Evaluating allocations of opportunities*

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Abstract

This paper provides a robust criterion for comparing lists of probability distributions - interpreted as allocations of opportunities - faced by different social groups. We axiomatically argue in favor of comparing those lists of probability distributions on the basis of a uniform - among groups - valuation of their expected utility. We identify an empirically implementable criterion for comparing allocations of opportunities that coincides with the unanimity

*First submitted on December 1, 2023. Re-submitted after major revisions on December 6, 2024.

[1]Corresponding author. We gratefully acknowledge the funding of the research that led to this paper by the French *Agence Nationale de la Recherche* through three contracts: (*i*) Measurement of Ordinal and Multidimensional Inequalities (ANR-16-CE41-0005), (*ii*) Challenging Inequalities: A Indo-European perspective (ANR-18-EQUI-0003) and (*iii*) The European University of Research AMSE (ANR-17-EURE-020). Tista Kundu also acknowledges thankfully the Axa Research Fund Post-Doctoral Fellowship. Finally, and with the usual disclaiming qualification, we thank Rahul Deb, Francisco S. J. Ferreira, Marc Fleurbaey, Larry Kranich, Hervé Moulin, Debraj Ray, Arunava Sen, Ernesto Savaglio, Alain Trannoy, Rakesh Vohra and two anonymous referees for their useful comments and suggestions.

of all such uniform valuations of expected utility that exhibit aversion to inequality of opportunity. We illustrate our criterion by evaluating allocations of educational opportunities among castes and genders in 14 Indian states.

Keywords: Equalization of opportunity, probability distribution, groups, zonotope, caste, gender, education.

JEL classification numbers: D63, D81, I24.

Shortened title : Evaluating opportunities.

1 Introduction

Improving and equalizing opportunities are considered by many to be important social objectives. In the US, opinion surveys conducted by the Pew Research Center[2] in the last 25 years have consistently found that 90% of the respondents agree that "our society should do what is necessary to make sure that everyone has an equal opportunity to succeed". A natural way to interpret the "equal opportunity to succeed" ideal is through the requirement that individuals' probabilities (chances) of reaching outcomes of interest should not depend on *morally irrelevant* characteristics such as skin color, gender, national origin, family background and so on. This view therefore refers to a partition of the society into groups, formed on some exogenous moral irrelevant basis, who face different probabilities of achieving these outcomes. This paper proposes a general and robust methodology for comparing these lists of probability distributions - *allocations of opportunities* as we shall call them - in a way that is sensitive to both the average probability of reaching favorable outcomes offered to the members of the groups and the equalization of these probabilities among groups.

To illustrate and motivate our approach, consider Figure 1, that shows the probabilities that low and high caste adults aged between 30 and 40 in two neighboring states of India (West-Bengal and Odisha) achieve one of the six educational levels reported in the 68th round of the Employment-Unemployment survey of the Indian National Sample Survey Organization (NSSO). These Education levels are ranked from illiteracy (1) to upper tertiary education (6). Low caste status is defined as belonging to the official category of Scheduled Caste (SC) and Scheduled Tribe (ST), the remainder being considered high caste.

Figure 1 clearly indicates that low caste adults have significantly lower chances of achieving any level of education than high caste adults, whichever state they live in.

[2]See, e.g., https://www.pewresearch.org/2011/03/11/the-elusive-90-solution.

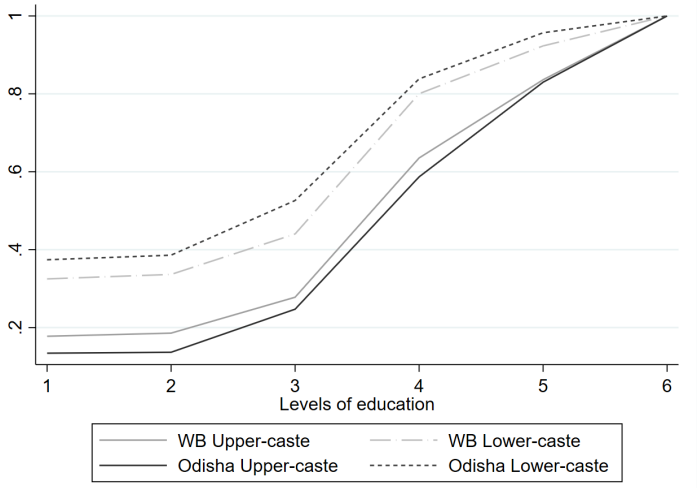

Figure 1: Cumulative distribution of education levels, low and high caste groups, Odisha and West Bengal, 2012.

Yet, one may want to go beyond the mere observation that educational opportunities are unequally distributed among caste groups in both states. In particular, one may want to make comparative statements on the extent to which this caste inequality in educational opportunities differs between West Bengal and Odisha. Figure 1 suggests indeed that the educational opportunities faced by low- and high-caste adults - clearly unequal in both West Bengal and Odisha - are "more unequal" in the latter than in the former. In effect, the underprivileged group performs better in West Bengal than in Odisha while the opposite is true for the favored group. Since the average distribution of educational opportunities is quite - albeit not perfectly - similar in both states, it could be concluded that caste-based inequalities of educational opportunities are larger in Odisha than in West Bengal. Of course, pure equalization may not be the only aim of opportunity-sensitive policies. It may also be important, as in the case of income distributions, to increase the opportunities offered to some or all individuals *in addition to* equalizing them.

Comparing alternative lists of probability distributions over outcomes is conceptually similar to comparing decisions under objective ambiguity examined, for example, in Ahn (2008), Gravel, Marchant, and Sen (2012) or Gravel, Marchant,

and Sen (2018). Exploiting this similarity, we propose to compare alternative lists of probability distributions by means of the *uniform expected valuation of expected utility* criterion. This criterion assigns to each outcome a utility level and evaluates a probability distribution according to its expected utility. The various lists of probability distributions are then compared on the basis of an expected valuation of their expected utility, under the assumption that all social groups are equally likely (or carry equal ethical worth). If the comparison of allocations of opportunities exhibits aversion to inequality of opportunity, the valuation must be a concave function of the expected utility. We provide in the appendix an axiomatic characterization of the family of uniform expected valuation of expected utility criteria.

The main contribution of this paper is the identification of an empirically implementable criterion that identifies when one allocation of opportunities is better than another for *all* inequality-of-opportunity-averse uniform expected valuations of expected utility criteria. Our contribution thus parallels the classical result in income inequality measurement theory in which the empirically implementable generalized Lorenz dominance criterion is known (see e.g. Shorrocks (1983)) to be equivalent to the non-implementable unanimity of all utilitarian comparisons of income distribution that assume that individuals convert income into utility by the same increasing and concave function. The implementable criterion identified in this article is the inclusion of the *quasi-ordering extended zonotope* uniquely associated with each allocation. The zonotope of any list of probability distributions is the set of all their Minkowski sums (see, e.g. Koshevoy (1995), Koshevoy (1998)). A quasi-ordering-extended zonotope is a zonotope that has been enlarged with translations that capture the assumptions made about the ranking of outcomes of interest to groups' members. Our approach is quite general in this respect and is applicable to any incompletely or completely ordered outcomes, including the possibility of total incomparability. Although the quasi-ordering-extended zonotope inclusion test is empirically implementable with any number of groups and outcomes, its actual

implementation may sometimes be more difficult than the verification that two generalized Lorenz curves do not cross. However, we provide simple verification procedures in many cases, one of them being the two-group case discussed above. Another is the case, commonly considered in the inequality of opportunity literature, where distributions over outcomes are ordered across groups as per expected utility. In this case, our criterion works in a sequential fashion, almost like the generalized Lorenz criterion. In the complete ordering of outcomes case, the test consists in checking for first-order dominance between the worst-off - in expected utility - groups, then for the (uniform average) of the two worst-off groups, and so on.

We also illustrate the usefulness of our criterion by comparing allocations of educational opportunities offered to Indian adults depending on their caste and gender in 14 Indian states. While our criterion is, just like Lorenz dominance, *a priori* incomplete, it nonetheless succeeds in conclusively comparing a significant number of Indian states in terms of educational opportunities. Although the obtained ranking of the states often mirrors their ranking in terms of their average overall opportunities, there are a few states where inequality of educational opportunities is so great that even their relatively favorable average distribution of opportunities does not enable them to dominate other states with a lower but more equally distributed average. An interesting example of this is Kerala, an Indian state that is often portrayed in a favorable way on the education front[3]. Yet, unequal distribution of educational opportunities among castes prevents Kerala from dominating - as per our criterion - the state of Andhra Pradesh, whose average distribution of educational opportunities is stochastically dominated at the first-order by that of Kerala.

1.1 Related literature

There is an abundant literature on equality of opportunities that has been nicely surveyed in Roemer and Trannoy (2016) and, in its theoretical underpinnings, in

[3]Notably by Drèze and Sen (2020).

Fleurbaey (2008). This literature is based on what John Roemer (1996) calls the Dworkin (1981) "cut" between the characteristics that affect an individual's outcome for which the individual should be held *responsible* and the morally irrelevant characteristics that determine what this literature calls the individual's "type". The main creed of this literature is that opportunity equalization should be concerned with *equalizing outcomes* among individuals who share the same "responsibility characteristics". However, no attempt should be made to modify the outcomes that can be shown to result from the free exercise of responsibility alone. In recent years, the cut between the variables that affect individuals' achievements has been expanded to luck and randomness (see, e.g. Vallentyne (2002), Lefranc, Pistolesi, and Trannoy (2009) and Lefranc and Trannoy (2017)). Along with a few other contributions (e.g. Bénabou and Ok (2001) and Mariotti and Veneziani (2017)), our approach departs from this "cut"-inspired literature by being agnostic about individuals' degree of responsibility for some of their characteristics. In fact, responsibility does not play any role in our approach, even though individuals in each group can be considered "responsible" for their success in achieving the relevant outcomes. The only postulate of our approach is that the groups themselves are formed on the basis of characteristics - such as gender, caste belonging, or others - that are considered "morally irrelevant" for explaining or determining such a success. Put bluntly, there is no reason why men and women, say, should differ in their probability of achieving the various outcomes.

