mathcal{F}}$, then by (3.2) we have that

$$P\left(\hat{A}|\bar{\mathcal{F}}\right)(\tilde{\omega}_1):=P\left(\hat{A}\times\tilde{\Omega}|\bar{\mathcal{F}}\right)(\tilde{\omega}_1)=\int_{\tilde{\Omega}}\hat{P}^{\tilde{\omega}}(\hat{A})\tilde{P}\left(d\tilde{\omega}|\bar{\mathcal{F}}\right)(\tilde{\omega}_1)=\hat{P}^{\tilde{\omega}_1}(\hat{A})$$

for every $\tilde{\omega}_1\in\tilde{\Omega}$. Hence, for fixed $\tilde{\omega}\in\tilde{\Omega}$, the probabilities $\hat{P}^{\tilde{\omega}}(\cdot)$ which appear in Definition 3.5 might be regarded as conditional probabilities on the product space $\Omega=\hat{\Omega}\times\tilde{\Omega}$ endowed with the $\sigma$-algebra $\bar{\mathcal{F}}=\{\emptyset,\Omega\}\otimes\tilde{\mathcal{F}}$.

*Remark* 3.7. The random matching mechanism we propose describes the *directed search* for a suitable counterpart by a continuum of agents. The word "directed" refers to the fact that agents are motivated to meet among each other intentionally. However, in general we don't make any specific assumption on the motivation for a meeting. Partnerships can immediately break up or last until a given random break-up time. Every agent is described by a type which may change at any time step, possibly as a consequence of a new match. In particular, we allow probabilities governing matching, type change, and break-ups to be driven by some stochastic exogenous factors. Information or opinion sharing can be viewed as possible stochastic factors inducing type change after a match. In our application of section 4, for example, an agent involved in a new match might become optimistic about the evolution of the asset's price if her new partner communicates having earned good returns after buying it, as in the empirical findings in [5]. This can feed a contagion mechanism, also described in [33], making people switch to optimistic views. Note that, differently from [4], information or opinion sharing is not necessarily about the final payoff of the asset, but can also concern the short-term returns.

In the following definition we describe a dynamical system $\mathbb{D}$ which satisfies additional conditional independence assumptions.

**Definition 3.8.** *A dynamical system* $\mathbb{D}$ *as in Definition* 3.3 *is* Markov conditionally independent given $\tilde{\omega} \in \tilde{\Omega}$ *if for* $\lambda$-*almost every $i$ and $j$, for* $\hat{P}^{\tilde{\omega}}$-*almost every $\hat{\omega} \in \hat{\Omega}$, for every period* $n \geq 1$, *and for all* $k_1, k_2 \in S$, $l_1, l_2 \in S \cup \{J\}$, *the following properties hold:*

1. *initial dependence:* $\beta_i^0$ *and* $\beta_j^0$ *are independent.*
2. *Markov and independent mutation, conditional to* $\tilde{\omega}$:

(3.19)
$$\hat{P}^{\tilde{\omega}}\left(\bar{\beta}_i^n(\tilde{\omega}, \cdot) = (k_1, l_1), \ \bar{\beta}_j^n(\tilde{\omega}, \cdot) = (k_2, l_2)\Big|(\beta_i^t(\tilde{\omega}, \cdot))_{t=0}^{n-1}, \ (\beta_j^t(\tilde{\omega}, \cdot))_{t=0}^{n-1}\right)(\hat{\omega})$$
$$= \hat{P}^{\tilde{\omega}}\left(\bar{\beta}_i^n(\tilde{\omega}, \cdot) = (k_1, l_1)\Big|\beta_i^{n-1}(\tilde{\omega}, \cdot)\right)(\hat{\omega})\hat{P}^{\tilde{\omega}}\left(\bar{\beta}_j^n(\tilde{\omega}, \cdot) = (k_2, l_2)\Big|\beta_j^{n-1}(\tilde{\omega}, \cdot)\right)(\hat{\omega}).$$

3. *Markov and independent random matching, conditional to* $\tilde{\omega}$:

$$\hat{P}^{\tilde{\omega}}\left(\bar{\bar{\beta}}_i^n(\tilde{\omega}, \cdot) = (k_1, l_1), \ \bar{\bar{\beta}}_j^n(\tilde{\omega}, \cdot) = (k_2, l_2)\Big|\right.$$
$$\left.\bar{\beta}_i^n(\tilde{\omega}, \cdot), \ \bar{\beta}_j^n(\tilde{\omega}, \cdot), \ (\beta_i^t(\tilde{\omega}, \cdot))_{t=0}^{n-1}, \ (\beta_j^t(\tilde{\omega}, \cdot))_{t=0}^{n-1}\right)(\hat{\omega})$$
(3.20)
$$= \hat{P}^{\tilde{\omega}}\left(\bar{\bar{\beta}}_i^n(\tilde{\omega}, \cdot) = (k_1, l_1)\Big|\bar{\beta}_i^n(\tilde{\omega}, \cdot)\right)(\hat{\omega})\hat{P}^{\tilde{\omega}}\left(\bar{\bar{\beta}}_j^n(\tilde{\omega}, \cdot) = (k_2, l_2)\Big|\bar{\beta}_j^n(\tilde{\omega}, \cdot)\right)(\hat{\omega}).$$

4. *Markov and independent matched-agent-type changes with break-ups, conditional to* $\tilde{\omega}$:

$$\hat{P}^{\tilde{\omega}}\left(\beta_i^n(\tilde{\omega}, \cdot) = (k_1, l_1), \beta_j^n(\tilde{\omega}, \cdot) = (k_2, l_2)\Big|\right.$$
$$\left.\bar{\bar{\beta}}_i^n(\tilde{\omega}, \cdot), \bar{\bar{\beta}}_j^n(\tilde{\omega}, \cdot), (\beta_i^t(\tilde{\omega}, \cdot))_{t=0}^{n-1}, (\beta_j^t(\tilde{\omega}, \cdot))_{t=0}^{n-1}\right)(\hat{\omega})$$
(3.21)
$$= \hat{P}^{\tilde{\omega}}\left(\beta_i^n(\tilde{\omega}, \cdot) = (k_1, l_1)\Big|\bar{\bar{\beta}}_i^n(\tilde{\omega}, \cdot)\right)(\hat{\omega})\hat{P}^{\tilde{\omega}}\left(\beta_j^n(\tilde{\omega}, \cdot) = (k_2, l_2)\Big|\bar{\bar{\beta}}_j^n(\tilde{\omega}, \cdot)\right)(\hat{\omega}).$$

We now prove the existence of a Markov conditionally independent random matching by using the same arguments as in [15]. The proof relies on the product structure of the space $\Omega$ in (3.1) and the Markov kernel $\hat{P}$ in (3.2), as well as on concepts from nonstandard analysis. Note here that an object with an upper left star means the transfer of a standard object to the nonstandard universe. For a detailed overview of the necessary tools of nonstandard analysis and a definition of *hyperfinite internal probability space* we refer to Appendces D.2 and D.3 in [15], respectively.

From now on, we work under the following assumption.

*Assumption* 3.9. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be the probability space in (3.1). We assume that there exists its corresponding hyperfinite internal probability space, which we denote from now on also by $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ by slight notational abuse.

*Remark* 3.10. It is possible to construct such a space whose corresponding Loeb space can be transferred to a classical standard probability space, as can be seen, for example, in [15], where $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ is the space of trajectories of a multidimensional Markov process.

In the outlined setting we now prove the existence of a rich Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, on which a dynamical system $\mathbb{D}$ described in Definition 3.5 for given input probabilities is defined.

*Remark* 3.11. Before we state the main results, we give an intuition on how the product structure and the Markov kernel allow us to use the same arguments as in [15]. If we fix

the state of the world $\tilde{\omega}$, the input functions $(\eta, \theta, \xi, \sigma, \varsigma)$ are deterministic and we are in the setting of [15]. Thus, by the results in [15] the existence of a space $\hat{\Omega}$ and $\hat{P}$ follows directly. Motivated by this, we fix $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and iteratively construct a suitable space $(\hat{\Omega}, \hat{\mathcal{F}})$, together with a probability measure $\hat{P}^{\tilde{\omega}}$ for any $\tilde{\omega} \in \tilde{\Omega}$. By using the definition of the Markov kernel, we can then define a measure $P$ on the product space $(\Omega, \mathcal{F}) = (\tilde{\Omega} \times \hat{\Omega}, \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}})$.

In a first step, we focus on the random matching step and show the existence of a suitable hyperfinite probability space and partial matching. This is a generalization of Lemma 7 in [15].

**Proposition 3.12.** *Let $(I, \mathcal{I}_0, \lambda_0)$ be a hyperfinite counting probability space with Loeb space $(I, \mathcal{I}, \lambda)$. Then, there exists a hyperfinite internal set $\Omega$ with internal power set $\mathcal{F}_0$ such that for any initial internal type function $\alpha^0 : I \to S$, any initial internal partial matching $\pi^0 : I \to I$ with*

$$g^0(i) = \begin{cases} \alpha^0(\pi^0(i)) & \text{if } \pi^0(i) \neq i, \\ J & \text{if } \pi^0(i) = i, \end{cases}$$

*and any matching probability function $\theta_{kl} : \tilde{\Omega} \times \mathbb{N} \times \hat{\Delta} \to [0,1]$ and $\hat{p} \in {}^*\hat{\Delta}$, there exist an internal random matching $\pi$ from $I \times \Omega$ to $I$ and an internal probability measure $P_0$ with the following properties.*

1. *It holds that*

$$\Omega := \tilde{\Omega} \times \hat{\Omega} \quad and \quad \mathcal{F}_0 := \tilde{\mathcal{F}} \times \hat{\mathcal{F}}_0,$$

   *where $\hat{\Omega}$ is a hyperfinite internal set, $\hat{\mathcal{F}}_0$ is its internal power set, and $(\tilde{\Omega}, \tilde{\mathcal{F}})$ is the hyperfinite internal probability space which exists by Assumption 3.9.*

2. *We have $P_0 := \tilde{P} \ltimes \hat{P}_0^{\hat{p}}$, where $\hat{P}_0^{\hat{p}}$ is a Markov kernel from $\tilde{\Omega}$ to $\hat{\Omega}$. From now on, we denote $\hat{P}_0^{\hat{p}, \tilde{\omega}} := \hat{P}_0^{\hat{p}}(\tilde{\omega})$.*

3. *The internal random matching $\pi : I \times \Omega \to I$ is defined as*

$$\pi(i, (\tilde{\omega}, \hat{\omega})) := \hat{\pi}(i, \hat{\omega}),$$

   *where $\hat{\pi} : I \times \hat{\Omega} \to I$ is an internal random matching. We use the notation*

   (3.22) $$\pi_{\hat{\omega}}(i) := \hat{\pi}(i, \hat{\omega}) = \pi(i, (\tilde{\omega}, \hat{\omega}))$$

   *for any $\omega = (\tilde{\omega}, \hat{\omega}) \in \Omega$.*

4. *Let $H = \{i : \pi^0(i) \neq i\}$. Then*

$$\hat{P}_0^{\hat{p}, \tilde{\omega}} \left( \{\hat{\omega} \in \hat{\Omega} : \pi_{\hat{\omega}}(i) = \pi^0(i) \text{ for any } i \in H \} \right) = 1$$

   *for any $\hat{p} \in {}^*\hat{\Delta}$ and $\tilde{\omega} \in \tilde{\Omega}$.*

5. *The internal mapping from $I \times \Omega$ to $S \cup \{J\}$ is defined by the immersion*

$$g(i, (\tilde{\omega}, \hat{\omega})) := \hat{g}(i, \hat{\omega}),$$

*where $\hat{g}$ is the internal mapping from $I \times \hat{\Omega}$ to $S \cup \{J\}$, given by*

$$\hat{g}(i,\hat{\omega}) = \begin{cases} \alpha^0(\hat{\pi}(i,\hat{\omega})) & \text{if } \hat{\pi}(i,\hat{\omega}) \neq i, \\ J & \text{if } \hat{\pi}(i,\hat{\omega}) = i, \end{cases}$$

*for any $(i,\hat{\omega}) \in I \times \hat{\Omega}$. Then for any $k,l \in S$ and fixed $\hat{p} \in {}^*\hat{\Delta}$ and $\tilde{\omega} \in \tilde{\Omega}$ we have*

$$\hat{P}_0^{\hat{p},\tilde{\omega}}(\hat{g}_i = l) \simeq \theta_{kl}(\tilde{\omega}, 0, \hat{p})$$

*for $\lambda$-almost every agent $i \in I$ satisfying $\alpha^0(i) = k$ and $\pi^0(i) = i$.*

6. *For any $\hat{p} \in {}^*\hat{\Delta}$ and $\tilde{\omega} \in \tilde{\Omega}$, denote the corresponding Loeb probability spaces of the internal probability spaces $(\hat{\Omega}, \hat{\mathcal{F}}_0, \hat{P}_0^{\hat{p},\tilde{\omega}})$ and $(I \times \hat{\Omega}, \mathcal{I}_0 \otimes \hat{\mathcal{F}}_0, \lambda_0 \otimes \hat{P}_0^{\hat{p},\tilde{\omega}})$ by $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}^{\hat{p},\tilde{\omega}})$ and $(I \times \hat{\Omega}, \mathcal{I} \boxtimes \hat{\mathcal{F}}, \lambda \boxtimes \hat{P}^{\hat{p},\tilde{\omega}})$, respectively. Moreover, denote the corresponding Loeb probability spaces of the internal probability spaces $(\Omega, \mathcal{F}_0, P_0)$ and $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$ by $(\Omega, \mathcal{F}, P)$ and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, respectively. The mapping $\hat{g}$ is an essentially pairwise independent random variable from $(I \times \hat{\Omega}, \mathcal{I} \boxtimes \hat{\mathcal{F}}, \lambda \boxtimes \hat{P}^{\hat{p},\tilde{\omega}})$ to $S \cup \{J\}$ for any $\hat{p} \in {}^*\hat{\Delta}$ and $\tilde{\omega} \in \tilde{\Omega}$.*

*Proof.* See Appendix B for the proof. ∎

We are now ready to give the following theorem, which is the main result of the section.

**Theorem 3.13.** *Let Assumption 3.9 hold and fix the input functions*

$$(\eta_{kl}, \theta_{kl}, \xi_{kl}, \sigma_{kl}[r,s], \varsigma_{kl}[r])_{k,l,r,s \in S \times S \times S \times S}$$

*defined above. Then for any extended type distribution $\ddot{p} \in \hat{\Delta}$ and any deterministic initial condition $\Pi^0 = (\alpha^0, \pi^0)$ there exists a rich Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which a discrete dynamical system $\mathbb{D} = (\Pi^n)_{n=1}^{\infty}$ with parameters $(\eta, \theta, \xi, \sigma, \varsigma)$ as in Definition 3.5 can be constructed. In particular,*

$$\Omega = \tilde{\Omega} \times \hat{\Omega}, \quad \mathcal{F} = \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}}, \quad P = \tilde{P} \ltimes \hat{P},$$

*where $(\hat{\Omega}, \hat{\mathcal{F}})$ is a measurable space and $\hat{P}$ a Markov kernel from $\tilde{\Omega}$ to $\hat{\Omega}$. The dynamical system $\mathbb{D}$ is also Markov conditionally independent according to Definition 3.8 and with initial cross-sectional extended type distribution $\hat{p}^0$ equal to $\ddot{p}^0$ with probability one.*

*Proof.* See section 2 of [9]. ∎

We now state some properties of the dynamical system $\mathbb{D}$ with input processes, which is a generalization of the results in Appendix C in [15]. In particular, given $\tilde{\omega} \in \tilde{\Omega}$ the following result allows us to recursively calculate the expectation under $\hat{P}^{\tilde{\omega}}$ of $\hat{p}^n, \check{p}^n, \check{p}^n$ for every $n \geq 1$, which will be useful for applications.

For each time $n \geq 1$ we define a map $\Gamma^n : \tilde{\Omega} \times \hat{\Delta} \to \hat{\Delta}$ as follows:

$$
\begin{aligned}
\Gamma_{kl}^n(\tilde{\omega}, \hat{p}) = &\sum_{k_1, l_1 \in S} \left(1 - \xi_{k_1 l_1}\left(\tilde{\omega}, n, \tilde{\check{p}}^n\right)\right) \sigma_{k_1 l_1}[k,l]\left(\tilde{\omega}, n, \tilde{\check{p}}^n\right) \tilde{p}_{k_1 l_1}^n \\
(3.23) \qquad &+ \sum_{k_1, l_1 \in S} \left(1 - \xi_{k_1 l_1}\left(\tilde{\omega}, n, \tilde{\check{p}}^n\right)\right) \sigma_{k_1 l_1}[k,l]\left(\tilde{\omega}, n, \tilde{\check{p}}^n\right) \theta_{k_1 l_1}\left(\tilde{\omega}, n, \tilde{p}^n\right) \tilde{p}_{k_1 J}^n,
\end{aligned}
$$

and

$$\Gamma_{kJ}^n(\tilde{\omega}, \hat{p}) = b_k\left(\tilde{\omega}, n, \tilde{p}^n\right) \tilde{p}_{kJ}^n + \sum_{k_1, l_1 \in S} \xi_{k_1 l_1}\left(\tilde{\omega}, n, \tilde{p}^n\right) \varsigma_{k_1 l_1}[k]\left(\tilde{\omega}, \tilde{p}^n\right) \tilde{p}_{k_1 l_1}^n$$

(3.24)
$$+ \sum_{k_1, l_1 \in S} \xi_{k_1 l_1}\left(\tilde{\omega}, n, \tilde{p}^n\right) \varsigma_{k_1 l_1}[k]\left(\tilde{\omega}, n, \tilde{p}^n\right) \theta_{k_1 l_1}\left(\tilde{\omega}, n, \tilde{p}^n\right) \tilde{p}_{k_1 J}^n$$

with

$$\tilde{p}_{kl}^n = \sum_{k_1, l_1 \in S} \eta_{k_1 k}\left(\tilde{\omega}, n, \hat{p}\right) \eta_{l_1 l}\left(\tilde{\omega}, n, \hat{p}\right) \hat{p}_{k_1 l_1}$$

$$\tilde{p}_{kJ}^n = \sum_{l \in S} \hat{p}_{lJ} \eta_{lk}\left(\tilde{\omega}, n, \hat{p}\right),$$

and

$$\tilde{\tilde{p}}_{kl}^n = \tilde{p}_{kl}^n + \theta_{kl}\left(\tilde{\omega}, n, \tilde{p}^n\right) \tilde{p}_{kJ}^n,$$
$$\tilde{\tilde{p}}_{kJ}^n = b_k\left(\tilde{\omega}, n, \tilde{p}^n\right) \tilde{p}_{kJ}^n.$$

**Theorem 3.14.** *Assume that the discrete dynamical system $\mathbb{D}$ introduced in Definition 3.5 is Markov conditionally independent given $\tilde{\omega} \in \tilde{\Omega}$ according to Definition 3.8. Given $\tilde{\omega} \in \tilde{\Omega}$, the following holds:*

1. *For each $n \geq 1$, $\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n] = \Gamma^n(\tilde{\omega}, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])$.*

2. *For each $n \geq 1$,*
$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kl}^n] = \sum_{k_1, l_1 \in S} \eta_{k_1, k}\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]\right) \eta_{l_1, l}\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]\right) \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}_{k_1, l_1}^{n-1}]$$

   *and*
$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kJ}^n] = \sum_{k_1 \in S} \eta_{k_1, k}\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]\right) \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}_{k_1, J}^{n-1}].$$

3. *For each $n \geq 1$,*
$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kl}^n] = \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kl}^n] + \theta_{kl}\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n]\right) \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kJ}^n]$$

   *and*
$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kJ}^n] = b_k\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n]\right) \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kJ}^n].$$

4. *For $\lambda$-almost every agent $i$, the extended type process $\{\beta_i^n\}_{n=0}^\infty$ is a Markov chain in $\hat{S}$ on $(I \times \hat{\Omega}, \mathcal{I} \boxtimes \hat{\mathcal{F}}, \lambda \boxtimes \hat{P}^{\tilde{\omega}})$, whose transition matrix $z^n$ at time $n-1$ is given by*
$$z_{(k'J)(kl)}^n(\tilde{\omega}) = \sum_{k_1, l_1, k' \in S} \left(1 - \xi_{k_1 l_1}(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega}, n})\right) \sigma_{k_1 l_1}[k, l]\left(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega}, n}\right)$$

   (3.25)
$$\cdot \theta_{k_1 l_1}\left(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega}, n}\right) \eta_{k' k_1}\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]\right),$$

$$z_{(k'l')(kl)}^n(\tilde{\omega}) = \sum_{k_1, l_1, k', l' \in S} \left(1 - \xi_{k_1 l_1}\left(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega}, n}\right)\right) \sigma_{k_1 l_1}[k, l]\left(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega}, n}\right)$$

   (3.26)
$$\cdot \eta_{k' k_1}\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]\right) \eta_{l' l_1}\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]\right),$$

$$z^n_{(k'l')(kJ)}(\tilde{\omega}) = \sum_{k_1,l_1 \in S} \xi_{k_1 l_1}\left(\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n}\right) \varsigma_{k_1 l_1}[k]\left(\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n}\right)$$

$$(3.27) \qquad \cdot \eta_{k'k_1}\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]\right) \eta_{l'l_1}\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]\right),$$

$$z^n_{(k'J)(kJ)}(\tilde{\omega}) = b_k\left(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega},n}\right) \eta_{k'k}\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]\right)$$

$$+ \sum_{k_1,l_1 \in S} \xi_{k_1 l_1}\left(\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n}\right) \varsigma_{k_1 l_1}[k]\left(\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n}\right) \theta_{k_1 l_1}\left(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega},n}\right)$$

$$(3.28) \qquad \cdot \eta_{k'k_1}\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]\right).$$

5. For $\lambda$-almost every $i$ and $j$, the Markov chains $\{\beta^n_i\}^\infty_{n=0}$ and $\{\beta^n_j\}^\infty_{n=0}$ are independent of $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P}^{\tilde{\omega}})$.

6. For $\hat{P}^{\tilde{\omega}}$-almost every $\hat{\omega} \in \hat{\Omega}$, the cross-sectional extended type process $\{\beta^n_{\hat{\omega}}\}^\infty_{n=0}$ is a Markov chain on $(I, \mathfrak{I}, \lambda)$ with transition matrix $z^n$ at time $n-1$, which is defined in (3.25)–(3.28).

7. For any $(k,l) \in S \times (S \cup \{J\})$ we have

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n_{kl}] = \breve{p}^n_{kl}, \quad \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n_{kl}] = \check{p}^n_{kl}, \quad and \quad \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n_{kl}] = \hat{p}^n_{kl},$$

$\hat{P}^{\tilde{\omega}}$-a.s.

*Proof.* See section 3 of [9]. ∎

*Remark* 3.15. By using Remark 3.6, the results in Theorem 3.14 also hold on the product space $\Omega = \hat{\Omega} \times \tilde{\Omega}$ by conditioning on $\bar{\mathfrak{F}}$ with respect to the probability measure $P$.

## 4. Application: A dynamic directed random matching model for the evolution of asset price bubbles.

We now use the random matching mechanism described in section 3 to describe interactions among investors in the setting of section 2. We consider an atomless probability space $(I, \mathfrak{I}, \lambda)$ representing the space of investors. We introduce the space of investors' types $S = \{1, 2, 3\}$, where investors of type 1, 2, 3 are respectively optimistic, neutral, and pessimistic, and the space $\hat{\Delta} = (\hat{p}_{ij})_{i=1,2,3,j=1,2,3,J}$ of processes with values in the space of matrices with three rows and four columns[3] representing the extended type distributions. We denote with $\hat{p}^n_{ij}$ the fraction of investors of type $i = 1, 2, 3$ at time $t_n$ matched with a partner of type $j = 1, 2, 3$, and with $\hat{p}^n_{iJ}$ the fraction of unmatched agents of type $i$ at time $t_n$. For any $k = 0, \ldots, N$, we then have

$$(4.1) \qquad \hat{p}^k_i = \sum_{j=1}^3 \hat{p}^k_{ij} + \hat{p}^k_{iJ}.$$

Let $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{F}} = (\tilde{\mathfrak{F}}^i)_{i=0,\ldots,N}, \tilde{P})$ be a filtered probability space from subsection 2.1 on which the stochastic processes $(F^i)_{i=0,\ldots,N}$, $(M^i)_{i=0,\ldots,N}$, $(\Lambda^i)_{i=0,\ldots,N}$, $(\kappa^i)_{i=0,\ldots,N}$, $(\Theta^i)_{i=0,\ldots,N}$ are defined. Then, Theorem 3.13 provides the existence of a rich Fubini extension $(I \times \Omega, \mathfrak{I} \boxtimes \mathfrak{F},$

---

[3]This space can be endowed with a topology on the space of matrices.

$\lambda \boxtimes P$) for a discrete dynamical system $\mathbb{D} = (\Pi^n)_{n=1}^{\infty}$ as in Definition 3.5 with

$$\Omega = \tilde{\Omega} \times \hat{\Omega}, \quad \mathcal{F} = \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}}, \quad P = \tilde{P} \ltimes \hat{P},$$

where $(\hat{\Omega}, \hat{\mathcal{F}})$ is a measurable space on which the stochastic process $\hat{p}^i = (\hat{p}_i^k)_{k=0,\dots,N}$, $i = 1, 2, 3$, can be constructed as in the proof of Theorem 3.13, and $\hat{P}$ is a Markov kernel from $\tilde{\Omega}$ to $\hat{\Omega}$ according to Definition 3.1.

By Theorem 3.14, the dynamics of the fraction of optimistic, neutral, and pessimistic agents are identified by the conditional transition probabilities

$$(\eta_{kl}, \theta_{kl}, \xi_{kl}, \sigma_{kl}[r,s], \varsigma_{kl}[r])_{k,l,r,s \in S \times S \times S \times S}.$$

Next, we specify the form of these functions in this setting.

We do not make any particular assumptions on $(\theta_{kl}, \xi_{kl}, \varsigma_{kl}[r])_{k,l,r,s \in S \times S \times S}$, which identify the conditional probabilities of matching, break-up, and post-break-up change of type. We just emphasize that their evolution depends on some state of $\tilde{\Omega}$. We focus instead on modeling the functions $\eta_{k_1,k_2}$ and $\sigma_{k_1 k_2}[l_1, l_2]$, $k_1, k_2, l_1, l_2 = 1, 2, 3$, appearing in (3.8) and (3.16), respectively. Such quantities represent the probabilities of type change, independent of matching or after a new match, conditional to the state of $\tilde{\Omega}$ and to the current values of the fractions of investors' types. They depend on the state of $\tilde{\Omega}$, on the values of the current fractions of optimistic, neutral, and pessimistic investors, and on the type of the agent to whom they are matched in the case of postmatch type change, as follows:

(i) A match with an optimistic agent may induce an upgrade in type (i.e., from 3 to 1 or 2, or from 2 to 1).

(ii) A match with a pessimistic agent may induce a downgrade of type (i.e., from 1 to 2 or 3, or from 2 to 3).

(iii) The propensity of an agent to become *more optimistic,* i.e., to change type from 3 to 1, 2 or from 2 to 1, is increasing with respect to $\hat{p}^1 - \hat{p}^3$. On the contrary, the propensity of an agent to become *more pessimistic* is increasing with respect to $\hat{p}^3 - \hat{p}^1$. This holds for both the type changes, conditional or unconditional to the match.

Before giving an example of such conditional probabilities, we comment on the above model.

*Remark* 4.1. The main features of the model described above are the following:

(i) All the processes determining the matching, breaking-up, and type change probabilities depend on some underlying stochastic processes on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. These factors may represent socioeconomic indicators as well as the fundamental price of the assets or the impact of public news. In this way, the birth of the bubble may be determined by changes in the driving factors, leading to a higher probability of matching with optimistic traders or of changing type from pessimistic to optimistic and vice versa.

(ii) Investors can change their views with a probability depending on the number of optimistic and pessimistic agents in the market. This assumption reflects that, when observing a large number of optimistic agents in the market, an investor may be prone to thinking she will have the opportunity to profit from other market participants'

overvaluation at least in the short or medium term. For example in the model of [39] agents are willing to pay prices exceeding their own valuation of the fundamental value of a bubbly asset because of speculation opportunities in the near future.

(iii) Traders can also influence each other when they meet. This reflects some phenomena which are typically observed during both the blowing-up and the bursting phase. When the bubble grows, herd behavior may be induced by a tendency to mimic the actions of acquaintances making gains and by word-of-mouth spread of information regarding the fast increase of the stock price. This will eventually fuel further up the prices; see [33] for a description and formalization of this phenomenon and [5], where such mechanisms are documented in the US 2007 housing bubble. Similar attitudes, but in a different direction, characterize the investors' behavior starting the burst and speeding up the decrease of the price after the burst.

We conclude the section by giving a concrete example of a possible choice of $\eta_{k_1,k_2}$ and $\sigma_{k_1 k_2}[l_1, l_2]$, $k_1, k_2, l_1, l_2 = 1, 2, 3$.

*Example* 4.2. Fix $\tilde{\omega} \in \tilde{\Omega}$ and $j \in \{1, \dots, N\}$, representing time. We set

$$\eta_{k_1,k_2}^{\tilde{\omega},j} := \eta_{k_1 k_2}(\tilde{\omega}, j, \cdot)$$

and

$$\sigma_{k_1 k_2}^{\tilde{\omega},j}[l_1, l_2] := \sigma_{k_1 k_2}[l_1, l_2](\tilde{\omega}, j, \cdot).$$

We model such conditional probabilities in the following way.

(i) Consider an increasing function $f^j : \mathbb{R}_+ \to [0, 1/2]$ such that $f^j(0) = 0$ and set

$$(4.2) \qquad f_+^j = f^j \left( (\hat{p}_1^j - \hat{p}_3^j)^+ \right), \qquad f_-^j = f^j \left( (\hat{p}_3^j - \hat{p}_1^j)^+ \right).$$

Given the random variables $F_{ijk} : (\tilde{\Omega}, \tilde{\mathcal{F}}) \to ([0, 1/2], \mathcal{B}([0, 1/2]))$, $i, j, k = 1, 2, 3$, the type change probabilities after a match are then defined as follows:

• After a match of two agents of the same type who stay in a relationship we have

$$\sigma_{kk}^{\tilde{\omega},j}(r, s) = \delta_r(k)\delta_s(k),$$

i.e., they both maintain their types.

• After a match of two agents who are respectively optimistic and neutral and stay in a relationship, we assume

$$\sigma_{12}^{\tilde{\omega},j}(k, \ell) = 0 \text{ if } k = 3 \text{ or } \ell = 3 \quad \text{and} \quad \sigma_{12}^{\tilde{\omega},j}(2, 1) = 0,$$

i.e., none of the agents changes his type to pessimistic and they do not switch types. Moreover, we set

$$\sigma_{12}^{\tilde{\omega},j}(1, 1) = F_{121}(\tilde{\omega}) + f_+^j, \quad \sigma_{12}^{\tilde{\omega},j}(2, 2) = F_{122}(\tilde{\omega}) + f_-^j,$$

$$\sigma_{12}^{\tilde{\omega},j}(1, 2) = 1 - \sigma_{12}^{\tilde{\omega},j}(1, 1) - \sigma_{12}^{\tilde{\omega},j}(2, 2),$$

i.e., the probability that the neutral agent changes to optimistic due to the match is given by a random term depending only on $\tilde{\Omega}$ plus a term which is strictly positive only if the number of buyers is higher than the number of sellers, and increasing with respect to their difference. The opposite holds for a possible change of the optimistic agent to neutral.