Another important difference between our approach and those surveyed by Roemer and Trannoy (2016) is that we provide a robust dominance-based definition of opportunity equalization and improvement from an *ex ante* perspective while many other authors adopt an *ex post* "outcome-based" approach in which they either try to identify allocation rules that satisfy axiomatic "compensation" and "natural reward" principles (see Fleurbaey (2008)) or propose specific rules for comparing allocations of opportunities in a setting where the agents' responsibility characteristics play an explicit role. Two important such rules, introduced respectively by VandeGaer

(1993) and Roemer (1993) are the "Min of Means" and the "Mean of Mins" (using Fleurbaey (2008) terminology). When applied to the framework considered in this paper, the Min of Means compares allocations of opportunities on the basis of the average outcome of the group who has, in the considered allocation, the lowest such average outcome. The Mean of Mins, under the assumption that agents within any group are "responsible" for their outcome, compares allocations on the basis of the average - over the groups - of their lowest outcomes. The family of uniform expected concave valuations of expected utility criteria on which we base our dominance has been proposed as a possible social objective in some of the theoretical literature on equality of opportunity, notably the one inspired by the *ex ante* approach examined by VandeGaer (1993), Ooghe, Schokkaert, and VandeGaer (2007) and, in the context of intergenerational mobility measurement, by Martinez, Schockkaert, and VandeGaer (2001). However, these authors have not provided dominance criteria over the family of all uniform expected concave valuations of expected utility. They have simply proposed specific members of this family such as the one - inspired by the classical work of Atkinson (1970) - in which the concave function can be written as some power transformation of the expected utility.

Most contributions to the literature that define opportunity equalization in terms of the Dworkin (1981) cut (like Peragine (2004) and other contributions surveyed by Ramos and Gaer (2016) and Brunori, Ferreira, and Peragine (2021)) proceed by decomposing total outcome inequality (measured by some index) into *within* and *between* group inequality (see, e.g., Ferreira and Gignoux (2011)), defining between-group inequality, in the tradition of Shorrocks (1984), with respect to the group's mean outcome. A focus on group mean outcome is also a feature of the approach developed by Bénabou and Ok (2001) in the specific context of mobility measurement. However, focusing on the mean outcomes of the groups is restrictive because it ignores all information related to the possibly varying riskiness of those outcomes between the groups. A recent paper proposing an aggregate index of inequality of

opportunity based in part on the group mean is Hufe, Kanbur, and Peichl (2022). The aggregate index is defined, in the tradition of Cowell (1985) and Magdalou (2021), as an aggregation of differences between the current distribution of outcomes and some ideal one that would result from an absence of inequality of opportunity – defined as the requirement that the average income be the same in every group – and an elimination of poverty. Except for poverty reduction, the approach does not, however, exhibit sensitivity to improving opportunities for some or all groups. The approach is also based on a specific inequality index and, therefore, lacks the robustness that is at the heart of the approach put forth in this paper.

The contributions to the literature that we find closest to ours are, for different reasons, Andreoli, Havne, and Lefranc (2019) and Mariotti and Veneziani (2017). Andreoli, Havne, and Lefranc (2019) propose a robust definition of opportunity *equalization* across types based on a sequential (if more than two types are involved) comparison of the absolute value of the "gaps" between probability distributions faced by individuals of two types. Their criterion is robust because it applies to societies where the probability distributions faced by different types are ordered by a large class of preferences. However, by requiring such a unanimous agreement on the ranking of the probability distributions of different types (for example, by first-order stochastic dominance), Andreoli, Havne, and Lefranc (2019) restrict the applicability of their criterion to those societies where this unanimous agreement holds. We do not impose this restriction on the criterion examined here. Moreover, while the criterion proposed by Andreoli, Havne, and Lefranc (2019) is sensitive to welfare *gaps* between distributions, it is not sensitive to welfare *levels*. As a result, their criterion may promote a policy that actually reduces the opportunities offered to every group if the reduction in opportunities is not uniform across groups and reduces the gap between them. By contrast, our criterion is sensitive to both the "levels" of opportunities offered to the groups and the gap between them, and incorporates in its very definition a trade-off between aggregate improvement in opportunities and

unequal sharing of those improvements among groups. This trade-off is similar in spirit, here again, to that underlying the generalized Lorenz curve in conventional one-dimensional income inequality measurement. Mariotti and Veneziani (2017) propose a justification for comparing allocations of opportunities with only two ordered outcomes (say "hell" and "heaven") on the basis of the product - over all groups - of the probabilities of occurrence of the good outcome. When applied to such a restricted setting, their complete ranking of opportunities allocations is compatible with our incomplete one. However our criterion, which happens to be the intersection of a very large class of rankings of which the Mariotti and Veneziani (2017) ranking is only one member, applies also to all allocations of opportunities and is therefore not restricted to those whose outcomes are binary.

1.2 Organization of the paper

The remainder of this paper proceeds as follows. The next section describes the general setting in which allocations of opportunities are evaluated and provides a normative foundation for this evaluation from the stand point of an ethical observer placed behind the veil of ignorance. Section 3 introduces the extended zonotope criterion and establishes its equivalence with the ranking of allocations of opportunities made by all opportunity-inequality-averse uniform expected valuations of expected utility. Section 3 also indicates how the criterion can be empirically implemented in a large number of cases. Section 4 presents the results of the empirical implementation of the criterion to assess gender-based and caste-based inequalities of educational opportunities in India. Section 5 concludes.

2 A framework for evaluating allocations of opportunities

We are interested in comparing allocations of opportunities to members of some exogenously given groups. These groups may be based on caste, religion, race, gender, family background, etc. We do not assume that the number of such groups is the same across allocations. For instance, we may consider allocations of opportunities among one group alone. Our approach would then view all such allocations as achieving (trivially) perfect equality of opportunities, even though they may differ in the "average level" of those opportunities. At the other extreme, we may consider cases where every individual forms a distinct group and faces an individual-specific distribution of opportunities. If this latter interpretation is favored, we could then interpret a collection of individuals with identical observable characteristics (gender, background, etc.) - what is usually considered as a "group" - as the replication of as many identical individuals as there are members in this collection. The demographic size of the groups can hence be taken care of by resorting to this replication principle.

The opportunities offered to a group are described from an *ex ante* point of view[4] by the *probabilities* of its members achieving *relevant outcomes*. We assume specifically that there are k such outcomes taken from the set $\{1, ..., k\}$. Outcomes are interpreted as anything observable that people have a reason to value, like income, education, or health levels (expressed in discrete units). Outcomes may also be formed of pairs of, say, education and health levels. Hence, we do not require outcomes to be completely ordered. We can even take the extreme view that they are not ordered at all. For example, if we were to address the allocation, among men and women, of the opportunity to join the army, there would be only two outcomes (joining or not) in no obvious order. Formally, the ranking of outcomes may be viewed as

[4]See Fleurbaey (2010) and Fleurbaey (2018) for discussions of, and alternative approaches to, the normative analysis of "socially risky situations".

resulting from some linear quasi-ordering $\geq_{QO}$ on $\{1, ..., k\}$ with the interpretation that $j \geq_{QO} h$ if and only if outcome j is "clearly better" for an agent than outcome h.[5] The (extreme) case of incomplete quasi-ordering where *none* of the outcomes can be compared with one another is denoted by $\geq_{\varnothing}$. The other extreme case of *complete* linear ordering commonly assumed in the equality of opportunity literature surveyed in Roemer and Trannoy (2016) is denoted by $\geq_C$. Given the quasi-ordering $\geq_{QO}$, we assume that the outcomes $\{1, ..., k\}$ are labeled in a way compatible with $\geq_{QO}$. That is, if $h < i$, then $h \geq_{QO} i$ *does not* hold. We also denote by $\mathcal{U}^{\geq_{QO}} \subseteq \mathbb{R}^k$ the set of all lists of k (utility) numbers compatible with the quasi-ordering $\geq_{QO}$ defined by:

$$(1) \qquad \mathcal{U}^{\geq_{QO}} = \{(u_1, ..., u_k) \in \mathbb{R}^k : \ j \geq_{QO} h \Longrightarrow u_j \geq u_h, \ \forall j, h \in \{1, ..., k\}\}$$

Any allocation of opportunities $\mathbf{p}$ is depicted as an $n(\mathbf{p}) \times k$ row-stochastic matrix:

$$\mathbf{p} = \begin{bmatrix} p_{11} & \cdots & p_{1k} \\ \cdots & \cdots & \cdots \\ p_{n(\mathbf{p})1} & \cdots & p_{n(\mathbf{p})k} \end{bmatrix}$$

where p_{ij}, for $i = 1, ..., n(\mathbf{p})$ and $j = 1, ..., k$, denotes the probability that an agent of group i will achieve outcome j in allocation $\mathbf{p}$ and $n(\mathbf{p})$ denotes the number of groups in $\mathbf{p}$. Let $\mathbb{A}$ be the set of all conceivable allocations of opportunities, i.e. $\mathbb{A} = \bigcup_{n \geq 1} \mathbf{S}^n$, where $\mathbf{S} := \left\{\mathbf{x} = (x_1, ..., x_k) \in \mathbb{R}^k_+ : \ \sum_{h=1}^k x_h = 1\right\}$ denotes the $k-1$-dimensional simplex.

Allocations of opportunities are compared by an *ethical observer* who agrees with the ranking of the outcomes provided by $\geq_{QO}$ and who is placed behind a veil of ignorance as to the group to which she would belong if she lived in the societies described by these allocations. This ethical observer can thus be thought

[5] All mathematical notations and concepts are defined in Appendix A.1.

of as a decision maker under objective ambiguity who ranks lists of "objective" probability distributions studied in papers such as Ahn (2008), Olszewski (2007), Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2018). The ethical observer would compare the allocations by means of some ordering $\succsim$, with asymmetric and symmetric factors $\succ$ and $\sim$. Since the ordering $\succsim$ is defined on the whole set $\mathbb{A}$, it is in particular defined on the set of all conceivable one-group allocations (i.e. on all probability distributions over the k outcomes). Hence the ethical observer is also a decision maker under risk who is capable of ordering all probability distributions over outcomes. We specifically focus on ethical observers who use an ordering $\succsim$ that can be represented by a (continuous) increasing function $\Phi : \mathbb{R} \to \mathbb{R}$ and a list $(u_1, ..., u_k)$ of real numbers in $\mathcal{U}^{\geq_{QO}}$ in such a way that:

$$\mathbf{q} \succsim \mathbf{p} \iff \frac{1}{n(\mathbf{q})} \sum_{i=1}^{n(\mathbf{q})} \Phi \left(\sum_{j=1}^{k} q_{ij} u_j \right) \geq \frac{1}{n(\mathbf{p})} \sum_{i=1}^{n(\mathbf{p})} \Phi \left(\sum_{j=1}^{k} p_{ij} u_j \right) \tag{2}$$

for any two allocations $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{A}$. An ordering satisfying this property *could* therefore be thought of as resulting from the following three-step procedure:

1. In the first step, every group in the compared allocations is given an *expected utility* that results from the assignment of utility numbers $u_1, ..., u_k$ to outcomes in a way that reflects the ranking of the outcomes by the quasi-ordering.

2. In the second step, the ethical observer assigns a *valuation* to this expected utility through some function Φ.

3. In the third step, the ethical observer ranks the allocations on the basis of the *expected valuation* of their groups' expected utilities, under the assumption that the ethical observer is equally likely to fall into any group.

We therefore call *Uniform expected valuator of expected utility* (UEVEU) any such ethical observer. There are actually quite a few of them, as many in fact as there are logically conceivable valuation functions Φ and logically conceivable ways

of assigning utility to outcomes in a manner consistent with $\geq_{QO}$.[6] While we believe that evaluating allocations of opportunities by means of an ordering numerically representable as per (2) is plausible, it is possible thanks to results in Gravel, Marchant, and Sen (2011) and Gravel, Marchant, and Sen (2012) to single out this family of rankings *as the only one* that satisfy a collection of (reasonably) natural axioms. These axioms, as well as the characterization result, are provided in Appendix A.3.

We take the view that UEVEU ethical observers exhibit aversion to inequality of opportunity, which we define as a preference for an allocation exhibiting *no* inequality of opportunity - say because it consists of one single group - over allocations exhibiting some inequality of opportunity. In order to define this notion precisely, we first introduce the following definition of comparative aversion to opportunity inequality among ethical observers.

Definition 1 *We say that the ordering* $\succsim_1$ *on* $\mathbb{A}$ *exhibits* at least as much aversion to inequality of opportunity *as the ordering* $\succsim_2$ *if, for every probability distribution* $\rho \in \mathbf{S}$ *and allocation* $\mathbf{p} \in \mathbb{A}$*, we have* $\rho \succsim_2 \mathbf{p} \implies \rho \succsim_1 \mathbf{p}$.

With this notion of comparative aversion to inequality of opportunity, we can define aversion to inequality of opportunity *tout court* as the fact of being more averse to inequality of opportunity than an ethical observer who is neutral with respect to it. This requires, however, a definition of neutrality with respect to opportunity inequality. In the usual income inequality setting, neutrality to income equality is defined as the fact of considering as equivalent all income distributions with the same per capita income. Although we are not aware of the existence of a well-accepted standard of neutrality toward inequality of opportunity, we believe that a plausible

[6] Evaluating opportunities by means of a UEVEU criterion has been suggested by Martinez, Schockkaert, and VandeGaer (2001) (see their Equations (1)-(3) p. 528) in connection with mobility measurement. It can also be observed that comparing alternative lists of probability distributions on the basis of inequality (2) is what the criterion characterized by Ahn (2008) in the context of decision making under objective ambiguity would recommend, under the additional assumption that all social groups are equally likely to form. This criterion can also be viewed as a particular member of the class of smooth ambiguity criteria studied by Klibanoff, Marinacci, and Mukerji (2005).

candidate would consider as equivalent all allocations that distribute among their groups the same (symmetric) average probability distribution over outcomes.

For any allocation $\mathbf{p}$, we denote by $p_i \in \mathbf{S}$ the probability distribution faced by group i in $\mathbf{p}$, and by $\overline{p} := \frac{1}{n(\mathbf{p})} \sum_{i=1}^{n(\mathbf{p})} p_i$ its (symmetric across group) average. We say that $\succsim$ exhibits *neutrality to inequality of opportunity* if $\mathbf{p} \sim \mathbf{q}$ for any two allocations $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{A}$ such that $\overline{p} = \overline{q}$. With this standard of neutrality, we can define aversion to inequality of opportunity as follows.

Definition 2 *An ordering $\succsim$ on $\mathbb{A}$ is said to exhibit* aversion to inequality of opportunity *if it exhibits at least as much aversion to inequality of opportunity as some ordering $\succsim_0$ exhibiting neutrality to inequality of opportunity.*

As shown in Gravel, Marchant, and Sen (2012) (Proposition 5), the notion of comparative aversion to opportunity inequality can translate, when expressed for orderings represented by (2), into a statement of "comparative concavity" applied to the function Φ of that expression.[7] Combining this fact with Definition 2, we can therefore establish the following.

Proposition 1 *An ordering $\succsim$ on $\mathbb{A}$ that can be numerically represented as per (2) exhibits aversion to inequality of opportunity if and only if Φ is concave.*

The normative dominance approach that we use for comparing allocations is based on a consensus among all opportunity inequality-averse UEVEU ethical observers. This dominance as defined as follows.

Definition 3 *Given allocations $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{A}$, we say that allocation $\mathbf{q}$ UEVEU-dominates allocation $\mathbf{p}$, which we denote by $\mathbf{q} \succsim^{QO}_{UEVEU} \mathbf{p}$, iff*

$$\frac{1}{n(\mathbf{q})} \sum_{i=1}^{n(\mathbf{q})} \Phi\left(\sum_{h=1}^{k} q_{ih} u_h\right) \geq \frac{1}{n(\mathbf{p})} \sum_{i=1}^{n(\mathbf{p})} \Phi\left(\sum_{h=1}^{k} p_{ih} u_h\right) \tag{3}$$

[7] In the simple setting of two groups and two ordered outcomes, say "hell" and "heaven", an intuitive interpretation of the role played by concavity is that an opportunity inequality averse ethical observer should prefer the allocation of opportunities where each group has a $1/2$ probability of achieving either outcomes to the allocation where one group is sure to end up in heaven while the other is sure to be damned to hell.

for all increasing and concave functions $\Phi : \mathbb{R} \to \mathbb{R}$ *and all* $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$.

We find worth emphasizing that, except when no assumption is made on the ranking of outcomes, this dominance criterion combines a concern for reducing opportunity inequality with an objective of improving the opportunities offered to some or to all of the groups. This is particularly clear when outcomes are completely ordered, in which case any allocation of opportunities - regardless of how unequal it is - will be considered weakly better than a perfectly egalitarian allocation in which the members of all groups are sure to obtain the worst outcome. In the other direction, any inequality-averse UEVEU criterion will consider the egalitarian allocation that gives to anyone the certainty to end in the best outcome to be better than any other conceivable allocation of opportunities.

3 An operational criterion for evaluating allocations of opportunities

Since the empirical applications considered herein compare allocations of opportunities $\mathbf{p}$ and $\mathbf{q}$ for which $n(\mathbf{p}) = n(\mathbf{q}) = n$ for some integer n (i.e. allocations with the same number of groups), we state our results in this particular case for convenience. However, the results generalize straightforwardly to the case where the number of groups vary between allocations.

Establishing directly whether or not UEVEU dominance as per Definition 3 holds between two allocations is not possible in practice, since the set of functions Φ and utility numbers $u_1, ..., u_k$ for which inequality (3) needs to be verified is uncountable. In this section, we identify an equivalent formulation of UEVEU dominance that makes this verification feasible with a number of operations that is polynomial with respect to k and n and, in many cases, with very simple protocols. A general result, Theorem 1, is stated in Section 3.1, while the implementation issues are examined in Section 3.3 (Lemma 4, Proposition 2, Theorem 2 and Theorem 3).

3.1 General formulation of the criterion

We start by observing that the set $\mathcal{U}^{\geq_{QO}}$ is a non-empty closed convex cone, the dual[8] of which is defined by:

$$(4) \qquad \mathcal{U}_*^{\geq_{QO}} := \left\{ (v_1, ..., v_k) \in \mathbb{R}^k : \sum_{j=1}^{k} v_j u_j \geq 0 \text{ for all } (u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}} \right\}.$$

Note that $\mathcal{U}_*^{\geq_{QO}} = \{\mathbf{0}^k\}$ if and only if none of the outcomes can be compared with one another. Another remark about the dual cone is that all the $k-$tuples $(v_1, ..., v_k)$ that it contains have components that sum to 0. We state this formally as follows.[9]

Remark 1 *If* $(v_1, ..., v_k) \in \mathcal{U}_*^{\geq_{QO}}$, *then* $v_1 + ... + v_k = 0$.

The dual cone associated with $\mathcal{U}^{\geq_{QO}}$ is the set of all changes in the probability distribution over outcomes that increase the expected utility for all utility functions compatible with $\geq_{QO}$. In plain English, it is the set of all indisputable improvements in the opportunities to achieve the outcomes. This interpretation is supported by the fact that the sum of these changes is zero, so that they produce a new probability distribution over outcomes that cumulates to 1, just like the initial distribution. The exact nature of these changes in the distribution depends on the precise definition of the quasi-ordering.

The proposed operational definition of improvement in allocating opportunities makes use of the zonotope set $\mathbf{Z}(\mathbf{p}) \subseteq \mathbb{R}_+^k$, defined by:

$$(5) \qquad \mathbf{Z}(\mathbf{p}) := \left\{ z \in \mathbb{R}_+^k : \; z = \sum_{i=1}^{n} \theta_i p_i \;\; \text{for some } \theta_i \in [0,1],\, i = 1, \ldots, n \right\}$$

A closely related set has been used by Koshevoy (1995) (see also Koshevoy and Mosler (2007)) to define a criterion called by this author *Lorenz majorization*. We

[8] Which is the negative of what Rockafellar (1970) p. 121 calls the *polar* of $\mathcal{U}^{\geq_{QO}}$.