- After a match of two agents who are respectively neutral and pessimistic and stay in a relationship, we assume that

$$\sigma_{23}^{\tilde{\omega},j}(k,\ell) = 0 \text{ if } k = 1 \text{ or } \ell = 1 \quad \text{and} \quad \sigma_{23}^{\tilde{\omega},j}(3,2) = 0,$$

i.e., we exclude the possibility that one of them may become optimistic. In analogy to the previous case, we fix

$$\sigma_{23}^{\tilde{\omega},j}(2,2) = F_{232}(\tilde{\omega}) + f_+^j, \quad \sigma_{23}^{\tilde{\omega},j}(3,3) = F_{233}(\tilde{\omega}) + f_-^j,$$

$$\sigma_{23}^{\tilde{\omega},j}(2,3) = 1 - \sigma_{23}^{\tilde{\omega},j}(2,2) - \sigma_{23}^{\tilde{\omega},j}(3,3).$$

- For a match of two agents who are respectively neutral and pessimistic and stay in a relationship, we put

$$\sigma_{13}^{\tilde{\omega},j}(3,1) = \sigma_{13}^{\tilde{\omega},j}(3,2) = \sigma_{13}^{\tilde{\omega},j}(2,2) = 0,$$

i.e., the agents do not switch their types and do not become neutral. We also set

$$\sigma_{13}^{\tilde{\omega},j}(1,1) = F_{131}(\tilde{\omega}) + (f_+^j)^2, \quad \sigma_{13}^{\tilde{\omega},j}(1,2) = F_{132}(\tilde{\omega}) + f_+^j(1 - f_+^j),$$
$$\sigma_{13}^{\tilde{\omega},j}(3,3) = F_{133}(\tilde{\omega}) + (f_-^j)^2, \quad \sigma_{13}^{\tilde{\omega},j}(2,3) = F_{132}(\tilde{\omega}) + f_-^j(1 - f_-^j),$$
$$(4.3) \qquad \sigma_{13}^{\tilde{\omega},j}(1,3) = 1 - \sigma_{13}^{\tilde{\omega},j}(1,1) - \sigma_{13}^{\tilde{\omega},j}(1,2) - \sigma_{13}^{\tilde{\omega},j}(3,3) - \sigma_{13}^{\tilde{\omega},j}(2,3)$$

consistently with the construction above.
- Condition (3.6) holds, i.e., $\sigma_{k\ell}^{\tilde{\omega},j}(r,s) = \sigma_{\ell k}^{\tilde{\omega},j}(s,r)$.

(ii) Consider an increasing function $g^j : \mathbb{R}_+ \to [0, 1/2]$ such that $g^j(0) = 0$, and introduce $g_+^j$ and $g_-^j$ analogously to (4.2). We define

$$B^j = \begin{pmatrix} 1 - g_-^j & g_-^j(1 - g_-^j) & (g_-^j)^2 \\ g_+^j & 1 - g_+^j - g_-^j & g_-^j \\ (g_+^j)^2 & g_+^j(1 - g_+^j) & 1 - g_+^j \end{pmatrix},$$

and then the matrix of the probabilities $\eta_{k,l}^{\tilde{\omega},j} = [B^j]_{kl} + [C(\omega)]_{kl}$, $k, l = 1, 2, 3$, where

$$C_{ij}(\omega) : (\tilde{\Omega}, \tilde{\mathcal{F}}) \to ([0, 1/2], \mathcal{B}([0, 1/2]))$$

are random variables, $i, j = 1, 2, 3$.

*Example* 4.3. We focus again on the postmatching change of type described as in Example 4.2: agents may change their views after a match because of the information from other traders. In particular, during the blowing-up phase of the bubble, upgrade of types may be

induced by matches with optimistic agents (i.e., buyers). In the current example we include the idea that the tendency of an agent to switch to more optimistic forecasts after a match also depends on the number of optimistic agents she has already met (and vice versa). We include these considerations as follows. We let again $N$ be the finite number of time periods. The type[4] of an agent is identified by a number $k \in \hat{\hat{S}} := \{0, 1, \ldots, 3(N+1)^3 - 1\}$ defined as

$$(4.4) \qquad k = n_o + n_n(N+1) + n_p(N+1)^2 + (v-1)(N+1)^3,$$

where $n_o$, $n_n$, $n_p$ are the number of optimistic, neutral, and pessimistic investors that the agent has met, respectively, and $v = 1, 2, 3$ indicates an optimistic, neutral, and pessimistic view, respectively.[5] This means that, after a match, an agent of type $k$ in (4.4) changes to type

$$\bar{k} = \bar{n}_o + \bar{n}_n(N+1) + \bar{n}_p(N+1)^2 + (\bar{v}-1)(N+1)^3,$$

where $\bar{n}_i = n_i + 1$ for $i = o, n, p$ and $\bar{n}_j = n_j$ for $j \neq i$. Moreover, $\bar{v} \neq v$ if and only if the investor changes forecasts.

Note that the views of an agent of type $k$ are immediately identified as optimistic if $k < (N+1)^3$, neutral if $k \in [(N+1)^3, 2(N+1)^3)$, and pessimistic if $k \geq 2(N+1)^3$. We can also recover the fractions of optimistic, neutral, and pessimistic investors at time $t_k$, denoted above by $\hat{p}_i^k$, $i = 1, 2, 3$, respectively, by

$$\hat{p}_i^k = \sum_{l=i(N+1)^3}^{(i+1)(N+1)^3 - 1} \hat{\hat{p}}_l^k, \quad i = 1, 2, 3, \quad k = 0, \ldots, N,$$

where $\hat{\hat{p}}_l^k$ is the fraction of investors of type $l \in \hat{\hat{S}}$ at time $t_k$. We then extend Example 4.2 by defining $\sigma_{k_1 k_2}^{\tilde{\omega}, j}[l_1, l_2]$, $k_1, k_2, l_1, l_2 \in \hat{\hat{S}}$, $l = 1, \ldots N$, by also including the number of optimistic, neutral, and pessimistic agents already met: pessimistic agents of type $l = n_o + n_n(N+1) + n_p(N+1)^2 + 2(N+1)^3$ which encounter an optimistic agent may switch to neutral or optimistic with a probability increasing with respect to $n_o$. This is in line with Remark 4.1(iii).

As an example, for two agents of types

$$k_1 = n_o + n_n(N+1) + n_p(N+1)^2 + 2(N+1)^3 \quad \text{and} \quad k_2 = m_o + m_n(N+1) + m_p(N+1)^2,$$

and defining

$$l_1 = (n_o + 1) + n_n(N+1) + n_p(N+1)^2 \quad \text{and} \quad l_2 = m_o + m_n(N+1) + (m_p + 1)(N+1)^2,$$

we can set

$$\sigma_{k_1 k_2}^{\tilde{\omega}, j}[l_1, l_2] = \frac{n_o}{n_o + n_n + n_p} F_{131}(\tilde{\omega}) + (f_+^j)^2,$$

---

[4]This is a small notational difference with respect to the setting of section 3, where 0 is not included in the indices set. In this way the representation in (4.4) is easier.

[5]Clearly, not all $k$ in (4.4) represent feasible types: for example, at the $n$th step, only types represented by $k = n_o + n_n(N+1) + n_p(N+1)^2 + v(N+1)^3$ with $n_o + n_n + n_p \leq n$ are possible. When a type is infeasible, the fraction of agents of that type is zero.

where $F_{131}(\cdot)$ and $f_+^j$ are defined in Example 4.2. Here, $\sigma_{k_1 k_2}^{\tilde{\omega}, j}[l_1, l_2](\tilde{\omega})$ is the probability that a pessimistic agent (note that $k_1 > 2(N+1)^3$) that has already met $n_o$ optimistic, $n_n$ neutral, and $n_p$ pessimistic investors and meets a further optimistic investor (note that $k_2 < (N+1)^3$), becomes optimistic (note that $l_1 < (N+1)^3$). This probability is increasing with respect to $n_o$, so that the more optimistic investors the agent has already met, the more she is prone to switch to optimistic views.

**4.1. Absence of arbitrage.** In this section, we provide a setting under which the financial market model considered in section 4 is arbitrage-free in the sense of the following definition.

*Definition 4.4. Let $(\Omega, \mathcal{G}, \mathbb{G}, \bar{P})$ be a filtered probability space with a risky asset $S$ and a risk-free asset $B = 1$, such that the corresponding (discounted) price process is given by a nonnegative adapted process $S = (S^i)_{i=0,\ldots,N}$. We call a self-financing strategy $\gamma = (\gamma(0), \gamma(1))$[6] an arbitrage opportunity if the associated value process $V = (V^i)_{i=0,\ldots N}$ given by*

$$V^0 := \gamma^1 \quad and \quad V^i := \gamma^i \cdot S^i$$

*satisfies*

$$V_0 \le 0, \quad V_T \ge 0 \ \bar{P}\text{-a.s.}, \quad and \quad \bar{P}[V_T > 0] > 0.$$

*Theorem 4.5 (Theorem 1.7 in [20]). There exists no arbitrage on the financial market introduced in section 4 if and only if there exists an equivalent martingale measure $Q$ with respect to $P$ for the (discounted) market price $S$.*

*Remark 4.6.* Definition 4.4 of arbitrage corresponds to the point of view of a price taker. This is in line with considering a continuum of agents in section 3. More specifically, in an atomless probability space $(I, \mathcal{I}, \lambda)$ representing the space of agents, each individual agent has no market-making power (see also, e.g., [36], [44]) and cannot influence the price process which is in contrast determined by the impact of the group of all agents.

To ease the computations, we assume that the agents break up immediately, by setting $\xi_{kl} = 1$ for any $k, l = 1, 2, 3$. Thus, we only consider the probability $\varsigma_{kl}^n[r]$ that when two agents of types $k, l$ break up immediately after matching, the first agent becomes of type $r$. Again for the sake of simplicity, we also assume that $\kappa^k = 0$ for all $k = 0, \ldots, N$. Then, by (2.3) and (2.5) it follows that

$$S^k = S^{k-1} + F^k - F^{k-1} + 2\Lambda^k M^k (X^k - X^{k-1})$$
$$(4.5) \qquad = S^{k-1} + F^k - F^{k-1} + 2\Lambda^k M^k \left[ \Theta^k (\hat{p}_1^k - x\hat{p}_3^k) - \Theta^{k-1}(\hat{p}_1^{k-1} - \hat{p}_3^{k-1}) \right]$$

for any $k = 1, \ldots, N$. For $P = \tilde{P} \ltimes \hat{P}$ given as in (3.2), we define a measure $Q$ of the form

$$(4.6) \qquad\qquad Q := \tilde{Q} \ltimes \hat{P},$$

where $\tilde{Q} \sim \tilde{P}$ is a probability measure on $\tilde{\Omega}$.

The next lemma will be useful in the following and shows that the Markov kernel $\hat{P}$ does not impact the expectation of $(S^k)_{k=0,\ldots,N}$: this follows from the exact law of large numbers stated at point 7 in Theorem 3.14, which permits us to obtain (4.8) in the proof of the lemma.

---

[6]A trading strategy $\gamma = (\gamma(0), \gamma(1))$ with $\gamma(j) = (\gamma^i(j))_{i=1,\ldots,N}$ for $j = 0, 1$ is called self-financing if $\gamma^i \cdot S^i = \gamma^{i+1} \cdot S^{i+1}$ for $i = 1, \ldots, N-1$ with $\gamma^i \cdot S^i := \gamma^i(0) + \gamma^i(1)S^i$.

**Lemma 4.7.** *Consider a random matching mechanism with immediate break-up. Let $S = (S^k)_{k=0,\ldots,n}$ be of the form (4.5) and let $Q$ be given as in (4.6). Then it holds that*

$$\mathbb{E}^Q\left[S^k - S^{k-1}|\mathcal{F}^{k-1}\right] = \mathbb{E}^{\tilde{Q}}\left[F^k - F^{k-1}|\tilde{\mathcal{F}}^{k-1}\right]$$
$$+ 2\mathbb{E}^{\tilde{Q}}\left[\Lambda^k M^k \Theta^k \left(\Gamma_{1J}^k(\cdot, \hat{p}^{k-1}) - \Gamma_{3J}^k(\cdot, \hat{p}^{k-1})\right)|\tilde{\mathcal{F}}^{k-1}\right]$$
$$- 2\Theta^{k-1}(\hat{p}_1^{k-1} - \hat{p}_3^{k-1})\mathbb{E}^{\tilde{Q}}\left[\Lambda^k M^k|\tilde{\mathcal{F}}^{k-1}\right], \quad k = 1, \ldots, n,$$

*for $\Gamma$ defined in (3.24).*

*Proof.* Fix $k = 1, \ldots, n$. Since $(F^i)_{i=0,\ldots N}$, $(M^i)_{i=0,\ldots N}$, $(\Lambda^i)_{i=0,\ldots N}$, $(\Theta^i)_{i=0,\ldots,N}$ are defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{Q})$, by (4.5) we have

$$\mathbb{E}^Q\left[S^k - S^{k-1}|\mathcal{F}^{k-1}\right] = \mathbb{E}^{\tilde{Q}}\left[F^k - F^{k-1}|\tilde{\mathcal{F}}^{k-1}\right] + 2\mathbb{E}^Q\left[\Lambda^k M^k \Theta^k \left(\hat{p}_1^k - \hat{p}_3^k\right)|\mathcal{F}^{k-1}\right]$$

(4.7)
$$- 2\Theta^{k-1}(\hat{p}_1^{k-1} - \hat{p}_3^{k-1})\mathbb{E}^{\tilde{Q}}\left[\Lambda^k M^k|\tilde{\mathcal{F}}^{k-1}\right].$$

From the definition of $Q$ in (4.6) we get

$$\mathbb{E}^Q\left[\Lambda^k M^k \Theta^k \hat{p}^k\right] = \int_{\tilde{\Omega}} \Lambda^k(\tilde{\omega}) M^k(\tilde{\omega}) \Theta^k(\tilde{\omega}) \mathbb{E}^{\hat{P}^{\tilde{\omega}}}\left[\hat{p}^k\right] d\tilde{Q}(\tilde{\omega})$$
$$= \int_{\tilde{\Omega}} \Lambda^k(\tilde{\omega}) M^k(\tilde{\omega}) \Theta^k(\tilde{\omega}) \Gamma^k\left(\tilde{\omega}, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{k-1}]\right) d\tilde{Q}(\tilde{\omega})$$

(4.8)
$$= \int_{\tilde{\Omega}} \Lambda^k(\tilde{\omega}) M^k(\tilde{\omega}) \Theta^k(\tilde{\omega}) \Gamma^k\left(\tilde{\omega}, \hat{p}^{k-1}\right) d\tilde{Q}(\tilde{\omega})$$

(4.9)
$$= \mathbb{E}^{\tilde{Q}}\left[\Lambda^k M^k \Theta^k \Gamma^k\left(\cdot, \hat{p}^{k-1}\right)\right]$$

by points 1 and 7 in Theorem 3.14. By (4.1) we have

$$\hat{p}_1^k - \hat{p}_3^k = \sum_{j=1}^3 \hat{p}_{1j}^k + \hat{p}_{1J}^k - \sum_{j=1}^3 \hat{p}_{3j}^k - \hat{p}_{3J}^k = \hat{p}_{1J}^k - \hat{p}_{3J}^k,$$

as the agents immediately break up. Then by (4.9) it follows that

(4.10)  $$\mathbb{E}^Q\left[\Lambda^k M^k \Theta^k \left(\hat{p}_1^k - \hat{p}_3^k\right)|\mathcal{F}^{k-1}\right] = \mathbb{E}^{\tilde{Q}}\left[\Lambda^k M^k \Theta^k \left(\Gamma_{1J}^k(\cdot, \hat{p}^{k-1}) - \Gamma_{3J}^k(\cdot, \hat{p}^{k-1})\right)|\tilde{\mathcal{F}}^{k-1}\right],$$

where we use that $\Gamma_{ij}^k(\tilde{\omega}, \hat{p}) = 0$ for all $i, j = 1, 2, 3, \tilde{\omega} \in \tilde{\Omega}, \hat{p} \in \hat{\Delta}$ by (3.23), as $\xi_{k_1 l_1} = 1$ for any $k_1, l_1 = 1, 2, 3$.

Putting together (4.7) and (4.10) we get the result.  ∎

**Proposition 4.8.** *Consider a random matching mechanism with immediate break-up. Let $\tilde{Q}$ be a probability measure equivalent to $\tilde{P}$ such that*
- *the stochastic processes $(F^k)_{k=1,\ldots,N}$ and $(\Theta^k)_{k=1,\ldots,N}$ are $(\tilde{Q}, \tilde{\mathbb{F}})$-martingales;*
- *for any $k = 1, \ldots, n$ it holds that*

(4.11)  $$\mathbb{E}^{\tilde{Q}}\left[\Gamma_{1J}^k(\cdot, \hat{p}^{k-1}) - \Gamma_{3J}^k(\cdot, \hat{p}^{k-1})|\tilde{\mathcal{F}}^{k-1}\right] = \hat{p}_1^{k-1} - \hat{p}_3^{k-1};$$

- *the stochastic processes $(\Gamma_{1J}^k(\cdot,\hat{p}^{k-1}) - \Gamma_{3J}^k(\cdot,\hat{p}^{k-1}))_{k=1,\ldots,N}$, $(\Lambda^k M^k)_{k=1,\ldots,N}$, and $(\Theta^k)_{k=1,\ldots,N}$ are pairwise conditionally independent under $\tilde{Q}$ given $\tilde{\mathcal{F}}^{k-1}$.*
  *Then $Q := \tilde{Q} \ltimes \hat{P}$ is an equivalent martingale measure for the discounted asset price $S$. Hence, the market model introduced in section 4 is arbitrage-free.*

*Proof.* The measure $Q := \tilde{Q} \ltimes \hat{P}$ is clearly equivalent to $P$ since $\tilde{Q} \sim \tilde{P}$. By Lemma 4.7 we have

$$
\begin{aligned}
\mathbb{E}^Q \left[ S^k - S^{k-1} | \mathcal{F}^{k-1} \right] &= \mathbb{E}^{\tilde{Q}} \left[ F^k - F^{k-1} | \tilde{\mathcal{F}}^{k-1} \right] \\
&\quad + 2\mathbb{E}^{\tilde{Q}} \left[ \Lambda^k M^k \Theta^k \left( \Gamma_{1J}^k(\cdot,\hat{p}^{k-1}) - \Gamma_{3J}^k(\cdot,\hat{p}^{k-1}) \right) | \tilde{\mathcal{F}}^{k-1} \right] \\
&\quad - 2\Theta^{k-1}(\hat{p}_1^{k-1} - \hat{p}_3^{k-1}) \mathbb{E}^{\tilde{Q}} \left[ \Lambda^k M^k | \tilde{\mathcal{F}}^{k-1} \right] \\
&= 2\mathbb{E}^{\tilde{Q}} \left[ \Lambda^k M^k | \tilde{\mathcal{F}}^{k-1} \right] \mathbb{E}^{\tilde{Q}} \left[ \Theta^k | \tilde{\mathcal{F}}^{k-1} \right] \\
&\quad \cdot \mathbb{E}^{\tilde{Q}} \left[ \left( \Gamma_{1J}^k(\cdot,\hat{p}^{k-1}) - \Gamma_{3J}^k(\cdot,\hat{p}^{k-1}) \right) | \tilde{\mathcal{F}}^{k-1} \right] \\
&\quad - 2\Theta^{k-1}(\hat{p}_1^{k-1} - \hat{p}_3^{k-1}) \mathbb{E}^{\tilde{Q}} \left[ \Lambda^k M^k | \tilde{\mathcal{F}}^{k-1} \right] \\
&= 0,
\end{aligned}
$$

where the second equality comes from the martingale property of $F$ and from the conditional independency assumption, whereas the last one follows from the martingale property of $(\Theta^k)_{k=1,\ldots,N}$ and from (4.11). ∎

In the following, we analyze under which conditions there exists a probability measure $\tilde{Q} \sim \tilde{P}$ satisfying the assumptions of Proposition 4.8. Similarly to Example 4.2 we choose suitable functions $\tilde{\eta}_{ij}, \theta_{ij}, \tilde{\varsigma}_{ilj} : \tilde{\Omega} \times \mathbb{N} \to [0,1]$, for any $i,j,l = 1,2,3$, $i \neq j$, and define

$$
\text{(4.12)} \qquad \eta_{ij}(\tilde{\omega},k,\hat{p}^{k-1}) := \tilde{\eta}_{ij}(\tilde{\omega},k) + f_{ij}\left( \hat{p}_{1J}^{k-1} - \hat{p}_{3J}^{k-1} \right),
$$

$$
\text{(4.13)} \qquad \theta_{il}(\tilde{\omega},k,\tilde{p}^k) := \theta(\tilde{\omega},k)\tilde{p}_{lJ}^k,
$$

$$
\text{(4.14)} \qquad \varsigma_{il}[j](\tilde{\omega},k,\tilde{p}^k) := \delta_{\{l\}}(j)\left( \tilde{\varsigma}_{ilj}(\tilde{\omega},k) + g_{ilj}\left( \tilde{p}_{1J}^k - \tilde{p}_{3J}^k \right) \right),
$$

for $k = 1,\ldots,N$, where $f_{ij}, g_{ilj} : \mathbb{R} \to [0,1]$, in such a way that (3.5) holds and that $\eta_{ij}, \varsigma_{il}[j] \in [0,1]$. Here $\hat{p}^{k-1}$ is the distribution of agent types after the $(k-1)$th time step and $\tilde{p}^k$ is the distribution of types after the random change at the $k$th time step. Finally, in view of (3.4) and (3.7), we define

$$
\text{(4.15)} \qquad \eta_{ii}(\tilde{\omega},k,\hat{p}^{k-1}) = 1 - \sum_{j=1,j\neq i}^{3} \eta_{ij}(\tilde{\omega},k,\hat{p}^{k-1})
$$

and

$$
\text{(4.16)} \qquad \varsigma_{il}[i](\tilde{\omega},k,\tilde{p}^k) = 1 - \sum_{j=1,j\neq i}^{3} \varsigma_{il}[j](\tilde{\omega},k,\tilde{p}^k).
$$

Note that (4.14) and (4.16) imply that

$$(4.17) \qquad \varsigma_{ii}[i](\tilde{\omega}, k, \tilde{p}^k) = 1 - \sum_{j=1, j\neq i}^{3} \varsigma_{ii}[j](\tilde{\omega}, k, \tilde{p}^k) = 1,$$

that is, when two agents of the same type meet, they keep their type.

*Remark* 4.9. In the construction (4.12)–(4.14), type changes are governed by a stochastic driver defined on the space $\tilde{\Omega}$ plus a term which only depends on the former distribution of types. Also note that $\varsigma_{ij}[l]$ depends in general on the distribution $\tilde{\tilde{p}}$ immediately after the break-up, which coincides with the distribution $\tilde{p}$ before the matching, as agents break up immediately.

*Lemma* 4.10. *Consider a random matching mechanism with immediate break-up, where the functions $\eta_{ij}, \theta_{ij}, \varsigma_{ij}[l]$ for $i, j, l = 1, 2, 3$ are defined in (4.12)–(4.14). Let $Q$ be given as in (4.6). Then it holds that*

$$\Gamma_{iJ}^k(\tilde{\omega}, \hat{p}^{k-1}) = (1 - \theta(\tilde{\omega}, k)) F_i^{\tilde{\omega}}(\hat{p}^{k-1}) + \theta(\tilde{\omega}, k) F_i^{\tilde{\omega}}(\hat{p}^{k-1}) F_i^{\tilde{\omega}}(\hat{p}^{k-1})$$

$$+ \sum_{k_1=1, k_1 \neq i}^{3} \left( \tilde{\varsigma}_{k_1 ii}(\tilde{\omega}, k) + g_{k_1 ii} \left( F_1^{\tilde{\omega}}(\hat{p}^{k-1}) - F_3^{\tilde{\omega}}(\hat{p}^{k-1}) \right) \right.$$

$$\left. + 1 - \tilde{\varsigma}_{ik_1 k_1}(\tilde{\omega}, k) - g_{ik_1 k_1} \left( F_{\tilde{\omega}}^1(\hat{p}^{k-1}) - F_3^{\tilde{\omega}}(\hat{p}^{k-1}) \right) \right)$$

$$(4.18) \qquad \cdot F_i^{\tilde{\omega}}(\hat{p}^{k-1}) F_{k_1}^{\tilde{\omega}}(\hat{p}^{k-1}) \theta(\tilde{\omega}, k),$$

*for $i = 1, 2, 3$ and $k \geq 2$, where*

$$F_i^{\tilde{\omega}}(\hat{p}^{k-1}) := \sum_{l=1, l\neq i}^{3} \hat{p}_{lJ}^{k-1} \left( \tilde{\eta}_{li}(\tilde{\omega}, k) + f_{li} \left( \hat{p}_{1J}^{k-1} - \hat{p}_{3J}^{k-1} \right) \right)$$

$$(4.19) \qquad + \hat{p}_{iJ}^{k-1} \left( 1 - \sum_{l=1, l\neq i}^{3} \left( \tilde{\eta}_{il}(\tilde{\omega}, k) + f_{il} \left( \hat{p}_{1J}^{k-1} - \hat{p}_{3J}^{k-1} \right) \right) \right).$$

*Proof.* By (3.24), (4.12)–(4.14), and (4.16)–(4.17), we have

$$\Gamma_{iJ}^k(\tilde{\omega}, \hat{p}^{k-1})$$

$$= (1 - \theta(\tilde{\omega}, k)) \tilde{p}_{iJ}^k + \theta(\tilde{\omega}, k) \tilde{p}_{iJ}^k \tilde{p}_{iJ}^k$$

$$+ \sum_{k_1=1, k_1 \neq i}^{3} \left( \tilde{\varsigma}_{k_1 ii}(\tilde{\omega}, k) + g_{k_1 ii} \left( \tilde{p}_{1J}^k - \tilde{p}_{3J}^k \right) + 1 - \tilde{\varsigma}_{ik_1 k_1}(\tilde{\omega}, k) - g_{ik_1 k_1} \left( \tilde{p}_{1J}^k - \tilde{p}_{3J}^k \right) \right)$$

$$(4.20) \qquad \cdot \tilde{p}_{iJ}^k \tilde{p}_{k_1 J}^k \theta(\tilde{\omega}, k),$$

with $\tilde{p}_{iJ}^k = F_i^{\tilde{\omega}}(\hat{p}^{k-1})$, where $F_i^{\tilde{\omega}}(\hat{p}_{t_{k-1}})$ is defined in (4.19). ∎

We now give an example where the functions $(\tilde{\eta}_{ij})_{i,j=1,2,3}, \theta, (\tilde{\varsigma}_{ijk})_{i,j,k=1,2,3}$ can be chosen to guarantee the existence of a probability measure $\tilde{Q} \sim \tilde{P}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ satisfying the assumptions of

Proposition 4.8. Proposition 4.8 then implies that the associated $Q := \tilde{Q} \ltimes \hat{P}$ is an equivalent martingale measure for the market price $S$ of the asset, and consequently that the market model is arbitrage-free.

*Example* 4.11. Let

$$\tilde{\Omega} := \prod_{k=1}^{N} \left( \tilde{\Omega}_k \times \tilde{\tilde{\Omega}}_k \right), \tag{4.21}$$

with $\tilde{\Omega}_k := \{\tilde{\omega}_1^k, \tilde{\omega}_2^k\}$ for $k = 1, \ldots, N$ and $\tilde{\tilde{\Omega}}_k := \prod_{r=1}^{3} \tilde{\tilde{\Omega}}_{k,r}$ where $\tilde{\tilde{\Omega}}_{k,r} = \{\tilde{\tilde{\omega}}_{i_r}^k, \ldots, \tilde{\tilde{\omega}}_{i_{r+1}}^k\}$ with $i_1 = 1$, $i_2 = l_1$, $i_3 = l_2$, $i_4 = l$ for finite numbers $1 < l_1 < l_2 < l$, endowed with $\sigma$-algebras $\tilde{\mathcal{F}}^k$ and $\tilde{\tilde{\mathcal{F}}}^k$, respectively. Without mentioning any further, we assume that each $\sigma$-algebra is generated by the subsets of the corresponding space. Moreover, we denote by $\omega, \tilde{\omega}^k$ elements of $\Omega, \tilde{\Omega}^k$ for $k = 1, \ldots, N$, respectively. Introduce $P := \tilde{P} \ltimes \hat{P}$ by choosing

$$\tilde{P} := \bigotimes_{k=1}^{N} \left( \tilde{P}_k \otimes \tilde{\tilde{P}}_k \right)$$

with $\tilde{P}_k$ probability measure on $\tilde{\Omega}_k$ and $\tilde{\tilde{P}}_k = \prod_{r=1}^{3} \tilde{\tilde{P}}_{k,r}$ where $\tilde{\tilde{P}}_{k,1}$, $\tilde{\tilde{P}}_{k,2}$, and $\tilde{\tilde{P}}_{k,3}$ are probability measures on $\tilde{\tilde{\Omega}}_{k,1}$, $\tilde{\tilde{\Omega}}_{k,2}$, and $\tilde{\tilde{\Omega}}_{k,3}$, respectively, for any $k = 1, \ldots, N$. Moreover, we assume that

$$\tilde{P}_k(\tilde{\omega}_i^k) > 0 \quad \text{for any } i = 1, 2, \ k = 1, \ldots, N, \tag{4.22}$$

and $\tilde{\tilde{P}}_k(\tilde{\tilde{\omega}}_i^k) > 0$ for any $i = 1, \ldots, l$, $k = 1, \ldots, N$. Let the processes $\Theta$ and $F$ be defined on $\prod_{k=1}^{N} \tilde{\tilde{\Omega}}_{k,1}$ and $\prod_{k=1}^{N} \tilde{\tilde{\Omega}}_{k,2}$, respectively. We assume that there exist $\tilde{\tilde{Q}}_{k,1} \sim \tilde{\tilde{P}}_{k,1}$, $\tilde{\tilde{Q}}_{k,2} \sim \tilde{\tilde{P}}_{k,2}$ under which $\Theta$ and $F$ are martingales, respectively. Analogously, let $\Lambda$ and $M$ be defined on $\prod_{k=1}^{N} \tilde{\tilde{\Omega}}_{k,3}$ and $\tilde{\tilde{Q}}_{k,3} \sim \tilde{\tilde{P}}_{k,3}$ be a martingale measure for $\Lambda M$. Set

$$\tilde{\tilde{Q}}_k = \prod_{r=1}^{3} \tilde{\tilde{Q}}_{k,r}, \quad k = 1, \ldots, n. \tag{4.23}$$

We introduce

$$\theta(\tilde{\omega}, k) := \theta_1(k) \mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_1^k\}} + \theta_2(k) \mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_2^k\}} \tag{4.24}$$

and

$$\tilde{\eta}_{ij}(\tilde{\omega}, k) = \begin{cases} \tilde{\eta}_1^{13}(k) \mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_1^k\}} + \tilde{\eta}_2^{13}(k) \mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_2^k\}} & \text{for } i = 1, \ j = 3, \\ \tilde{\eta}_1^{31}(k) \mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_1^k\}} + \tilde{\eta}_2^{31}(k) \mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_2^k\}} & \text{for } i = 3, \ j = 1, \\ \tilde{\eta}_1^{21}(k) \mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_1^k\}} + \tilde{\eta}_2^{21}(k) \mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_2^k\}} & \text{for } i = 2, \ j = 1, \\ \tilde{\eta}_1^{23}(k) \mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_1^k\}} + \tilde{\eta}_2^{23}(k) \mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_2^k\}} & \text{for } i = 2, \ j = 3, \\ 0 & \text{for all other indices,} \end{cases} \tag{4.25}$$

for some functions $\tilde{\eta}_1^{13}, \tilde{\eta}_2^{13}, \tilde{\eta}_1^{31}, \tilde{\eta}_2^{31}, \tilde{\eta}_1^{21}, \tilde{\eta}_2^{21}, \tilde{\eta}_1^{23}, \tilde{\eta}_2^{23} : \{1, \ldots, N\} \to [0, 1/2]$. The functions (4.14) are defined by

$$(4.26) \qquad \tilde{\varsigma}_{ijl}(\tilde{\omega}, k) = \begin{cases} \tilde{\varsigma}_1^{13}(k)\mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_1^k\}} + \tilde{\varsigma}_2^{13}(k)\mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_2^k\}} & \text{for } i = 1, \ j = 3, \ l = 3, \\ \tilde{\varsigma}_1^{31}(k)\mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_1^k\}} + \tilde{\varsigma}_2^{31}(k)\mathbf{1}_{\{\tilde{\omega}^k = \tilde{\omega}_2^k\}} & \text{for } i = 3, \ j = 1, \ l = 1, \\ 0 & \text{for all other indices,} \end{cases}$$

for some functions

$$(4.27) \qquad \tilde{\varsigma}_1^{31}, \tilde{\varsigma}_2^{31}, \tilde{\varsigma}_1^{13}, \tilde{\varsigma}_2^{13} : \{1, \ldots, N\} \to [0, 1/2].$$

We assume that the functions $f_{ij} : \mathbb{R} \to [0, 1]$, $i, j = 1, 2, 3$, $i \neq j$, which appear in (4.12) satisfy

$$(4.28) \qquad f_{2,1}(x), f_{3,1}(x), f_{3,2}(x) \in (0, 1/2], \quad f_{1,2}(x) = f_{2,3}(x) = f_{1,3}(x) = 0,$$

$$(4.29) \qquad f_{3,1}(x) + f_{3,2}(x) \leq 1/2,$$

for any $x > 0$.

Moreover, we assume that $g_{ijj} : \mathbb{R} \to [0, 1]$, $i, j = 1, 2, 3$, $i \neq j$, appearing in (4.14) is such that

$$(4.30) \qquad g_{211}(x), g_{311}(x), g_{322}(x) \in (0, 1/2],$$

$$(4.31) \qquad g_{133}(x), g_{122}(x), g_{233}(x) = 0$$

for any $x > 0$.