[9] See the Appendix A.2 for all proofs.

use the zonotope defined by (5) to define what we call *Quasi-Ordering Extended Zonotope* (QOEZ) dominance as follows.

Definition 4 *Given* $\mathbf{p}, \mathbf{q} \in \mathbf{S}^n$*, we say that allocation* $\mathbf{q}$ *QOEZ-dominates allocation* $\mathbf{p}$*, which we write as* $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$*, if* $\mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq QO} \subseteq \mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq QO}$.

In plain English, $\mathbf{q}$ QOEZ-dominates $\mathbf{p}$ if any Minkowski sum of distributions of probabilities observed in the groups in $\mathbf{q}$, possibly modified by a transformation in $\mathcal{U}_*^{\geq QO}$ that unambiguously improves expected utility, is also a Minkowski sum of distributions of probabilities in $\mathbf{p}$, again possibly modified by an expected utility improving transformation in $\mathcal{U}_*^{\geq QO}$. While Minkowski sums of distributions may superficially resemble weighted averages of those distributions - and therefore be evocative of an "equalizing operation" - this is not really so because the weights of expression θ_i do not sum to 1 and may, in particular, only consist of 0 and 1. There is, however, an alternative definition of the zonotope, stated in the following lemma, that makes the equalization operation underlying extended Zonotope dominance more apparent.

Lemma 1 *Let* $\mathbf{p} \in \mathbf{S}^n$*. Then* $\mathbf{Z}(\mathbf{p}) = Co\left\{\sum_{i=1}^n \alpha_i p_i : (\alpha_1, ..., \alpha_n) \in \{0,1\}^n\right\}$.

Hence, the zonotope of an allocation of opportunities $\mathbf{p}$ is the *convex hull* of all the possible *partial sums* of the probability distributions over outcomes offered to the members of the groups. This zonotope contains therefore all weighted averages of these partial sums. Viewed in this way, the QOEZ dominance of allocation $\mathbf{p}$ by allocation $\mathbf{q}$ roughly means that all probability distributions in $\mathbf{q}$, possibly modified by expected utility improvements in $\mathcal{U}_*^{\geq QO}$ or all their partial sums, or all averages of the same are improvements over the weighted average of (partial sums of) probability distributions in $\mathbf{p}$. Koshevoy (1998) and Koshevoy and Mosler (2007) have also introduced, in a different context, a criterion based on the inclusion of extended zonotope sets. However, the extensions they consider, based on the multiplications of k-dimensional commodity bundles by their prices, are quite different from ours.

We now establish the main equivalence between the ranking of two allocations of opportunities as per QOEZ dominance and the ranking of those allocations agreed upon by all opportunity-inequality-averse UEVEU ethical observers who compare outcomes by means of the quasi-ordering $\geq_{QO}$.

Theorem 1 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be in* $\mathbf{S}^n$*. Then, the two following statements are equivalent:*

(i) $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$*;*

(ii) $\mathbf{q} \succsim_{UEVEU}^{QO} \mathbf{p}$*.*

Theorem 1 thus provides a rather solid justification for comparing allocations of opportunities on the basis of QOEZ dominance. It should be noted, in view of the impossibility of verifying Inequality (3) for all concave and increasing Φ and all utility numbers $u_1, ..., u_k$ compatible with the quasi-ordering of the outcomes, that the computational complexity to check $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ is polynomial in n and k. This verification consists indeed in solving n linear programming problems. It can be shown using interior-point methods (see e.g. Karmarkar (1984), Schrijver (1998) and Grótschel, Lovász, and Schriver (2012)) that the complexity of each such problem is $O((n+k)^{3.5})$.

QOEZ dominance bears some relationships with other notions of matrix majorization discussed in the literature (see e.g. Marshall, Olkin, and Arnold (2011) or Dahl, Guterman, and Shteyner (2018)). QOEZ dominance is indeed implied by, but is not equivalent to, *directional majorization* as defined by Dahl, Guterman, and Shteyner (2018). The discussion in the next subsection will also make clear that QOEZ dominance is implied by (but not equivalent to) what is called by Dahl, Guterman, and Shteyner (2018) *doubly stochastic majorization*.

A simple, but interesting, implication of the QOEZ dominance of one allocation by another is the corresponding domination, by all conceivable expected utilities, of their respective symmetric average distributions.

Remark 2 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be in* $\mathbf{S}^n$. *If* $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ *then* $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq_{QO}}$.

We can interpret Theorem 1 in the extreme, but instructive, case where no outcomes are comparable. In this case, we can combine standard results on one-dimensional inequality measurement and Theorem 3.1 in Koshevoy and Mosler (1996) and easily obtain the following remark.

Remark 3 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be in* $\mathbf{S}^n$. *The two following statements are equivalent:*

(*i*) $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p})$ *and* $\overline{p} = \overline{q}$;

(*ii*) $\mathbf{q} \succsim_{UEVEU}^{\emptyset} \mathbf{p}$.

Hence, non-extended Zonotope inclusion between two allocations of opportunities with the same symmetric mean is the only instance of unanimous agreement among all opportunity inequality averse UEVEU ethical observers that do not make any assumption whatsoever on the ranking of outcomes. The verification of this inclusion between two compact and convex combinations of a finite set of points in $\mathbf{S}^k$ - thanks to Lemma 1 - is very simple. It amounts to checking if every partial sum of the probability distribution in one set is a convex combination of the partial sum of the probability distribution of the other.

Before discussing, in Section 3.3, empirical methods for verifying extended zonotope inclusion in the case where outcomes are - at least partially - ordered, we briefly discuss the *elementary transformations* in the allocations of opportunities that all UEVEU ethical observers consider worth doing.

3.2 Elementary transformations

Although we do not identify all elementary transformations that are considered worth doing by the QOEZ dominance criterion in the general case with n groups – see, however, the results of the next subsection on allocations of opportunities among two groups – we identify some of them.

We start with the following, also discussed by Kolm (1977) in the more general setting of multidimensional inequality measurement.

Definition 5 (Uniform Averaging) *Given* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbf{S}^n$*, we say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through a* uniform averaging *if there exists an* $n \times n$ *bistochastic matrix* $\mathbf{b}$ *such that* $\mathbf{q} = \mathbf{b}.\mathbf{p}$

This elementary transformation consists of *uniformly averaging* the various distributions of the outcomes of the different groups. Specifically, if $\mathbf{q}$ is obtained from $\mathbf{p}$ through a *uniform averaging*, then for every group i, the probability q_{ih} of someone from that group achieving the outcome h is a weighted average of the probabilities of people from the different groups in $\mathbf{p}$ achieving that outcome. This averaging is *uniform* in the sense that, for any group i, the weights used in the calculation of the average are the same for all outcomes.

The second elementary transformation that we examine is what we call a *bilateral equalizing transfer*. Contrary to uniform averaging - which does not use information on the ranking of outcomes - a bilateral equalizing transfer relies heavily on such information. The formal definition of this elementary transformation is as follows.

Definition 6 (Bilateral Equalizing Transfer) *Given* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbf{S}^n$*, we say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through a bilateral equalizing transfer if there exist indices* i_1*,* i_2*,* $i'_1, i'_2 \in \{1, ..., n\}$ *and* $v \in \mathcal{U}_*^{\geq_{QO}}$ *such that* $p_j = q_j$ *for all* $j \notin \{i_1, i_2, i'_1, i'_2\}$*, and*

$$q_{i'_1} = p_{i_1} + v,\ q_{i'_2} = p_{i_2} - v,\ p_{i_2} - p_{i_1} - v \in \mathcal{U}_*^{\geq_{QO}}.$$

A bilateral equalizing transfer is an operation that improves (through some change v) the opportunities faced by a group and that worsens (through the *same* v applied in reverse) the opportunities faced by another group, in the case where the latter group's opportunities are unambiguously better than the former's regarding the quasi-ordering $\geq_{QO}$ or, equivalently thanks to Donaldson and Weymark (1998),

regarding all complete rankings of outcomes whose intersection is $\geq_{QO}$. In the case of a complete quasi-ordering of outcomes, an equalizing transfer is a particular form of an operation called "opportunity dominance" by Fleurbaey (2008). We observe that such a transformation only concerns two distributions of outcomes in each of the two allocations and leaves the other groups' distributions unchanged. Hence, compared to a uniform averaging which concerns the *entire matrix*, a bilateral equalizing transfer is a *local operation* that concerns only two rows of each of the matrices under comparison.

Example 1 *We illustrate this elementary transformation in the case of an incomplete ranking of outcomes. Consider for this purpose the following binary health-education example where the outcomes are* $(0,0)$ *(bad health, low education),* $(0,1)$ *(bad health, high education)* $(1,0)$ *(good health, low education) and* $(1,1)$ *(good health, high education) with the quasi-ordering* $\geq_{QO}$ *defined by* $(1,1) \geq_{QO} (0,1) \geq_{QO} (0,0)$ *and* $(1,1) \geq_{QO} (1,0) \geq_{QO} (0,0)$ *(*$(0,1)$ *and* $(1,0)$ *being incomparable). Assume that there are only two groups, and consider the two allocations:*

$\mathbf{p} =$

	$0,0$	$0,1$	$1,0$	$1,1$
gr 1	$1/2$	$1/6$	$1/6$	$1/6$
gr 2	$1/4$	$1/4$	$1/4$	$1/4$

$\mathbf{q} =$

	$0,0$	$0,1$	$1,0$	$1,1$
gr 1	$7/16$	$19/96$	$1/6$	$19/96$
gr 2	$5/16$	$7/32$	$1/4$	$7/32$

Observe first that in allocation $\mathbf{p}$*, the probabilities of achieving the four outcomes in group 1 provide a lower expected utility than those of group 2. This can be seen by the fact that, for any of the two complete rankings of the four outcomes that are consistent with* $\geq_{QO}$*, the distribution of outcomes faced by group 2 first-order stochastically dominates that faced by group 1. A second observation that can be made is that the move from* $\mathbf{p}$ *to* $\mathbf{q}$ *has been done by improving the probability distribution of group 1 by the vector* $v = (-1/16, 1/32, 0, 1/32)$ *and by worsening the probability distribution of group 2 by the corresponding vector* $-v = (1/16, -1/32, 0, -1/32)$*. Note that these two balanced offsetting changes in the distributions of outcomes have*

preserved the dominance of group 2 over group 1. Yet, the spread of this difference has shrunk.