Finally, we assume that (4.28) and (4.30)–(4.31) hold switching the indices 1 and 3 when $x < 0$.

*Remark* 4.12. In (4.24)–(4.27) the probabilities governing type changes which are defined on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ are identified only by two possible states at every time. Moreover, we only allow for random type changes before the matching to pessimistic and optimistic type. For post matchings type change, we assume that neutral investors cannot change their type. Such assumptions allow for less lengthy computations in the following. An extension to a more general case (see, for example, subsection 4.2), can be easily provided. In particular, when the probabilities are identified by a higher number of states, we have more degrees of freedom for defining the measure $\tilde{Q}$.

Assumptions (4.28) and (4.30)–(4.31) are in line with the model in section 4. In particular, under (4.28), if there are more optimistic than pessimistic investors, agents can switch to more pessimistic views only due to some exogenous, stochastic effects modeled by $\tilde{\eta}$. Furthermore, there is instead a strictly positive term, depending on the difference between optimistic and pessimistic investors, which increases the probability that pessimistic or neutral traders switch to more optimistic forecasts. The reverse happens if there are more pessimistic than optimistic traders.

Similar considerations hold for the upgrade/downgrade probabilities in (4.14). Under the hypothesis $\hat{p}_1 > \hat{p}_3$, there is always a strictly positive term which increases the probability of upgrade of type (see (4.26)), whereas the probability downgrade of type after matching is

only given by a stochastic, exogenous effect represented by $\tilde{\varsigma}$, see (4.26). Again, the reverse holds when $\hat{p}_1 < \hat{p}_3$.

Further bounds on conditions on the involved functions are necessary to guarantee that the type change probabilities remain in $(0,1)$.

The following lemma is a straightforward consequence of the construction of $\tilde{\Omega}$ and $\tilde{P}$, together with the definition of the processes $\Theta$, $F$, $\Lambda$, and $M$ in Example 4.11 and with (4.22).

**Lemma 4.13.** *In the setting of Example* 4.11, *introduce the probability measure* $Q := \tilde{Q} \ltimes \hat{P}$ *with*

$$(4.32) \qquad \tilde{Q} := \bigotimes_{k=1}^{N} \left( \tilde{Q}_k \otimes \tilde{\tilde{Q}}_k \right),$$

*where* $\tilde{\tilde{Q}}_k$ *is defined in* (4.23) *and* $\tilde{Q}_k(\tilde{\omega}_i^k) > 0$ *for any* $i = 1, 2, k = 1, \ldots, N$. *Then* $Q$ *satisfies Proposition* 4.8 *and is equivalent to* $P$.

We then get the following result.

**Proposition 4.14.** *In the setting of Example* 4.11, *there exist functions* $\tilde{\eta}_i^{31}$, $\tilde{\eta}_i^{13}$, $\tilde{\eta}_i^{21}$, $\tilde{\eta}_i^{23}$, $\theta_i$, $\tilde{\varsigma}_i^{31}$, $\tilde{\varsigma}_i^{13}$, $i = 1, 2$, *appearing in* (4.24)–(4.26) *such that the market is arbitrage-free.*

*Proof.* Let $\tilde{Q}$ be of the form (4.32). Moreover, define

$$(4.33) \qquad q(k) := \tilde{Q} \left( \prod_{l=1}^{k-1} \tilde{\Omega}_l \times \tilde{\omega}_1^k \times \prod_{l=k+1}^{N} \tilde{\Omega}_l \times \prod_{l=1}^{N} \tilde{\tilde{\Omega}}_l \right) = \tilde{Q}_k(\tilde{\omega}_1^k)$$

for any $k = 1, \ldots, N$. By Proposition 4.8 and Lemma 4.13, we need to find $q(k)$ for $k \in \{1, \ldots, N\}$ such that

$$
\begin{aligned}
& \hat{p}_1^{k-1} - \hat{p}_3^{k-1} \\
&= \mathbb{E}^Q \left[ \hat{p}_1^k - \hat{p}_3^k | \mathcal{F}^{k-1} \right] \\
(4.34) \qquad &= \mathbb{E}^{\tilde{Q}} \left[ \Gamma_{1J}^k(\cdot, \hat{p}^{k-1}) - \Gamma_{3J}^k(\cdot, \hat{p}^{k-1}) | \tilde{\mathcal{F}}^{k-1} \right] \\
&= \mathbb{E}^{\tilde{Q}_k} \left[ \Gamma_{1J}^k(\cdot, \hat{p}^{k-1}) - \Gamma_{3J}^k(\cdot, \hat{p}^{k-1}) | \tilde{\mathcal{F}}^{k-1} \right] \\
(4.35) \qquad &= q(k) a_1 + (1 - q(k)) a_2,
\end{aligned}
$$

where

$$
\begin{aligned}
(4.36) \quad a_i =\ & (1 - \theta_i(k)) \left[ F_i^1(\hat{p}^{k-1}) - F_i^3(\hat{p}^{k-1}) \right] \\
& + \theta_i(k) \left( F_i^1(\hat{p}^{k-1}) F_i^1(\hat{p}^{k-1}) - F_i^3(\hat{p}^{k-1}) F_i^3(\hat{p}^{k-1}) \right) \\
& + \theta_i(k) \Big( \Big[ g_{211} \left( F_i^1(\hat{p}^{k-1}) - F_i^3(\hat{p}^{k-1}) \right) + \Big( 1 - g_{122} \left( F_i^1(\hat{p}^{k-1}) - F_i^1(\hat{p}^{k-1}) \right) \Big) \Big] \\
& \quad \cdot F_i^1(\hat{p}^{k-1}) F_i^2(\hat{p}^{k-1}) \\
& \quad - \Big[ g_{233} \left( F_i^1(\hat{p}^{k-1}) - F_i^3(\hat{p}^{k-1}) \right) + \Big( 1 - g_{322} \left( F_i^1(\hat{p}^{k-1}) - F_i^1(\hat{p}^{k-1}) \right) \Big) \Big]
\end{aligned}
$$

$$\cdot F_i^3(\hat{p}^{k-1})F_i^2(\hat{p}^{k-1})$$
$$+\left[\tilde{\varsigma}_i^{31}(k)+g_{311}\left(F_i^1(\hat{p}^{k-1})-F_i^3(\hat{p}^{k-1})\right)-\tilde{\varsigma}_i^{13}(k)-g_{113}\left(F_i^1(\hat{p}^{k-1})-F_i^3(\hat{p}^{k-1})\right)\right]$$
$$\cdot F_i^1(\hat{p}^{k-1})F_i^3(\hat{p}^{k-1})\Big),$$

with

$$F_i^1(\hat{p}^{k-1})=\hat{p}_{2J}^{k-1}\left(\tilde{\eta}_i^{21}(k)+f_{21}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)\right)+\hat{p}_{3J}^{k-1}\left(\tilde{\eta}_i^{31}(k)+f_{31}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)\right)$$

$$(4.37)\qquad +\hat{p}_{1J}^{k-1}\left(1-f_{12}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)-\tilde{\eta}_i^{13}(k)-f_{13}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)\right)$$

and

$$F_i^3(\hat{p}^{k-1})=\hat{p}_{2J}^{k-1}\left(\tilde{\eta}_i^{23}(k)+f_{23}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)\right)+\hat{p}_{1J}^{k-1}\left(\tilde{\eta}_i^{13}(k)+f_{13}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)\right)$$

$$(4.38)\qquad +\hat{p}_{3J}^{k-1}\left(1-f_{32}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)-\tilde{\eta}_i^{31}(k)-f_{31}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)\right)$$

for $i=1,2$. Note that (4.34) comes from Lemma 4.7, whereas (4.35) follows by (4.18) and (4.24)–(4.26).

By (4.35) it follows that $q(k)$ has to satisfy

$$(4.39)\qquad q(k)=\frac{\hat{p}_1^{k-1}-\hat{p}_3^{k-1}-a_2}{a_1-a_2}=\frac{a_2+\hat{p}_3^{k-1}-\hat{p}_1^{k-1}}{a_2-a_1}$$

and that

$$(4.40)\qquad a_1<\hat{p}_1^{k-1}-\hat{p}_3^{k-1}\quad\text{and}\quad a_2>\hat{p}_1^{k-1}-\hat{p}_3^{k-1}$$

or

$$(4.41)\qquad a_1>\hat{p}_1^{k-1}-\hat{p}_3^{k-1}\quad\text{and}\quad a_2<\hat{p}_1^{k-1}-\hat{p}_3^{k-1}$$

to guarantee that $q(k)\in(0,1)$. Without loss of generality we assume from now on that $\hat{p}_1^{k-1}-\hat{p}_3^{k-1}>0$, as identical considerations hold with opposite sign if $\hat{p}_1^{k-1}-\hat{p}_3^{k-1}<0$. The goal is to find conditions on $\tilde{\eta}_i^{31}(k),\tilde{\eta}_i^{13}(k),\tilde{\eta}_i^{21}(k),\tilde{\eta}_i^{23}(k),\theta_i(k),\tilde{\varsigma}_i^{31}(k),\tilde{\varsigma}_i^{13}(k)$, $i=1,2$, such that (4.40) holds. In order to do that, we focus on $a_2$.

We first derive conditions which guarantee that $a_2>\hat{p}_1^{k-1}-\hat{p}_3^{k-1}$. Choose $\tilde{\eta}_2^{31}(k)$ and $\tilde{\eta}_2^{13}(k)$ such that

$$(4.42)\qquad \frac{\tilde{\eta}_2^{13}(k)}{\tilde{\eta}_2^{31}(k)}<\frac{\hat{p}_3^{k-1}}{\hat{p}_1^{k-1}},$$

and (to simplify the following computations) $\tilde{\eta}_2^{21}(k)=\tilde{\eta}_2^{23}(k)=0$. Also fix $\tilde{\varsigma}_2^{31}(k)>\tilde{\varsigma}_2^{13}(k)$. Then by (4.37)–(4.38), by assumption (4.28), and since $\hat{p}_1^{k-1}-\hat{p}_3^{k-1}>0$, we have that

$$F_2^1(\hat{p}^{k-1})-F_2^3(\hat{p}^{k-1})=\hat{p}_{2J}^{k-1}f_{21}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)+\hat{p}_{3J}^{k-1}\left(\tilde{\eta}_2^{31}(k)+f_{31}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)\right)$$
$$+\hat{p}_{1J}^{k-1}\left(1-\tilde{\eta}_2^{13}(k)\right)-\hat{p}_{1J}^{k-1}\tilde{\eta}_2^{13}(k)$$
$$(4.43)\qquad -\hat{p}_{3J}^{k-1}\left(1-f_{32}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)-\tilde{\eta}_2^{31}(k)-f_{31}\left(\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1}\right)\right)$$
$$(4.44)\qquad \geq\hat{p}_{1J}^{k-1}\left(1-2\tilde{\eta}_2^{13}(k)\right)-\hat{p}_{3J}^{k-1}\left(1-2\tilde{\eta}_2^{31}(k)\right)>\hat{p}_{1J}^{k-1}-\hat{p}_{3J}^{k-1},$$

where the last inequality follows from (4.42).

Similarly, by (4.36), (4.44), by assumptions (4.30)–(4.31), and again since $\hat{p}_1^{k-1} - \hat{p}_3^{k-1} > 0$, we get

$$
\begin{aligned}
a_2 &\geq (1 - \theta_2(k)) \left[ F_2^1(\hat{p}^{k-1}) - F_2^3(\hat{p}^{k-1}) \right] \\
&\quad + \theta_2(k)\left( F_2^2(\hat{p}^{k-1}) \left( F_2^1(\hat{p}^{k-1}) - F_2^3(\hat{p}^{k-1}) \right) + \left( \tilde{\varsigma}_2^{31}(k) - \tilde{\varsigma}_2^{13}(k) \right) F_2^1(\hat{p}^{k-1}) F_2^3(\hat{p}^{k-1}) \right), \\
&> (1 - \theta_2(k)) \left[ F_2^1(\hat{p}^{k-1}) - F_2^3(\hat{p}^{k-1}) \right].
\end{aligned}
$$

In order to guarantee that $a_2 > \hat{p}_1^{k-1} - \hat{p}_3^{k-1}$, we then choose $\theta_2(k)$ such that

$$
1 - \theta_2(k) > \frac{\hat{p}_1^{k-1} - \hat{p}_3^{k-1}}{F_2^1(\hat{p}^{k-1}) - F_2^3(\hat{p}^{k-1})}.
$$

By (4.44) this is possible if $\theta_2$ is small enough.

Next, we derive conditions which guarantee that $a_2 < \hat{p}^{k-1} - 1 - \hat{p}_3^{k-1}$. Choose

(4.45)                    $\tilde{\eta}_2^{31}(k) = \tilde{\eta}_2^{21}(k) = 0$ and $\tilde{\eta}_2^{13}(k) = \tilde{\eta}_2^{23}(k) = 1/2$.

From (4.37)–(4.38) we derive

$$
\begin{aligned}
F_2^1(\hat{p}^{k-1}) - F_2^3(\hat{p}^{k-1}) &= \hat{p}_{2J}^{k-1} f_{21}\left( \hat{p}_{1J}^{k-1} - \hat{p}_{3J}^{k-1} \right) + \hat{p}_{3J}^{k-1} f_{31}\left( \hat{p}_{1J}^{k-1} - \hat{p}_{3J}^{k-1} \right) \\
&\quad + \hat{p}_{1J}^{k-1}\left( 1 - \frac{1}{2} \right) - \frac{1}{2}\hat{p}_{1J}^{k-1} - \frac{1}{2}\hat{p}_{2J}^{k-1} \\
&\quad - \hat{p}_{3J}^{k-1}\left( 1 - f_{31}\left( \hat{p}_{1J}^{k-1} - \hat{p}_{3J}^{k-1} \right) - f_{32}\left( \hat{p}_{1J}^{k-1} - \hat{p}_{3J}^{k-1} \right) \right)
\end{aligned}
$$

(4.46)                          $\leq 0,$

where the first equality and (4.46) follow from (4.45) and (4.28)–(4.29), respectively. Looking now at (4.36) with $i = 2$, we can then choose $\theta_2(k)$ small enough so that $a_2 < \hat{p}_1^{k-1} - \hat{p}_3^{k-1}$. This concludes the proof. ∎

*Remark* 4.15. The parameters $\tilde{\eta}_2^{31}(k)$, $\tilde{\eta}_2^{13}(k)$, $\tilde{\eta}_2^{21}(k)$, $\tilde{\eta}_2^{23}(k)$, $\theta_2(k)$, $\tilde{\varsigma}_2^{31}(k)$, $\tilde{\varsigma}_2^{13}(k)$ given in the proof of Proposition 4.14 guarantee the existence of an equivalent martingale measure in the setting of Example 4.11. Other choices are of course possible under which an equivalent martingale measure still exists: for example, one can show that (4.45) can be relaxed.