The last elementary transformation that we discuss is not related to reducing inequalities of opportunity. It is rather concerned with improving opportunities for some, or all, of the groups (up to a permutation of them thanks to the anonymity principle that underlies all UEVEU criteria). It is defined as follows.

Definition 7 (Anonymous and Unanimous Expected Utility Improvement) *Given* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbf{S}^n$*, we say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through an* anonymous and unanimous expected utility improvement *if there exists a one-to-one function* $\pi : \{1, ..., n\} \longrightarrow \{1, ..., n\}$ *such that, for every* $i \in \{1, ..., n\}$*,* $q_{\pi(i)} = p_i + v_i$ *is true for some* $v_i \in \mathcal{U}_*^{\geq_{QO}}$.

In words, $\mathbf{q}$ is obtained from $\mathbf{p}$ through an anonymous and unanimous expected utility improvement if there is a permutation of the groups such that every permuted group in $\mathbf{q}$ faces opportunities that provide a greater expected utility than those in $\mathbf{p}$ for all lists of utility numbers compatible with the underlying quasi-ordering. Any such anonymous and unanimous expected utility improvement will clearly be appraised favorably by any UEVEU ethical observer (irrespective of her attitude to opportunity inequality) and, thanks to Theorem 1, by the QOEZ dominance criterion. In the following lemma, we establish formally that performing a bilateral equalizing transfer *or* a uniform averaging will also be considered worthwhile elementary operations by UEVEU observers that are averse to inequality of opportunity or, equivalently, by QOEZ dominance.

Lemma 2 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be in* $\mathbf{S}^n$*. If* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through either* uniform averaging *or* equalizing transfer *then* $\mathbf{q} \succsim^{QO}_{UEVEU} \mathbf{p}$.

3.3 Implementation of the criterion

This subsection provides operational procedures - some of them very simple - for verifying QOEZ dominance in practice.

We start by levying a possible difficulty with the QOEZ criterion invoked in Theorem 1. This criterion consists in verifying if an inclusion holds between two sets obtained by adding to the convex hull of a finite set of points (thanks to Lemma 1) the dual cone of the set of all lists of k utility numbers $u_1, ..., u_k$ compatible with the quasi-ordering $\geq_{QO}$. Yet, there are uncountably many such lists. The dual cone of an uncountable set may thus be difficult to manipulate. In the following lemma, we show, however, that the dual cone considered in the QOEZ dominance test is actually the dual cone of a finite set.

Lemma 3 *Let $\mathcal{U}_*^{\geq_{QO}}$ be defined as per (4). Then we have:*

$$\mathcal{U}_*^{\geq_{QO}} = \left\{ v \in \mathbb{R}^k : \sum_{j=1}^{k} v_j u_j \geq 0 \ \forall (u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}} \cap \{0,1\}^k \right\}.$$

Lemma 3 thus provides a very simple way to define the dual cone $\mathcal{U}_*^{\geq_{QO}}$. Consider, indeed, the family $\mathcal{F}^{\geq_{QO}}$ of subsets of outcomes whose elements form a chain with respect to the quasi-ordering $\geq_{QO}$. This family is formally defined by:

$$\mathcal{F}^{\geq_{QO}} = \{J \subseteq \{1, ..., k\} : h \in J \text{ and } j \geq_{QO} h \Longrightarrow j \in J\}$$

This family is closely related to the dual cone $\mathcal{U}_*^{\geq_{QO}}$ which can indeed be defined, thanks to Lemma 3, by:

$$\mathcal{U}_*^{\geq_{QO}} = \left\{ v \in \mathbb{R}^k : \sum_{h=1}^{k} v_h = 0 \text{ and } \sum_{h \in J} v_h \geq 0 \text{ for all } J \in \mathcal{F}^{\geq_{QO}} \right\}. \tag{6}$$

The family $\mathcal{F}^{\geq_{QO}}$ is important because it provides the complete (and finite) list of sets of outcomes whose increases in likelihood are indisputably perceived as improving op-

portunities. For example, if $\geq_{QO}$ is taken to be the complete ordering $\geq_C$ of outcomes, the family $\mathcal{F}^{\geq_C}$ is the (anti) cumulative lists of outcome $\{k\}$, $\{k-1,k\}$,...,$\{1,2,...,k\}$ used to check for first-order stochastic dominance, and Lemma 3 implies that the dual cone of $\mathcal{U}^{\geq_C}$ corresponds to the set of changes $(v_1,...,v_k)$ in the probability distributions that produce *first-order stochastic* improvements over the probability distributions to which they are applied.[10] But the dual cone $\mathcal{U}_*^{\geq_{QO}}$ can be defined in terms of the family $\mathcal{F}^{\geq_{QO}}$ for any incomplete quasi-ordering. For instance, the family $\mathcal{F}^{\geq_{QO}}$ associated to the incomplete quasi-ordering $\geq_{QO}$ of Example 1 above involving the four outcomes $\{(0,0),(0,1),(1,0),(1,1)\}$ is easily seen to be $\mathcal{F}^{\geq_{QO}} = \{\{(1,1)\},\{(1,0),(1,1)\},\{(0,1),(1,1)\},\{(0,1),(1,0),(1,1)\},\{(0,0),(0,1),(1,0),(1,1)\}\}$

Given that the dual cone $\mathcal{U}_*^{\geq_{QO}}$is just a collection of k changes in probabilities of achieving the relevant outcomes that sum to zero and that satisfy a finite collection of inequalities, we are equipped to state the next lemma that opens the way to several finite procedures for verifying QOEZ dominance. The lemma requires the following additional notation. For any possible number of groups $l = 1,\ldots,n$, let $m(l)$ denote the number of collections of l groups, and label these collections as $J_l^1,...,J_l^{m(l)}$. Given this, we can state the following.

Lemma 4 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be in* $\mathbf{S}^n$. *Then* $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ *if and only if, for all* $l = 1,\ldots,n$ *and* $m = 1,\ldots,m(l)$, *there exists* $\tilde{p} \in Co\left\{\sum_{i\in J_l^1} p_i,\ldots,\sum_{i\in J_l^{m(l)}} p_i\right\}$ *such that* $\sum_{i\in J_l^m} q_i - \tilde{p} \in \mathcal{U}_*^{\geq_{QO}}$.

This lemma says that testing for QOEZ dominance of $\mathbf{q}$ over $\mathbf{p}$ amounts to testing that, for any particular collection of l groups, one can find a convex combination of the symmetric average of the probability distributions in all collections of l groups in $\mathbf{p}$ that is dominated - as per expected utility for all utility functions in $\mathcal{U}^{\geq_{QO}}$- by the symmetric average probability distribution in the considered collection in $\mathbf{q}$. The test

[10]For any two distributions p and $q \in \mathbf{S}$, we say that q dominates p with respect to the first-order stochastic dominance, denoted $q \succsim^{1st} p$, if and only if $\sum_{h=j}^k q_h \geq \sum_{h=j}^k p_h$ for any $j = 1,...,k$.

provided by Lemma 4 really amounts to verifying - thanks to Lemma 3 - if, for every collection of groups - and there are finitely many of those - every list of inequalities associated with the dual cone of the considered quasi-ordering admits a solution that lies in the simplex associated with the considered number of groups. For example, if we consider the complete ordering $\geq_C$ of outcomes and, therefore, the set of changes in probability that generate first-order stochastic dominance as the dual cone, the test of Lemma 4 amounts to verifying if, for any $l = 1, \ldots, n$ and $m^l = 1, \ldots, m(l)$, there are solutions $\lambda_1^{m^l}, ..., \lambda_{m(l)}^{m^l} \in \mathbf{S}^{m(l)-1}$ to the following collection of inequalities, indexed by $h = 2, ..., k$:

$$\sum_{m=1}^{m(l)} \lambda_m^{m^l} \sum_{i \in J_l^m} \sum_{g=h}^{k} p_{ig} \leq \sum_{i \in J_l^{m^l}} \sum_{g=h}^{k} q_{ig} \tag{7}$$

Several linear programming solvers (available, for instance, in Python) can be used to make this verification. If the solver does not find solutions to system (7), then a strong suspicion of non-existence of those solutions - and of not dominance of $\mathbf{p}$ by $\mathbf{q}$ - would emerge. This suspicion could be further confirmed by resorting to the Farkas' lemma and verifying the existence of a solution to the (dual) set of linear inequalities described as "Farkas's Alternative 38" in Border (2020). An illustration of the use of the procedure of Lemma 4 and inequalities (7) using Indian data is provided in Appendix A.5.

We now discuss procedures for verifying QOEZ dominance in specific, but empirically relevant, cases that are much simpler than the verification of solutions to finite systems of inequalities such as (7). Moreover, conclusions drawn in such specific cases could be supported by well-known statistical inference criteria (see the empirical application for a discussion), which are not readily available to test the solutions to the inequalities such as (7).

Situations of important empirical interest for this purpose are those where the distributions faced by the different groups can all be ordered by any expected utility

consistent with the quasi-ordering of outcomes. For example, in the Indian illustration discussed below where the complete ordering of education levels is assumed, most of the various gender- and caste-based groups can be compared by first-order stochastic dominance. This case has also been considered in some of the empirical literature on measurement of equality of opportunity (see, e.g., Andreoli, Havne, and Lefranc (2019) and Lefranc, Pistolesi, and Trannoy (2009)). If the distributions faced by the different groups can all be compared by any expected utility consistent with the quasi-ordering of outcomes, then testing for extended Zonotope inclusion amounts to performing the simple sequential procedure described in the following proposition.

Proposition 2 *Let* $\mathbf{q}$ *and* $\mathbf{p} \in \mathbf{S}^n$ *be such that* $q_{i+1} - q_i \in \mathcal{U}_*^{\geq_{QO}}$ *and* $p_{i+1} - p_i \in \mathcal{U}_*^{\geq_{QO}}$ *for all* $i = 1, ..., n-1$. *Then* $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ *if and only if, for any* $l = 1, \ldots, n$*:*

$$\sum_{i=1}^{l} q_i - \sum_{i=1}^{l} p_i \in \mathcal{U}_*^{\geq_{QO}} \tag{8}$$

The procedure of Proposition 2 is reminiscent of the sequential logic underlying generalized Lorenz dominance. Indeed, the procedure starts by first comparing the worst-off group's distribution in the two allocations. If these distributions are not comparable, then the test fails and the two allocations cannot be compared. If one worst-off distribution dominates the other, then one moves to comparing the average cumulative distributions of the two worst-off groups, and so on.

When applied to the case where outcomes are completely ordered, the procedure described in Proposition 2 bears some resemblance to the test proposed by Dardanoni (1993) (Theorem 1) in the context of mobility measurement. Dardanoni (1993) has proposed his test in the context of ranking (square) mobility matrices with fixed margins in which children from low-background parents face worse opportunities - as appraised by first-order dominance - than children from high-background parents. Proposition 2 can be seen as an extension of his test to the case where the joint

distributions represented by the matrices do not have the same margins and where, possibly, $n \neq k$.

An interesting particular case, considered in Mariotti and Veneziani (2017), to which Proposition 2 applies is when $k = 2$ ("hell" and "heaven"). In such a case, there can only be either a complete ordering of the two outcomes, or no ordering at all. When outcomes are ordered, all distributions can be compared by first-order dominance. Our criterion would then evaluate allocations of opportunities by sequentially comparing the symmetric average probability of ending in hell in the j groups for which this probability is the largest in their respective allocation, for any $j = 1, \ldots, n$. An allocation of opportunities in which this symmetric average probability of ending in hell calculated over the j groups where it is the highest is lower than in another allocation for all j groups, would be said to dominate this latter allocation. Non-comparability of the two allocations would be obtained if the symmetric average probability of ending in hell calculated over the j groups where it is the highest is lower in one allocation than in another for some j, but when an opposite conclusion holds for some other number - l say - of groups with the highest probability of ending in hell. The incomplete ordering of two allocations generated by this criterion is a sub-relation of the complete criterion of Mariotti and Veneziani (2017).

The "hell-heaven" case enables a geometric depiction of QOEZ dominance, such as the one of Figure 2 involving three groups. In this figure, allocation $\mathbf{q}$ is perfectly egalitarian and provides every group with a 1/3 probability of ending in heaven. The zonotope set of $\mathbf{q}$ is the convex hull of all partial sums of the (identical) probability distributions (including the "empty" zero-sum). It is thus nothing else than the line segment connecting thrice the symmetric average probability distribution $\overline{q}$ over the three groups to the origin. The extended zonotope of this allocation of opportunities is the gray-lined area at the north-west of this line segment. Allocation $\mathbf{p}$, on the other hand, provides to the three groups a larger average probability of

$\mathbf{q} =$

	Hell	Heaven
gr 1	2/3	1/3
gr 2	2/3	1/3
gr 3	2/3	1/3

$\mathbf{p} =$

	Hell	Heaven
gr 1	1/10	9/10
gr 2	1/2	1/2
gr 3	9/10	1/10

$\mathbf{r} =$

	Hell	Heaven
gr 1	3/10	7/10
gr 2	1/2	1/2
gr 3	7/10	3/10

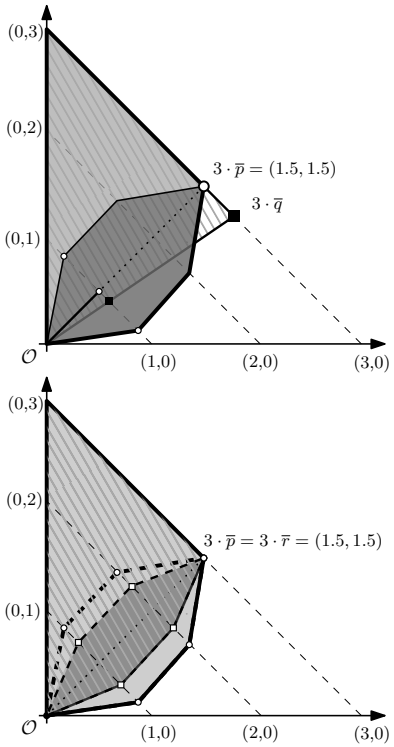

Figure 2: Zonotopes and their extensions for $\mathbf{p}$ (circles), $\mathbf{q}$ (squares) and $\mathbf{r}$ (boxes).

ending in heaven (1/2) but distributes this probability unequally among them. The zonotope set of $\mathbf{p}$ is depicted in the picture as the (dark shaded) convex hull of all the partial sums (over 0, 1, 2 and 3 groups) of their three probability distributions. The extension of this zonotope by first-order stochastic dominance improvements is the pale gray shaded area at the north-west of the zonotope and the total extended zonotope of $\mathbf{p}$ is the area corresponding to the union of the zonotope and the pale gray shaded area at its north-west. Observe that the extended zonotopes of $\mathbf{p}$ and $\mathbf{q}$ are not related by set inclusion. Hence $\mathbf{p}$ and $\mathbf{q}$ are not comparable by extended zonotope dominance. The reason behind this incomparability is the usual conflict between "efficiency" (improving for all the average probability of ending in heaven) and "equity" (distributing equally this average among groups). Allocation $\mathbf{q}$ scores better than $\mathbf{p}$ on the latter account but scores worse than $\mathbf{p}$ on the former. The graph at the bottom of Figure 2 reports a case of QOEZ dominance of allocation $\mathbf{p}$ by allocation $\mathbf{r}$, the latter being obtained from the former through uniform averaging.

Because of this, the zonotope of $\mathbf{r}$ and its extension (in shaded gray) lie within the space delimited by the extended zonotope of $\mathbf{p}$, in solid gray.

Another class of situations to which the procedure of Lemma 4 can be easily applied are those involving only *two* groups. In this case, the procedure of Lemma 4 works as follows. For any two allocations, it first checks if the symmetric average opportunities are better in one allocation than in the other. If such a dominance is observed, then the allocation with the dominating average is a candidate for a dominating allocation according to Remark 2. To verify that this is indeed so, the test examines, for *each* of the two distributions in the (possibly) dominating allocation, all mixtures of distributions in the (possibly) dominated allocation that yield the same probability of reaching outcomes in some members of the family $\mathcal{F}^{\geq_{QO}}$. There may not be any such mixtures, in which case it can be concluded that there is no dominance. If, however, such mixtures exist, then the verdict of dominance is obtained if each of the two distributions in the dominating allocation dominates at least one such mixture of the two distributions in the dominated allocation. The following theorem describes this procedure and states its equivalence to QOEZ dominance.

Theorem 2 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be in* $\mathbf{S}^2$*, and* $\mathbf{\Lambda}_i$ *(for* $i = 1, 2$*) be defined by:*

$$\mathbf{\Lambda}_i = \{1\} \cup \{\lambda \in [0,1] : \exists J \in \mathcal{F}^{\geq_{QO}} \ \ s.t.\ q_i(J) = \lambda p_1(J) + (1-\lambda)p_2(J)\}.$$

where, for any $J \subset \{1, ..., k\}$*,* $q_i(J) = \sum_{j \in J} q_{ij}$ *and* $q_i(J) = \sum_{j \in J} p_{ij}$ *for* $i = 1, 2$*. Then* $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ *if and only if* $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq_{QO}}$ *and there are* $\lambda_i \in \mathbf{\Lambda}_i$ *(for* $i = 1, 2$*) such that* $q_1 - (\lambda_1 p_1 + (1-\lambda_1)p_2) \in \mathcal{U}_*^{\geq_{QO}}$ *and* $q_2 - (\lambda_2 p_1 + (1-\lambda_2)p_2) \in \mathcal{U}_*^{\geq_{QO}}$*.*

The simplicity of the procedure described in Theorem 2 is illustrated in the following example of a dominance relationship between two distributions that is not immediately apparent.

Example 2 *Consider the following two allocations of opportunities to achieve four completely ordered outcomes between two groups:*

$\mathbf{p} =$

	1	*2*	*3*	*4*
gr 1	16/36	4/36	6/36	10/36
gr 2	13/36	3/36	12/36	8/36

, $\mathbf{q} =$

	1	*2*	*3*	*4*
gr 1	16/36	2/36	8/36	10/36
gr 2	13/36	5/36	9/36	9/36

Observe that $\overline{q} - \overline{p} = \frac{1}{72}(0, 0, -1, 1) \in \mathcal{U}_*^{\geq_C}$. *Hence* $\mathbf{q}$ *possibly QOEZ dominates* $\mathbf{p}$. *Let us use the procedure of Theorem 2 to verify that it is indeed the case. Here* $\mathcal{F}^{\geq_C} = \{\{1, 2, 3, 4\}, \{2, 3, 4\}, \{3, 4\}, \{4\}\}$ *and the sets* $\mathbf{\Lambda}_1$ *and* $\mathbf{\Lambda}_2$ *are defined as the union of the singleton* $\{1\}$ *and the set of solutions, in the* $[0, 1]$ *interval, of the following equations:*

$$10 = 10\lambda_{11} + 8(1 - \lambda_{11}) \Longleftrightarrow \lambda_{11} = 1$$

$$18 = 16\lambda_{12} + 20(1 - \lambda_{12}) \Longleftrightarrow \lambda_{12} = 1/2$$

$$20 = 20\lambda_{13} + 23(1 - \lambda_{13}) \Longleftrightarrow \lambda_{13} = 1/2$$

for $\mathbf{\Lambda}_1$ *and of the following equations:*

$$9 = 10\lambda_{21} + 8(1 - \lambda_{21}) \Longleftrightarrow \lambda_{21} = 1/2$$

$$18 = 16\lambda_{22} + 20(1 - \lambda_{22}) \Longleftrightarrow \lambda_{22} = 1/2$$

$$23 = 20\lambda_{23} + 23(1 - \lambda_{23}) \Longleftrightarrow \lambda_{23} = 0$$

for $\mathbf{\Lambda}_2$. *Hence* $\mathbf{\Lambda}_1 = \{1/2, 1\}$ *and* $\mathbf{\Lambda}_2 = \{0, 1/2, 1\}$. *Since* q_1 *first-order dominates* p_1, *have* $q_1 - \lambda p_1 - (1 - \lambda)p_2 \in \mathcal{U}_*^{\geq_C}$ *for* $\lambda = 1 \in \mathbf{\Lambda}_1$. *Since one can also observe that* $q_2 - \lambda p_1 - (1 - \lambda)p_2 \in \mathcal{U}_*^{\geq_C}$ *for* $\lambda = 1/2 \in \mathbf{\Lambda}_2$, *one has indeed that* $\mathbf{q}$ *QOEZ dominates* $\mathbf{p}$.