**4.2. Numerical simulations.** In this section we provide some numerical simulations of the model defined in (2.4) and (2.5), where $\hat{p}_1$ and $\hat{p}_3$ are governed by the dynamical system introduced in section 3 with matching and type change probabilities having dynamics given by (4.12)–(4.14). In particular, we consider the following setting. We discretize the time interval $[0, T]$ in $N$ subintervals. The processes $\Lambda = (\Lambda^n)_{i=0,\dots,N}$ and $M = (M^n)_{n=0,\dots,N}$ appearing in (2.3) are binomial models, defined on the probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{P})$, which approximate two geometric Brownian motions with drift equal to zero and volatilities $\sigma_\Lambda > 0$ and $\sigma_M > 0$. That is, $\Lambda^n := Y^n \Lambda^{n-1}$, $n = 1, \dots, N$ and constant $\Lambda^0 > 0$, where $Y^n$ is a random variable defined on $(\tilde{\Omega}, \tilde{\mathfrak{F}})$ such that

$$
\tilde{P}(Y^n = u) = p, \quad \tilde{P}(Y^n = 1/u) = 1 - p,
$$

where

$$u := e^{\sigma_\Lambda T/N}, \qquad p = \frac{1-d}{u-d},$$

and $M$ is defined analogously. The functions $f_{ij}$ and $g_{ijj}$, $i, j = 1, 2, 3$, and $i \neq j$, which appear in (4.12) and (4.14), respectively, are defined by

$$(4.47) \qquad f_{ij}(x) := g_{ijj}(x) := \begin{cases} \frac{1}{3}(x^+)^{0.4} & \text{if } i = 2, j = 1 \text{ or } i = 3, j = 2, \\ \frac{1}{3}(-x^-)^{0.4} & \text{if } i = 1, j = 2 \text{ or } i = 2, j = 3, \\ \left(\frac{1}{3}(x^+)^{0.4}\right)^2 & \text{if } i = 3, j = 1, \\ \left(\frac{1}{3}(-x^-)^{0.4}\right)^2 & \text{if } i = 1, j = 3 \end{cases}$$

for any $x \in \mathbb{R}$. Note that the choice of these functions is coherent with our framework, where investors might have a type upgrade or downgrade when $\hat{p}_1 - \hat{p}_3 > 0$ or $\hat{p}_1 - \hat{p}_3 < 0$, respectively. Moreover, since $|\hat{p}_1 - \hat{p}_3| \leq 1$, a direct switch from optimistic to pessimistic views or vice versa is more unlikely to happen. Also note that, since the processes $\tilde{\eta}_{ij}$ and $\tilde{\varsigma}_{ilj}$, $i, j, l = 1, 2, 3$, are bounded by $1/4$, (4.47) guarantees that the values in both (4.12) and (4.14) are bounded by $1/2$. In this way, the sum for fixed $i$ does not exceed 1. Furthermore, we have as follows:

1. The process $\Theta = (\Theta^n)_{n=0,\dots,N}$ in (2.5) is defined by $\Theta^n := 2/\pi \arctan(Z_\Theta^n)$, where $Z_\Theta = (Z_\Theta^n)_{n=0,\dots,N}$ is a binomial model on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ approximating a geometric Brownian motion with drift equal to zero and volatility $\sigma_\Theta > 0$. Note that this choice guarantees that the dynamics of $\Theta$ stay in $(0, 1)$.

2. The process $\theta = (\theta^n)_{n=0,\dots,N}$ in (4.13) is defined by $\theta^n := 2/\pi \arctan(Z_\theta^n)$, where $Z_\theta = (Z_\theta^n)_{n=0,\dots,N}$ is a binomial model on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ approximating a geometric Brownian motion with drift equal to zero and volatility $\sigma_\theta > 0$.

3. For $i, j = 1, 2, 3$, the processes $\tilde{\eta}_{ij} = (\tilde{\eta}_{ij}^n)_{n=0,\dots,N}$ in (4.12) are defined by $\tilde{\eta}_{ij}^n := 2/\pi \arctan(Z_{\eta,i,j}^n)$, where $Z_{\eta,i,j} = (Z_{\eta,i,j}^n)_{n=0,\dots,N}$ is a binomial model on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ approximating a geometric Brownian motion with drift equal to zero and volatility $\sigma_\eta > 0$.

4. For $i, j = 1, 2, 3$, the processes $\tilde{\varsigma}_{ij} = (\tilde{\varsigma}_{ij}^n)_{n=0,\dots,N}$ in (4.14) are defined by $\tilde{\varsigma}_{ij}^n := 2/\pi \arctan(Z_{\varsigma,i,j}^n)$, where $Z_{\varsigma,i,j} = (Z_{\varsigma,i,j}^n)_{n=0,\dots,N}$ is a binomial model on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ approximating a geometric Brownian motion with drift equal to zero and volatility $\sigma_\varsigma > 0$.

The processes $Z_\Theta, Z_\theta, Z_{\eta,i,j}, Z_{\varsigma,i,j}$, $i, j = 1, 2, 3$, introduced above, are all independent of each other. We choose parameters $N = 100$, $T = 1$, $\hat{p}_1^0 = \hat{p}_2^0 = \hat{p}_3^0 = 1/3$, $\Lambda^0 = M^0 = 1$, $\sigma_\Lambda = \sigma_M = 0.3$, $\Theta^0 = 5$, $\sigma_\Theta = 0.2$, $\eta_{ij}^0 = \varsigma_{ijl}^0 = 0.2$, $i, j, l = 1, 2, 3$, $\sigma_\eta = \sigma_\varsigma = 0.4$, $\theta = 0.5$, $\sigma_\theta = 0.2$. We also set the reversion process in (2.4) to be constant, i.e., specifically, $\kappa^n = 0.01$ for all $n = 1, \dots, N$. In Figure 4.1 we show some trajectories of the process $(\hat{p}_1^n - \hat{p}_3^n)_{n=0,\dots,N}$ on the right and the bubble process $\beta$ on the left. We decided to divide the trajectories into three couples of panels for the reader's convenience. We note the following:

- In our model, bubbles can be negative: this is a difference with respect to the classical martingale theory of bubbles introduced by Cox and Hobson [12] and Loewenstein and Willard [32] and mainly developed by Jarrow, Protter, and others [23], [24], [25], [26], [22], [7], where the fundamental value is always smaller than the market value. However, this is not in contrast with absence of arbitrages and is in agreement with real markets, where asset prices might sometimes be underestimated; see, for example, [18] and [40].
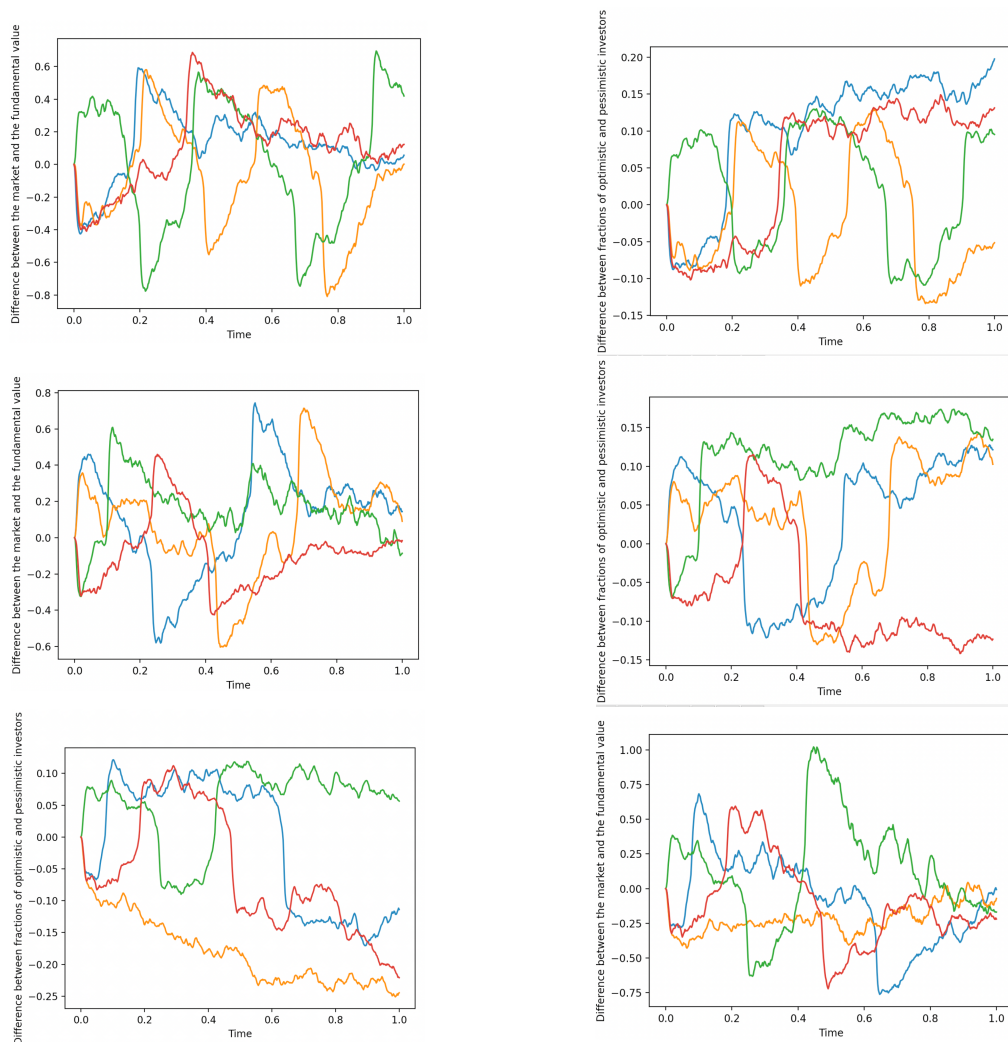
**Figure 4.1.** *Some trajectories of the process $(\hat{p}_1^n - \hat{p}_3^n)_{n=0,\dots,N}$, in the right panels, and of the bubble in* (2.4) *in the left panels. For any pair of panels, a trajectory of a given color on the right drives the bubble trajectory of the same color on the left, together with the realizations of the processes $\Lambda$, $M$, and $\theta$. Parameters are $N = 100$, $T = 1$, $\hat{p}_1^0 = \hat{p}_2^0 = \hat{p}_3^0 = 1/3$, $\Lambda^0 = M^0 = 1$, $\sigma_\Lambda = \sigma_M = 0.3$, $\Theta^0 = 5$, $\sigma_\Theta = 0.2$, $\eta_{ij}^0 = \varsigma_{ijl}^0 = 0.2$, $i, j, l = 1, 2, 3$, $\sigma_\eta = \sigma_\varsigma = 0.4$, $\theta = 0.5$, $\sigma_\theta = 0.2$, $\kappa = 0.01$.*

- Bubbles increase and also burst with different speeds. In particular, our model allows both for hard landing (i.e., steep and fast decrease) and soft landing (i.e., soft and slow decrease) after the burst of the bubble; see [6] and [45] for an analysis on hard and soft landing.

- Due to the presence of the functions $f_{ij}$ and $g_{ijj}$, $i, j, l = 1, 2, 3$, in our model investors may change type independently or after a match according to the current phase of the market, that is, if prices are rising, they have a higher probability to switch to more optimistic views, and vice versa. This drives both the ascending and the descending
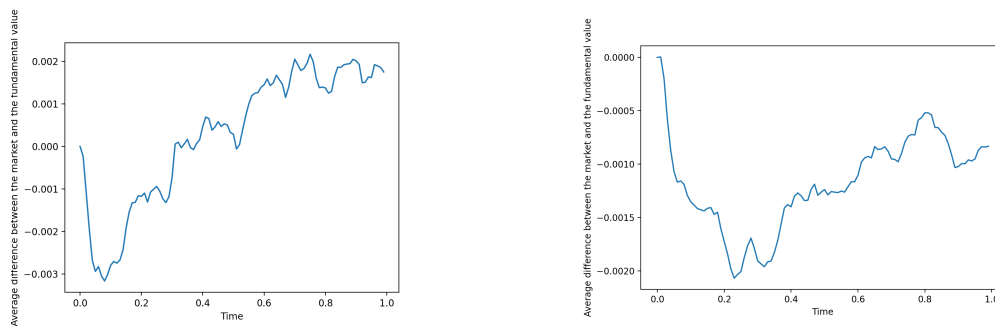
**Figure 4.2.** *Average of a sample of trajectories of the bubble process in* (2.4). *The left and right panels show the sample average for* 100,000 *and* 1,000,000 *simulations, respectively. Parameters are* $N = 100$, $T = 1$, $\hat{p}_1^0 = \hat{p}_2^0 = \hat{p}_3^0 = 1/3$, $\Lambda^0 = M^0 = 1$, $\sigma_\Lambda = \sigma_M = 0.3$, $\Theta^0 = 5$, $\sigma_\Theta = 0.2$, $\eta_{ij}^0 = \varsigma_{ijl}^0 = 0.2$, $i,j,l = 1,2,3$, $\sigma_\eta = \sigma_\varsigma = 0.4$, $\theta = 0.5$, $\sigma_\theta = 0.2$, $\kappa = 0.01$.

phase of the bubble. However, there are some exogenous factors, which we model on the space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, that may influence the bubble as well, and may anticipate or delay the burst. In particular, they can make the increase of the bubble before the burst (respectively, the decrease after the burst) more or less steep.

In Figure 4.2 we plot the function $t \to \frac{1}{n}\sum_{i=1}^n \beta_t^i$, where $\beta_t^i$ is the value for the $i$th simulated trajectory of the bubble, $i = 1, \ldots, n$. The left and right panels show the sample average for $n = 100,000$ and $n = 1,000,000$, respectively.

In Figure 4.3 we plot instead the function above for a sample of 1,000,000 trajectories in the case when the values of the fractions of investors are $\hat{p}_1^0 = 4/9, \hat{p}_2^0 = 2/9, \hat{p}_3^0 = 1/3$.[7] In this case, we see that at the beginning, the bubble blows up on average, because of the actions of the functions $f_{ij}$ and $g_{ijj}$, $i,j,l = 1,2,3$. However, as the number of pessimistic investors changing their views starts to decrease, the bubble slows down, and then bursts on average because of the action of the mean reverting term $-\kappa\beta$ in (2.4).

Figure 4.3 clearly shows that, when the fractions of pessimistic and optimistic investors are different, the bubble is not a martingale under the measure for which we simulate the processes. However, we find a measure under which the expectation of the bubble at time $t_1$ is very close to its value at $t_0$ by tuning the parameters. In particular, this measure is identified by letting the binomial processes driving $\tilde{\eta}_{13}$ and $\tilde{\varsigma}_{133}$ increasing with probability 0.95 and the ones driving $\tilde{\eta}_{31}$ and $\tilde{\varsigma}_{311}$ increasing with probability 0.1, at the first time step. We see that the average value of $\beta^1$ is 0.1 under the first measure and close to $10^{-5}$ under the new measure.

**5. Conclusions.** We have modeled the formation of asset price bubbles by introducing a random matching mechanism among agents in a discrete time version of the model in [27]. In order to do it, we extend results of [15] to a stochastic setting. In particular, via the introduction of a Markov kernel, we are able to construct the probability space where the asset price process is defined as the product of the space $\hat{\Omega}$ of the random matching and the

---

[7]We let anyway the bubble start from zero here: one can assume a sudden jump of $\hat{p}^1$ at initial time, or that $\theta = 0$ before time 0.
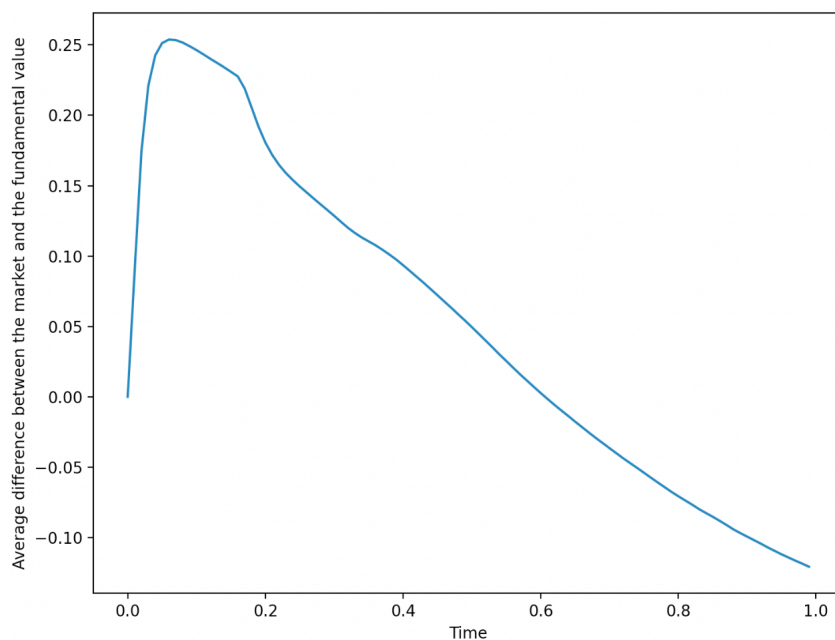
**Figure 4.3.** *Average of a sample of trajectories of the bubble process in* (2.4) *of size* 1,000,000. *Parameters are* $N = 100$, $T = 1$, $\hat{p}_1^0 = 4/9, \hat{p}_2^0 = 2/9, \hat{p}_3^0 = 1/3$, $\Lambda^0 = M^0 = 1$, $\sigma_\Lambda = \sigma_M = 0.3$, $\Theta^0 = 5$, $\sigma_\Theta = 0.2$, $\eta_{ij}^0 = \varsigma_{ijl}^0 = 0.2$, $i, j, l = 1, 2, 3$, $\sigma_\eta = \sigma_\varsigma = 0.4$, $\theta = 0.5$, $\sigma_\theta = 0.2$, $\kappa = 0.01$.

space $\tilde{\Omega}$ of the factors which may influence the transition probabilities. This approach allows us to isolate and model the self-exciting mechanism governing the blow-up of the bubble and the exogenous factors impacting the bursting phase of the bubble. In subsection 4.2 we present a numerical experiment showing how this approach is able to capture important behavioral features of asset price bubbles.