Remark 4 *Example 2 also shows that the three elementary transformations dis-*

cussed in Section 3.2 - namely uniform averaging, equalizing transfer and anonymous and unanimous expected utility improvements - are not the only ones that are implied by QOEZ dominance. The "global" transformation that enables one to go from $\mathbf{p}$ *to* $\mathbf{q}$ *in Example 2 consists in simultaneously adding* $1/36(0,-2,+2,0)$ *to* p_1 *and adding* $1/36(0,2,-3,1)$ *to* p_2*. It is shown in Appendix A.2 that it is actually impossible to perform on* $\mathbf{p}$ *any of the three elementary transformations of Section 3.2 while preserving the dominance of the newly constructed allocation by* $\mathbf{q}$*. Hence, the global transformation just described can not be decomposed into the three elementary transformations of Section 3.2.*

We end the discussion of allocations of opportunities involving only two groups by considering a particular - but theoretically important - case where the fact of performing two of the three types of elementary transformations of Section 3.2 a finite number of times *coincides* with QOEZ dominance. This case is when the two allocations offer the same (symmetric) average opportunity to the two groups and, therefore, only differ in the way by which this common average opportunity is split between them. In this case, QOEZ dominance actually coincides with the possibility of going from the dominated to the dominating allocation by a finite sequence of equalizing transfers and uniform averaging operations. The following theorem establishes that fact.

Theorem 3 *Let* $\mathbf{p},\mathbf{q}\in\mathbf{S}^2$ *be such that* $\overline{p}=\overline{q}$*. The following are equivalent:*

1. $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through a finite sequence of uniform averaging and/or bilateral equalizing transfers;*
2. $\mathbf{q}\succsim^{QO}_{UEVEU}\mathbf{p}$*;*
3. $\mathbf{q}\succsim^{QO}_{EZ}\mathbf{p}$*.*

Theorem 3 has a very simple implication when outcomes are completely ordered and when, as a result, the dual cone of the set of lists of utility numbers $(u_1,...,u_k)$

increasing with the outcomes is the set of changes v that generate first-order dominance between distributions. In this case, the following is an immediate consequence of Theorem 3.

Remark 5 *Let* $\mathbf{p}, \mathbf{q} \in \mathbf{S}^2$ *be such that* $\overline{p} = \overline{q}$. *Assume that*[11] $p_1 \succsim^{1st} p_2$. *Then* $\mathbf{q} \succsim^{C}_{EZ} \mathbf{p}$ *if and only if* $p_1 \succsim^{1st} q_1 \succsim^{1st} p_2$ *and* $p_1 \succsim^{1st} q_2 \succsim^{1st} p_2$.

This remark opens the way to a simple test of opportunity equalization in the case of two groups, that is applicable to any two allocations with identical average opportunities and where one group first-order stochastically dominates the other in one of the two allocations. The test amounts to verifying whether, in the other allocation, the distributions of outcomes of the two groups lie in between those of the two groups in the current allocation in terms of first-order stochastic dominance. This is, roughly, the situation that prevails in the West-Bengal Odisha comparison illustrated in Figure 1.

We conclude this section by exploring simple-to-verify *sufficient conditions* for QOEZ dominance. We also provide, for the case where the quasi-ordering of outcomes is complete, *necessary conditions* for this dominance. Hence, if the sufficient conditions are verified, we can be sure that QOEZ dominance holds. Similarly, if the necessary conditions are not verified, one can safely conclude that QOEZ dominance does not hold. A gray zone, amenable to the more difficult analysis based on Lemma 4, would of course remain if the sufficient condition was not verified and the necessary condition was.

The proposed sufficient condition for QOEZ dominance combines the test of Proposition 2 with the simple observation (see, e.g., Donaldson and Weymark (1998)) that a quasi-ordering is the intersection of a collection of orderings. Consider, indeed, an allocation of opportunities $\mathbf{p} \in \mathbf{S}^n$ among n groups. Denote by $Q^{\mathbf{p}}$ the quasi-

[11] Obviously, the result also holds if $p_2 \succsim^{1st} p_1$.

ordering defined on $\{p_1, ..., p_n\}$, by:

$$p_i \; Q^{\mathbf{p}} \; p_l \Leftrightarrow p_i - p_l \in \mathcal{U}_*^{\geq_{QO}}, \text{ for any } i, l \in \{1, ..., n\},$$

This quasi-ordering can be complete, as assumed in Proposition 2. It can also be empty if the probability distributions faced by any two different groups in $\mathbf{p}$ are incomparable as per expected utility for all lists of utility numbers in $\mathcal{U}^{\geq_{QO}}$. Let $\mathfrak{R}^{\mathbf{p}}$ denote the set of orderings over $\{p_1, ..., p_n\}$ whose intersection is $Q^{\mathbf{p}}$. The set $\mathfrak{R}^{\mathbf{p}}$ is not empty, as shown in Donaldson and Weymark (1998). For any $R \in \mathfrak{R}^{\mathbf{p}}$, let σ^R be the permutation of $\{1, ..., n\}$ satisfying $p_{\sigma(i)} \; R \; p_{\sigma(i-1)}$ for $i = 2, ..., n$. We can then easily establish the following.

Proposition 3 *Let* $\mathbf{q}$ *and* $\mathbf{p} \in \mathbf{S}^n$ *be two allocations of opportunities. Then* $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ *if, for any ordering* $R \in \mathfrak{R}^q$ *and* $R' \in \mathfrak{R}^p$ *and any* $l = 1, \ldots, n$*:*

$$\sum_{i=1}^{l} q_{\sigma^R(i)} - \sum_{i=1}^{l} p_{\sigma^{R'}(i)} \in \mathcal{U}_*^{\geq_{QO}} \tag{9}$$

The test of Proposition 3 works just like that of Proposition 2, but applied to all orderings whose intersection is the ranking of the distributions faced by the groups as per their expected utility in each of the two compared distributions. Hence, for any two orderings of the groups - one such ordering for each of the two allocations - the test requires the worst off group in $\mathbf{q}$ to be better off than the worst-off group in $\mathbf{p}$, the two worst-off groups in $\mathbf{q}$ to be better off than the two worst-off group in $\mathbf{p}$, and so on. Passing successfully the test is clearly a demanding task, especially if the number of orderings to be considered is large. Since this test is a sufficient condition for QOEZ dominance, failing it does not say much about the QOEZ dominance relationship that prevails between the two allocations.

If $\geq_{QO}$ is complete, we can also propose a simple-to-verify *necessary* condition for QOEZ dominance. This condition is inspired by a procedure commonly employed in conventional dominance analysis involving distributions of a single attribute such as

income, which consists in only considering the subset of concave maps $q \mapsto \min(q - \pi, 0)$. It also rides on the fact, closely related to Lemma 3, that the requirement that expected utility be increased for all list of k increasing utility numbers can actually be weakened to the requirement that it be increased only for those such utility numbers that take their value in the set $\{0, 1\}$. The condition that we propose, and that we call *2nd order dominance of the chances of success*, is defined as follows.

Definition 8 *Given* $\mathbf{p}, \mathbf{q} \in \mathbf{S}^n$*, we say that the distribution of chances of success among groups in* $\mathbf{q}$ *dominates at the 2nd order the distribution of those chances of success in* $\mathbf{p}$*, which we write as* $\mathbf{q} \succsim_2^{success} \mathbf{p}$*, if for any* $l \in \{1, ..., k\}$*,*

$$\sum_{i=1}^{n} \max \left[\pi - \sum_{h=l}^{k} q_{ih}, 0 \right] \leq \sum_{i=1}^{n} \max \left[\pi - \sum_{h=l}^{k} p_{ih}, 0 \right],$$

for any target probability of success $\pi \in [0, 1]$*.*

The test described in Definition 8, that applies only if the k outcomes are ordered, works as follows. Select an outcome $h \in \{1, ..., k\}$ and consider a target probability π of achieving, for any member of any group, at least this outcome. The allocation of opportunities $\mathbf{q}$ is unambiguously better than the allocation of opportunities $\mathbf{p}$ for this test if, for any outcome h and any target probability π of achieving at least this outcome, the sum of the shortfalls between the actual probability by which the groups achieve this outcome and the target is lower in $\mathbf{q}$ than in $\mathbf{p}$. In an even plainer English, $\mathbf{q}$ is a better allocation of opportunities than $\mathbf{p}$ if the shortfall by which groups fails in securing the probability of achieving any outcome is lower in $\mathbf{q}$ than in $\mathbf{p}$, whatever is the considered probability.

We establish in the following proposition that 2nd order dominance of the chances of success is necessary to QOEZ dominance in the case where the quasi-ordering of outcomes is complete.