**Appendix A. Fubini extension.** We here present the definition of a rich Fubini extension, which is needed for the setting in section 2. We start by recalling the concept of essentially pairwise independent random variables in the sense of Definition 2 in [15].

**Definition A.1.** *Consider the random variable* $f : (I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P) \to (Y, \mathcal{G})$, *where* $Y$ *is a Polish space endowed with the Borelian* $\sigma$-*algebra* $\mathcal{G}$. *We set* $f_i := f(i, \cdot)$ *for all* $i \in I$. *We say that* $f$ *is* essentially pairwise independent *if for* $\lambda$-*almost all* $j \in I$, $f_j$ *is independent of* $f_i$ *for* $\lambda$-*almost all* $i \in I$.

In Proposition 2.1 of [43] and Proposition 1.1 of [41] it is shown that an essentially pairwise independent random variable, which is also jointly measurable, is constant for $\lambda$-almost all $i \in I$. This is the so called "sample measurability problem," which has been studied in [14], [29]. To overcome this issue, the $\sigma$-algebra $\mathcal{I} \otimes \mathcal{F}$ needs to be enlarged to allow jointly measurable random variables to be essentially pairwise independent but not constant. This measurability problem can be solved by working with an extension of the product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ which still satisfies the Fubini property; see [43]. We here recall the definition of a Fubini extension; see, e.g., Definition 1 of [15].

**Definition A.2.** *A probability space* $(I \times \Omega, \mathcal{W}, Q)$ *is said to be a* Fubini extension *of the product probability space* $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ *if for any real-valued $Q$-integrable random variable $f$ on $(I \times \Omega, \mathcal{W}, Q)$ we have that*

1. *the functions $f_i(\cdot) := f(i, \cdot)$ and $f_\omega(\cdot) := f(\cdot, \omega)$ are integrable on $(\Omega, \mathcal{F}, P)$ for $\lambda$-almost all $i \in I$, and on $(I, \mathcal{I}, \lambda)$ for $P$-almost all $\omega \in \Omega$, respectively;*
2. *$\int_\Omega f_i dP$ and $\int_I f_\omega d\lambda$ are integrable on $(I, \mathcal{I}, \lambda)$ and on $(\Omega, \mathcal{F}, P)$, respectively, with*

$$\int_{I \times \Omega} f dQ = \int_I \left( \int_\Omega f_i dP \right) d\lambda = \int_\Omega \left( \int_I f_i d\lambda \right) dP.$$

*The Fubini extension is denoted by* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

Moreover, note that by definition it holds that $\lambda \boxtimes P|_{\mathcal{I} \otimes \mathcal{F}} = \lambda \otimes P$. In Theorem 6.2 in [42] and Proposition 5.6 in [43] it is shown that there exists a *rich* Fubini extension, which allows the construction of processes with essentially pairwise independent and jointly measurable random variables, which are not $\lambda$-almost surely constant.

**Appendix B. Proof of Proposition** 3.12. For every fixed $\tilde{\omega} \in \tilde{\Omega}$ we construct the measure $\hat{P}^{\tilde{\omega}}$ as in the proof of Lemma 7 in [15]. Furthermore, as the construction of the space $\hat{\Omega}$ in Lemma 7 in [15] is independent of the input functions, we can also follow their approach. Then, the definitions of $\Omega, \mathcal{F},$ and $P_0$ in points 1 and 2 in the proposition allow us to finish the proof. For the reader's convenience we show the proof in detail in the following.

*Proof.* Let $(I, \mathcal{I}_0, \lambda_0)$ be the hyperfinite counting probability space with its Loeb space $(I, \mathcal{I}, \lambda)$. The proof consists of four steps.

*Step* 1: For each $k \in S$, $\tilde{\omega} \in \tilde{\Omega}$ and $\hat{p} \in {}^*\hat{\Delta}$, let

$$b_k^0(\tilde{\omega}) = b_k(\tilde{\omega}, 0, \hat{p}) := 1 - \sum_{r \in S} \theta_{kr}(\tilde{\omega}, 0, \hat{p})$$

and $I_k = \{i \in I : \alpha^0(i) = k, \pi^0(i) = i\}$.

For each $i \in I_k$, $\tilde{\omega} \in \tilde{\Omega}$ and $\hat{p} \in {}^*\hat{\Delta}$ define a probability $\zeta_i^{\hat{p}, \tilde{\omega}}$ on $S \cup \{J\}$ such that

$$\zeta_i^{\hat{p}, \tilde{\omega}}(l) := \theta_{kl}(\tilde{\omega}, 0, \hat{p}) \text{ for } l \in S \text{ and } \zeta_i^{\hat{p}, \tilde{\omega}}(J) := \delta_J(l) \text{ for } l \in S \cup \{J\}.$$

Let $\hat{\Omega}_0 = (S \cup \{J\})^I$ be the internal set of all the internal functions from $I$ to $S \cup \{J\}$. For any $\tilde{\omega} \in \tilde{\Omega}$ and $\hat{p} \in {}^*\hat{\Delta}$, also let $\mu_0^{\hat{p}, \tilde{\omega}}$ be the internal product probability measure $\prod_{i \in I} \zeta_i^{\hat{p}, \tilde{\omega}}$ on $(\hat{\Omega}_0, \mathcal{A}_0)$, where $\mathcal{A}_0$ is the internal power set of $\hat{\Omega}_0$. For each fixed $\hat{\omega}_0 \in \hat{\Omega}_0$ and $k, l \in S$, the agents in the set $\bar{A}_{kl}^{\hat{\omega}_0} = \{i \in I_k : \hat{\omega}_0(i) = l\}$ are now supposed to be matched with agents in $\bar{A}_{lk}^{\hat{\omega}_0}$.

*Step* 2: The issue now is that $\bar{A}_{lk}^{\hat{\omega}_0}$ and $\bar{A}_{kl}^{\hat{\omega}_0}$ might fail to have the same internal cardinality, for $k \neq l$, and $\bar{A}_{kk}^{\hat{\omega}_0}$ may fail to have an even internal cardinality, which would allow an internal full matching on $\bar{A}_{kk}^{\hat{\omega}_0}$. The scope of the second step of the proof is to fix such a problem. For $k, l \in S$ with $k \neq l$, let

$$C_{kl}^{\hat{\omega}_0} = \{A_{kl} : A_{kl} \subseteq \bar{A}_{kl}^{\hat{\omega}_0}, A_{kl} \text{ is internal and } |A_{kl}| = \min\{|\bar{A}_{kl}^{\hat{\omega}_0}|, |\bar{A}_{lk}^{\hat{\omega}_0}|\}\}.$$

For any $k \in S$, let $C_{kk}^{\hat{\omega}_0}$ be the family of the sets of the form $\bar{A}_{kk}^{\hat{\omega}_0} \setminus \{i\}$ for $i \in \bar{A}_{kk}^{\hat{\omega}_0}$ if $|\bar{A}_{kk}^{\hat{\omega}_0}|$ is odd, and $C_{kk}^{\hat{\omega}_0}$ the set with one element $\bar{A}_{kk}^{\hat{\omega}_0}$ if $|\bar{A}_{kk}^{\hat{\omega}_0}|$ is even. Set $C^{\hat{\omega}_0} := \prod_{k,l \in S} C_{kl}^{\hat{\omega}_0}$. Define an

internal probability measure $\mu^{\hat{\omega}_0}$ on $C^{\hat{\omega}_0}$ with internal power set $\mathcal{C}^{\omega_0}$ by letting $\mu^{\hat{\omega}_0}(\mathbf{A}) = \frac{1}{|C^{\hat{\omega}_0}|}$ for $\mathbf{A} \in C^{\hat{\omega}_0}$.

Let
$$\hat{\Omega}_1 := \{(A_{kl})_{k,l \in S} : A_{kl} \subseteq I \text{ and } A_{kl} \text{ is internal, where } k,l \in S\}.$$

The probability measure $\mu^{\hat{\omega}_0}$ can be trivially extended to the common sample space $\hat{\Omega}_1$ with its internal power set by letting $\mu^{\hat{\omega}_0}(\mathbf{A}) = 0$ for $\mathbf{A} \in \hat{\Omega}_1 \setminus C^{\hat{\omega}_0}$.

Given the hyperfinite internal probability space $(\hat{\Omega}_0, \mathcal{A}_0, \mu_0^{\hat{p},\tilde{\omega}})$ and the internal transition probability $\mu^{\hat{\omega}_0}$, $\hat{\omega}_0 \in \hat{\Omega}_0$, we can define the internal probability measure $\mu_1^{\hat{p},\tilde{\omega}}$ on $\hat{\Omega}_0 \times \hat{\Omega}_1$ with its internal power set by letting $\mu_1^{\hat{p},\tilde{\omega}}(\hat{\omega}_0, \mathbf{A}) = \mu_0^{\hat{p},\tilde{\omega}}(\hat{\omega}_0) \times \mu^{\hat{\omega}_0}(\mathbf{A})$ for any $\hat{\omega}_0 \in \hat{\Omega}_0$ and $\mathbf{A} \in \hat{\Omega}_1$.

*Step* 3: For any fixed $\hat{\omega}_0 \in \hat{\Omega}_0$ and $\mathbf{A}^{\hat{\omega}_0} = (A_{kl})_{k,l \in S} \in C^{\hat{\omega}_0}$, we consider the internal partial matchings on $I$ that match agents from $A_{kl}$ to $A_{lk}$. Let $B_k^{\hat{\omega}_0} = I_k \setminus (\bigcup_{l \in S} A_{kl}^{\hat{\omega}_0})$, which is the set of initially unmatched agents who remain unmatched. Let $\bar{B}_k^{\hat{\omega}_0}$ denote the set $\{i \in I_k : \hat{\omega}_0(i) = J\}$; then it is clear that $B_k^{\hat{\omega}_0} = \bar{B}_k^{\hat{\omega}_0} \cup \bigcup_{l \in S}(\bar{A}_{kl}^{\hat{\omega}_0} \setminus \bar{A}_{kl}^{\hat{\omega}_0})$. Let $B^{\hat{\omega}_0} = \bigcup_{k=1}^K B_k^{\hat{\omega}_0}$. For each $k \in S$, let $\hat{\Omega}_{kk}^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$ be the internal set of all the internal full matchings on $A_{kk}^{\hat{\omega}_0}$. Let $\mu_{kk}^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$ be the internal counting probability measure on $\hat{\Omega}_{kk}^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$. For $k,l \in S$ with $k < l$, let $\mu_{kl}^{\hat{\omega}_0, A^{\hat{\omega}_0}}$ be the internal set of all the internal bijections from $A_{kl}^{\hat{\omega}_0}$ to $A_{lk}^{\hat{\omega}_0}$. Let $\mu_{kl}^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$ be the internal counting probability on $A_{kl}^{\hat{\omega}_0}$. Let $\hat{\Omega}_2$ be the internal set of all the internal partial matchings from $I$ to $I$. Define $\hat{\Omega}_2^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$ to be the set of $\phi \in \hat{\Omega}_2$ such that

1. the restriction $\phi|_H = \pi^0|_H$, where $H$ is the set $\{i : \pi^0(i) \neq i\}$ of initially matched agents;
2. $\{i \in I_k : \phi(i) = i\} = B_k^{\hat{\omega}_0}$ for each $k \in S$;
3. the restriction $\phi|_{A_{kk}^{\hat{\omega}_0}} \in \hat{\Omega}_{kk}^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$ for $k \in S$;
4. for $k,l \in S$ with $k < l$, $\phi|_{A_{kl}^{\hat{\omega}_0}} \in \hat{\Omega}_{kk}^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$.

We now define an internal probability measure $\mu_2^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$ on $\hat{\Omega}_2$ such that
1. for $\phi \in \hat{\Omega}_2^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$,
$$\mu_2^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}(\phi) = \prod_{1 \leq k \leq l \leq K, A_{kl}^{\hat{\omega}_0} \neq \emptyset} \mu_{kl}^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}(\phi|_{A_{kl}^{\hat{\omega}_0}});$$

2. for $\phi \notin \hat{\Omega}_2^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$, $\mu_2^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}(\phi) = 0$.

The probability measure $\mu_2^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$ can be trivially extended to the sample space $\hat{\Omega}_2$.

For any $\tilde{\omega} \in \tilde{\Omega}$, define an internal probability measure $\hat{P}_0^{\hat{p}}(\tilde{\omega})$ on $\hat{\Omega} = \hat{\Omega}_0 \times \hat{\Omega}_1 \times \hat{\Omega}_2$ with the internal power set $\hat{\mathcal{F}}_0$ by letting

(B.1)     $\hat{P}_0^{\hat{p}}(\tilde{\omega})((\hat{\omega}_0, \mathbf{A}, \hat{\omega}_2)) = \begin{cases} \mu_1^{\hat{p},\tilde{\omega}}(\hat{\omega}_0, \mathbf{A}) \times \mu_2^{\hat{\omega}_0, \mathbf{A}}(\hat{\omega}_2) & \text{if } \mathbf{A} \in C^{\hat{\omega}_0}, \\ 0 & \text{otherwise.} \end{cases}$

The construction in (B.1) provides the Markov kernel from $\tilde{\Omega}$ to $\hat{\Omega}$ as in point 2 of the proposition. From now on, denote $\hat{P}_0^{\hat{p},\tilde{\omega}} := \hat{P}_0^{\hat{p}}(\tilde{\omega})$ for any $\tilde{\omega} \in \tilde{\Omega}$.

For $(i, \hat{\omega}) \in I \times \hat{\Omega}$, let $\hat{\pi}(i, (\hat{\omega}_0, \mathbf{A}, \hat{\omega}_2)) = \hat{\omega}_2(i)$ and
$$\hat{g}(i, \hat{\omega}) = \begin{cases} \alpha^0(\hat{\pi}(i, \hat{\omega})) & \text{if } \hat{\pi}(i, \hat{\omega}) \neq i, \\ J & \text{if } \hat{\pi}(i, \hat{\omega}) = i. \end{cases}$$

Denote the corresponding Loeb probability spaces of the internal probability spaces $(\hat{\Omega}_0, \hat{\mathcal{F}}_0, \hat{P}_0^{\hat{p},\tilde{\omega}})$ and $(I \times \hat{\Omega}_0, \mathcal{I}_0 \otimes \hat{\mathcal{F}}_0, \lambda_0 \otimes \hat{P}_0^{\hat{p},\tilde{\omega}})$ by $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}^{\hat{p},\tilde{\omega}})$ and $(I \times \hat{\Omega}, \mathcal{I} \boxtimes \hat{\mathcal{F}}, \lambda \boxtimes \hat{P}^{\hat{p},\tilde{\omega}})$, respectively. Set

$$\bar{\Omega} = \{(\hat{\omega}_0, \mathbf{A}, \hat{\omega}_2) \in \hat{\Omega} : \hat{\omega}_0 \in \hat{\Omega}_0, \mathbf{A} \in C^{\hat{\omega}_0}, \hat{\omega}_2 \in \Omega_2^{\hat{\omega}_0, \mathbf{A}}\}.$$

Then by construction of $\hat{P}_0^{\hat{p},\tilde{\omega}}$, it is clear that $\hat{P}_0^{\hat{p},\tilde{\omega}}(\bar{\Omega}) = 1$. Moreover, $\hat{\pi}$ is by construction an internal matching and satisfies point 4 of the proposition. It is then possible to define $(\Omega, \mathcal{F}_0, P_0)$ as stated in points 1 and 2 of the proposition and to consider the corresponding Loeb probability space; see point 6. Furthermore, we can extend $\hat{\pi}$ and $\hat{g}$ to $\Omega$ as stated in points 3 and 5.

*Step* 4: We now prove points 5 and 6 of the proposition. Define an internal process $\hat{f}$ from $I \times \hat{\Omega}$ to $S \cup \{J\}$ such that for any $(i, \hat{\omega}) \in I \times \hat{\Omega}$ we have

$$\hat{f}(i, \hat{\omega}) = \begin{cases} \hat{\omega}^0(i) & \text{if } \pi^0(i) = i, \\ \alpha^0(\pi^0(i)) & \text{if } \pi^0(i) \neq i. \end{cases}$$

Fix from now on $\hat{p} \in {}^*\hat{\Delta}$ and $\tilde{\omega} \in \tilde{\Omega}$. It is clear that if $\alpha^0(i) = k$ and $\pi^0(i) = i$, then

$$\hat{P}^{\hat{p},\tilde{\omega}}(\hat{f}_i = l) \simeq \hat{P}_0^{\hat{p},\tilde{\omega}}(\hat{f}_i = l) = \mu_0^{\hat{p},\tilde{\omega}}(\hat{\omega}_0(i) = l) = \zeta_i^{\hat{p},\tilde{\omega}}(l) = \theta_{kl}(\tilde{\omega}, 0, \hat{p}),$$

which means that

$$\hat{P}^{\hat{p},\tilde{\omega}}(\hat{f}_i = l) = {}^\circ\theta_{kl}(\tilde{\omega}, 0, \hat{p}).$$

With similar arguments it follows that

$$\hat{P}^{\hat{p},\tilde{\omega}}(\hat{f}_i = J) = {}^\circ b_k(\tilde{\omega}, 0, \hat{p}).$$

Moreover, $\hat{f}_i$ and $\hat{f}_j$ are independent random variables on the sample space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}^{\hat{p},\tilde{\omega}})$ for any $i \neq j$ in $I$. The exact law of large numbers as in Lemma 1 in [15] implies that, under the scenario of a current distribution $\hat{p}$ and of a realization $\tilde{\omega} \in \tilde{\Omega}$, it holds that

$$\lambda(\{\alpha^0(i) = k, \pi^0(i) = i, \hat{\omega}_0(i) = l\}) = {}^\circ\hat{\rho}_{kJ} \cdot {}^\circ\theta_{kl}(\tilde{\omega}, 0, \hat{p})$$

and

$$\lambda(\{\alpha^0(i) = k, \pi^0(i) = i, \hat{\omega}_0(i) = J\}) = {}^\circ\hat{\rho}_{kJ} \cdot {}^\circ b_k(\tilde{\omega}, 0, \hat{p})$$

for $\hat{P}^{\hat{p},\tilde{\omega}}$-almost all $\omega = (\hat{\omega}_0, \mathbf{A}, \hat{\omega}_2) \in \hat{\Omega}$ and for any $k, l \in S$, which means that

$$(\text{B.2}) \qquad \frac{|\bar{A}_{kl}^{\hat{\omega}_0}|}{\hat{M}} \simeq \hat{\rho}_{kJ}\theta_{kl}(\tilde{\omega}, 0, \hat{p}) \simeq \hat{\rho}_{lJ}\theta_{lk}(\tilde{\omega}, 0, \hat{p}) \simeq \frac{|\bar{A}_{lk}^{\hat{\omega}_0}|}{\hat{M}} \text{ and } \frac{|\bar{B}_k^{\hat{\omega}_0}|}{\hat{M}} \simeq \hat{\rho}_{kJ}b_k(\tilde{\omega}, 0, \hat{p}).$$

Let $\tilde{\Omega}^{\hat{p},\tilde{\omega}}$ be the set of $\hat{\omega} = (\hat{\omega}_0, \mathbf{A}, \hat{\omega}_2) \in \hat{\Omega}$ such that (B.2) holds. Then $\hat{P}^{\hat{p},\tilde{\omega}}(\tilde{\Omega}^{\hat{p},\tilde{\omega}}) = 1$, and hence $\hat{P}^{\hat{p},\tilde{\omega}}(\tilde{\Omega}^{\hat{p},\tilde{\omega}} \cap \Omega) = 1$.

Fix any $\hat{\omega} = (\hat{\omega}_0, \mathbf{A}, \hat{\omega}_2) \in \tilde{\Omega}^{\hat{p},\tilde{\omega}} \cap \Omega$; then $\mathbf{A} = \mathbf{A}^{\hat{\omega}_0}$ for some $\mathbf{A}^{\hat{\omega}_0} \in C^{\hat{\omega}_0}$, so $\hat{\omega}_2 \in \hat{\Omega}_2^{\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}}$.

For any $k \neq l \in S$ we have

(B.3)
$$\frac{|A_{kl}^{\hat{\omega}_0}|}{\hat{M}} = \min\left(\frac{|\bar{A}_{kl}^{\hat{\omega}_0}|}{\hat{M}}, \frac{|\bar{A}_{lk}^{\hat{\omega}_0}|}{\hat{M}}\right) \simeq \hat{\rho}_{lJ}\theta_{lk}(\tilde{\omega}, 0, \hat{p}) = \frac{|\bar{A}_{kl}^{\hat{\omega}_0}|}{\hat{M}} \text{ and } \frac{|A_{kk}^{\hat{\omega}_0}|}{\hat{M}} \simeq \frac{|\bar{A}_{kk}^{\hat{\omega}_0}|}{\hat{M}} \simeq \hat{\rho}_{lJ}\theta_{kk}(\tilde{\omega}, 0, \hat{p}),$$

which also implies that

$$\frac{|B_k^{\hat{\omega}_0}|}{\hat{M}} \simeq \hat{\rho}_{kJ}b_k(\tilde{\omega}, 0, \hat{p}) \simeq \frac{|\bar{B}_k^{\hat{\omega}_0}|}{\hat{M}}.$$

For any $i \in I_k, i \in A_{kl}^{\hat{\omega}_0}$ if and only if $\pi(\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}, \hat{\omega}_2) = \hat{\omega}_2(i) \in A_{lk}^{\hat{\omega}_0}$, and $i \in B_k^{\hat{\omega}_0}$ if and only if $\pi(\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}, \hat{\omega}_2) = \hat{\omega}_2(i) = J$. Hence, for fixed $\hat{\omega} = (\hat{\omega}_0, \mathbf{A}^{\hat{\omega}_0}, \hat{\omega}_2)$, and for any $k, l \in S$, we can see that if $i \in A_{kl}^{\hat{\omega}_0} \subseteq \bar{A}_{kl}^{\hat{\omega}_0}$, then

$$\hat{f}(i, \hat{\omega}) = \hat{\omega}_0(i) = l = \alpha^0(\hat{\omega}_2(i)) = \hat{g}(i, \hat{\omega}),$$

and that if $i \in B_k^{\hat{\omega}_0} \subseteq \bar{B}_k^{\hat{\omega}_0}$, then

$$\hat{f}(i, \hat{\omega}) = \hat{\omega}_0(i) = J = \alpha^0(\hat{\omega}_2(i)) = \hat{g}(i, \hat{\omega}).$$

For any $i \in I \setminus (\cup_{k \in S} I_k)$, that is, for any $i \in I$ such that $\pi^0 \neq i$, we have that

$$\hat{f}(i, \hat{\omega}) = \alpha^0(\pi^0(i)) = \alpha^0(\pi(i, \omega)) = \hat{g}(i, \hat{\omega}).$$

It is clear that

$$\{i \in I : \hat{f}(i, \hat{\omega}) \neq \hat{g}(i, \hat{\omega})\} \subseteq \bigcup_{l \in S}\left(\bar{A}_{kl}^{\hat{\omega}_0} \setminus A_{kl}^{\hat{\omega}_0}\right),$$

which has $\lambda$-measure zero by (B.3). By the fact that $\hat{P}^{\hat{p},\tilde{\omega}}(\tilde{\Omega}^{\hat{p},\tilde{\omega}} \cap \Omega) = 1$, we know that

$$\lambda(i \in I : \hat{f}(i, \hat{\omega}) = \hat{g}(i, \hat{\omega})) = 1$$

for $\hat{P}^{\hat{p},\tilde{\omega}}$-almost all $\hat{\omega} \in \hat{\Omega}$.

    Since the Loeb product space $(I \times \hat{\Omega}, \mathcal{I} \boxtimes \hat{\mathcal{F}}, \lambda \boxtimes \hat{P}^{\hat{p},\tilde{\omega}})$ is a Fubini extension, the Fubini property implies that for $\lambda$-almost all $i \in I, \hat{g}(i, \hat{\omega})$ is equal to $\hat{f}(i, \hat{\omega})$ for $\hat{P}^{\hat{p},\tilde{\omega}}$-almost all $\hat{\omega} \in \hat{\Omega}$. Hence $g$ satisfies the second part of the lemma. Let $\tilde{I}$ be an $\mathcal{I}$-measurable set with $\lambda(\tilde{I}) = 1$ such that for any $i \in \tilde{I}, \hat{g}_i(\hat{\omega}) = \hat{f}_i(\hat{\omega})$ for $\hat{P}^{\hat{p},\tilde{\omega}}$-almost all $\hat{\omega} \in \hat{\Omega}$. Therefore, by the construction of $f$ we know that the collection of random variables $\{\hat{f}_i\}_{i \in \tilde{I}}$ is mutually independent in the sense that any finitely many random variables from that collection are mutually independent. This also implies point 6 of the proposition. ∎

### REFERENCES

[1] D. ABREU AND M. K. BRUNNERMEIER, *Bubbles and crashes*, Econometrica, 71 (2003), pp. 173–204.

[2] F. ALLEN AND D. GALE, *Bubbles and crisis*, Econ. J., 110 (2000), pp. 236–255.

[3] D. ANDOLFATTO, *Business cycles and labor-market search*, Amer. Econ. Rev., 86 (1996), pp. 112–132.

[4] D. ANDREI AND J. CUJEAN, *Information percolation, momentum and reversal*, J. Financ. Econ., 123 (2017), pp. 617–645.

[5] P. BAYER, K. MANGUM, AND J. W. ROBERTS, *Speculative fever: Investor contagion in the housing bubble*, Amer. Econ. Rev., 111 (2021), pp. 609–651.

[6] R. M. BECKER, U. FISCHBACHER, AND T. HENS, *Soft Landing of a Stock Market Bubble: An Experimental Study*, Working Paper 90, Zurich IEER, 2002.

[7] F. BIAGINI, H. FÖLLMER, AND S. NEDELCU, *Shifting martingale measures and the slow birth of a bubble*, Finance Stoch., 18 (2014), pp. 297–326.

[8] F. BIAGINI, A. MAZZON, AND T. MEYER-BRANDIS, *Liquidity induced asset bubbles via flows of ELMMs*, SIAM J. Financial Math., 9 (2018), pp. 800–834.

[9] F. BIAGINI, A. MAZZON, T. MEYER-BRANDIS, AND K. OBERPRILLER, *Supplement to Liquidity Based Modeling of Asset Price Bubbles via Random Matching*, preprint, https://arxiv.org/abs/2311.15793, 2023.

[10] Y. CHOI AND R. DOUADY, *Chaos and Bifurcation in* 2007–08 *Financial Crisis*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1522544, 2011.

[11] Y. CHOI AND R. DOUADY, *Financial crisis and contagion: A dynamical systems approach*, in Handbook on Systemic Risk, Cambridge University Press, 2013, pp. 453–480.

[12] A. M. COX AND D. G. HOBSON, *Local martingales, bubbles and option prices*, Finance Stoch., 9 (2005), pp. 477–492.

[13] J. DELONG, A. SHLEIFER, L. H. SUMMERS, AND R. J. WALDMANN, *Noise trader risk in financial markets*, J. Polit. Econ., 98 (1990), pp. 703–738.

[14] J. L. DOOB, *Stochastic processes depending on continuous parameter*, Trans. Amer. Math. Soc., 42 (1937).

[15] D. DUFFIE, L. QIAO, AND Y. SUN, *Dynamic directed random matching*, J. Econ. Theory, 174 (2018), pp. 124–183.

[16] D. DUFFIE, L. QIAO, AND Y. SUN, *Continuous Time Random Matching*, preprint, https://www.darrellduffie.com/uploads/working/DuffieQiaoSunCTRM_Dec2020.pdf, 2020.

[17] P. E. EARL, T.-C. PENG, AND J. POTTS, *Decision-rule cascades and the dynamics of speculative bubbles*, J. Econ. Psychol., 28 (2007), pp. 351–364.

[18] A. J. FILARDO, *The Impact of the International Financial Crisis on Asia and the Pacific: Highlighting Monetary Policy Challenges from a Negative Asset Price Bubble Perspective*, Working Paper 356, Bank for International Settlements, 2011.

[19] H. FÖLLMER, *Equilibria in financial markets with heterogeneous agents: A probabilistic perspective*, J. Math. Econom., 41 (2005), pp. 123–155.

[20] H. FÖLLMER AND A. SCHIED, *Stochastic Finance: An Introduction in Discrete Time*, De Gruyter, Berlin, 2016.

[21] J. M. HARRISON AND D. M. KREPS, *Speculative investor behavior in a stock market with heterogeneous expectations*, Quart. J. Econ., 92 (1978), pp. 323–336.

[22] R. JARROW, Y. KCHIA, AND P. PROTTER, *How to detect an asset bubble*, SIAM J. Financial Math., 2 (2011), pp. 839–865.

[23] R. JARROW AND P. PROTTER, *Forward and futures prices with bubbles*, Int. J. Theor. Appl. Finance, 12 (2009), pp. 901–924.

[24] R. JARROW AND P. PROTTER, *Foreign currency bubbles*, Rev. Deriv. Res., 14 (2011), pp. 67–83.

[25] R. JARROW, P. PROTTER, AND K. SHIMBO, *Asset price bubbles in complete markets*, in Advances in Mathematical Finance, Springer, New York, 2007, pp. 97–121.

[26] R. JARROW, P. PROTTER, AND K. SHIMBO, *Asset price bubbles in incomplete markets*, Math. Finance, 20 (2010), pp. 145–185.

[27] R. A. JARROW, P. PROTTER, AND A. F. ROCH, *A liquidity based model for asset price bubbles*, Quant. Finance, 12 (2012), pp. 1339–1349.

[28] T. KAIZOJI, *Speculative bubbles and crashes in stock markets: An interacting-agent model of speculative activity*, Phys. A, 287 (2000), pp. 493–506.

[29] J. H. KEISLER AND Y. E. SUN, *Loeb measures and Borel algebras*, in Reuniting the Antipodes - Constructive and Nonstandard Views of the Continuum, U. Berger, H. Osswald, and P. Schuster, eds., Kluwer, Dordrecht, 2001, pp. 111–117.

[30] N. Kiyotaki and R. Wright, *On money as a medium of exchange*, J. Polit. Econ., 97 (1989), pp. 927–954.

[31] A. Klenke, *Probability Theory: A Comprehensive Course*, Springer, New York, 2020.

[32] M. Loewenstein and G. A. Willard, *Rational equilibrium asset-pricing bubbles in continuous trading models*, J. Econ. Theory, 91 (2000), pp. 17–58.

[33] T. Lux, *Herd behaviour, bubbles and crashes*, Econom. J., 105 (1995), pp. 881–896.

[34] K. Matsuyama, N. Kiyotaki, and A. Matsui, *Toward a theory of international currency*, Rev. Econom. Stud., 60 (1993), pp. 283–307.

[35] E. M. Miller, *Risk, uncertainty, and divergence of opinion*, J. Finance, 32 (1977), pp. 1151–1168.

[36] L. Qiao, Y. Sun, and Z. Zhang, *Conditional exact law of large numbers and asymmetric information economies with aggregate uncertainty*, Econ. Theory, 62 (2016), pp. 43–64.

[37] J. Scheinkman and W. Xiong, *Overconfidence and speculative bubbles*, J. Polit. Econ., 111 (2003), pp. 1183–1219.

[38] J. A. Scheinkman, *Speculation, trading, and bubbles*, in Speculation, Trading, and Bubbles, Columbia University Press, New York, 2014, pp. 7–40.

[39] J. A. Scheinkman and W. Xiong, *Overconfidence and speculative bubbles*, J. Polit. Econ., 111 (2003), pp. 1183–1220.

[40] J. J. Siegel, *What is an asset price bubble? An operational definition*, Eur. Financ. Manag., 9 (2003), pp. 11–24.

[41] Y. Sun, *The almost equivalence of pairwise and mutual independence and the duality with exchangeability*, Probab. Theory Related Fields, 112 (1998), pp. 425–456.

[42] Y. Sun, *The theory of hyperfinite processes: The complete removal of individual uncertainty via exact LLN*, J. Econ. Theory, 29 (1998).

[43] Y. Sun, *The exact law of large numbers via Fubini extension and characterization of insurable risks*, J. Econ. Theory, 126 (2006), pp. 31–69.

[44] Y. Sun and Y. Zhang, *Individual risk and Lebesgue extension without aggregate uncertainty*, J. Econ. Theory, 144 (2009), pp. 432–443.

[45] J. Wan, *Prevention and landing of bubble*, Int. Rev. Econ. Finance, 56 (2018), pp. 190–204.

[46] W. Xiong, *Bubbles, Crises, and Heterogeneous Beliefs*, Technical report, Princeton University, Princeton, NJ, 2012.

[47] S. Zhuk, *Speculative Bubbles, Information Flow and Real Investment*, Information Flow and Real Investment, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2280304, 2013.