Proposition 4 *Let* $\mathbf{q}$ *and* $\mathbf{p} \in \mathbf{S}^n$ *be two allocations of opportunities. Then* $\mathbf{q} \succsim_{EZ}^{C} \mathbf{p}$ *only if* $\mathbf{q} \succsim_2^{success} \mathbf{p}$*.*

Hence any failure of 2nd order dominance of the chances of success of $\mathbf{q}$ over $\mathbf{p}$ indicates an absence of QOEZ dominance of $\mathbf{q}$ over $\mathbf{p}$. One could wonder about the reverse implication. After all, as is well-known from conventional dominance analysis (see e.g. Shaked and Shanthikumar (2007)), ranking two lotteries with monetary outcomes in terms of their expected utility for all increasing and concave utility functions is equivalent to their expected utility ranking based on the subclass of all increasing and and concave functions having the $min(x-t,0)$ form. Similarly, ranking two lotteries with monetary outcomes by their expected utility for all increasing utility function is equivalent to the first-order stochastic dominance of those lotteries, which is nothing else than the restriction of increasing utility functions to those taking their values in $\{0,1\}$. One could thus wonder if requiring agreement of all UEVEU ethical observers who use increasing and concave Φ and increasing lists of k real utility numbers is equivalent to requiring this agreement to the more limited class of those observers who use Φ having the $min(x-t,0)$ form and who constrain the k real utility numbers to take their value in $\{0,1\}$. The following example shows that this is *not* the case.

Example 3 *Assume that there are three (completely ordered) outcomes and two groups and consider the following two allocations of opportunities*

$$\mathbf{q} = \begin{pmatrix} 147/864 & 436/864 & 281/864 \\ 153/864 & 416/864 & 295/864 \end{pmatrix},$$

$$\mathbf{p} = \begin{pmatrix} 156/864 & 396/864 & 312/864 \\ 144/864 & 456/864 & 264/864 \end{pmatrix}.$$

Then $\mathbf{q} \succsim_2^{success} \mathbf{p}$*, while* $\mathbf{q} \not\succsim_{EZ}^{C} \mathbf{p}$*. See the appendix for details.*

The problem of ranking allocations of opportunities by all UEVEU criteria that use increasing and concave valuation functions of expected utility is thus fundamentally different from that of ranking probability distributions on monetary outcomes

by all expected increasing and concave utility of those outcomes.

4 Empirical Illustration

This section uses QOEZ dominance to compare allocations of educational opportunities in the 14 most economically important Indian *states* (thus excluding union territories as well as the North-Eastern states of India and Jammu-Kashmir where the NSSO data collection is knowingly imperfect). These states are listed, along with their acronym and some global descriptive statistics, in Table 1 of Appendix A.4. These states contain more than 80% of the Indian population. Some of these states, notably Kerala and Maharashtra, have a much higher rate of literacy than the national average while others, such as Bihar, Uttar Pradesh or Rajasthan, stand significantly below the national average on this front. In order to perform the evaluation, we first consider caste as the only (morally irrelevant) characteristic on which groups are formed and we accordingly distinguish between low caste and high caste. We then introduce gender as an alternative binary source of group differentiation. We finally combine caste and gender to create four groups among which we evaluate the allocations of opportunities of the considered states.

We base our empirical analysis on the most recent wave of the Employment-Unemployment Survey from the NSSO micro-database that covers 2011-2012, the last available wave. This survey provides information on the highest level of educational qualification achieved by every member of 101,718 households. This survey is one of the largest by NSSO that covers almost the entire country with a complex and multilayered sample design, which grants representativeness at the state level. We limit our attention to Indian adults between 30-40 years of age who do not attend any educational institution as a trainee or student to capture the most recent educational opportunities faced by Indian adults. Adults below 30 may not have completed their education. Adults over 40 years of age have on average completed their education

more than 15 years before 2012 and their educational achievements may in part be due to the education regime that was prevailing in India prior to 1968.

Educational achievements are measured on an ordered scale with six potential levels, described in Table 2 of Appendix A.4. These levels range from illiteracy to university degree or higher. We focus on the allocation of educational opportunities among gender- and caste-based groups. In our sample, 50% of adults are males, with the notable exception of Kerala (55%). The majority of the sample (73%) is from the upper caste group, with a peak in Kerala (90%) and the smallest proportion observed in Odisha (63%). In our sample, 8% of the respondents declare a high level of education, while a third have no formal education at all. Such percentages are vastly heterogeneous across states, with Kerala displaying the lowest incidence of low-education adults (7%) and one of the highest shares of graduates (10%), while other States, such as Uttar Pradesh and Rajasthan, display the broadest heterogeneity in education opportunities. The grouping of adults based on caste is done according to the official Indian categories of Scheduled Castes (SC) and Scheduled Tribes (ST) that we merged as the "low castes" (henceforth LC), while the remaining part of the adult population forms what we call the "high castes" (HC). This binary partition of the population into caste groups is obviously a very rough description of the complexities of the Indian caste system, described in details for example in Deshpande (2011). We do not adopt an "individual" interpretation of the group in this empirical application and we rather view the caste themselves as the groups. Hence, castes will not be weighted by their relative frequency in the population and will be treated symmetrically. Table 1 in Appendix A.4 shows the distribution of the population of every state into either the two caste groups or the two genders while Table 3 in the same appendix shows the distribution of the entire adult population of India in 2012 by education outcomes conditional on caste, gender and caste-gender, respectively.

Let us first look at the allocations of educational opportunities among caste-

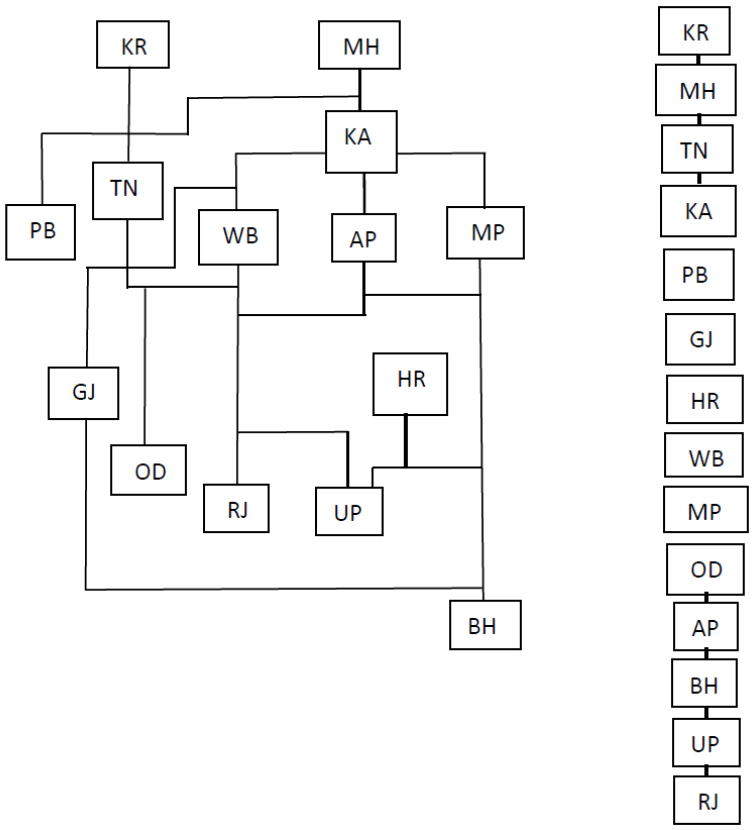

Figure 3: Caste-based QOEZ ranking of allocations of education opportunities in the Indian states (Hasse diagram).
Note: Left-hand side diagram is obtained by implementing the Dominance tests in Proposition 2. States on top of the diagram dominate states at the bottom of the diagram according to the UEVEU criterion. The right-hand side ranking is obtained by means of the welfare index in equation (10). Missing interconnections between states identify pairs for which a test of equality in the index value cannot be rejected at 95% confidence level.

groups in those 14 states depicted in Figure 6 of Appendix A.4. It should come as no surprise that the distribution of educational outcome in the HC group dominates the distribution of the LC group by first-order stochastic dominance in every state. Invoking Proposition 2, QOEZ dominance between any two states can therefore be tested by checking first for dominance between LC in the concerned states, and then by checking for dominance between the symmetric average distribution over the two caste groups. The conclusion of QOEZ dominance is obtained if each of these two first-order stochastic dominance comparisons are statistically significant.[12]

[12]All dominance conclusions in this section are based on a statistical inference methodology proposed in Davidson and Duclos (2000) that uses the union-intersection criterion of Bishop, Formby, and Thistle (1992). For details of the statistical inference methodology see Appendix D of Bennia, Gravel, Magdalou, and Moyes (2022).

The ranking of the states that results from those tests is provided in the Hasse diagram depicted on Figure 3. On this figure as in the two others depicted below, we contrast the *incomplete* ranking of states provided by QOEZ dominance with the *complete* ranking of those states that would result from the use of some opportunity-inequality sensitive numerical index.[13] There has been a few such indices proposed in the literature. Some of them have been used by Asadullah and Yalonetzky (2012) for comparing Indian states on the basis of their educational opportunities using data similar - but older - than those used herein. The latter indices are however not compatible with QOEZ dominance for two reasons. First, these indices are all constructed in such a way that they are unaffected by changes in the average probability of achieving any outcome. Hence, they only focus on inequality of opportunities across groups, and do not attach any value to improvements in the average probability of achieving the outcomes. The second reason for which the indices used by Asadullah and Yalonetzky (2012) are not compatible with QOEZ dominance is that none of them can be expressed as a UEVEU criterion for some valuation function and some utility function defined over the outcome. Because of this, we find more enlightening to compare Indian states on the basis of indices discussed by Martinez, Schockkaert, and VandeGaer (2001) (notably through their Equation (8)) that explicitly consider a specific UEVEU criterion in which the valuation function Φ is a power function of the expected utility and where the utility function is a power function of the outcome. For illustrative purpose, we specifically select the two powers of the two functions to be both $1/2$. Hence our chosen index assigns to every allocation of opportunities $\mathbf{p}$ the number $I(\mathbf{p})$ defined by:

$$I(\mathbf{p}) = \sum_{i=1}^{n}(\sum_{h=1}^{k} p_{ih} h^{1/2})^{1/2} \tag{10}$$

[13]The index-based complete ranking of the states is depicted with statistical inference based on a bootstrap estimation of the confidence interval of the values of the index. The lack of a bold line connecting two states indicate an absence of statistical significance of the comparison between the two states while the presence of such a line indicates the statistical significance of the comparison.

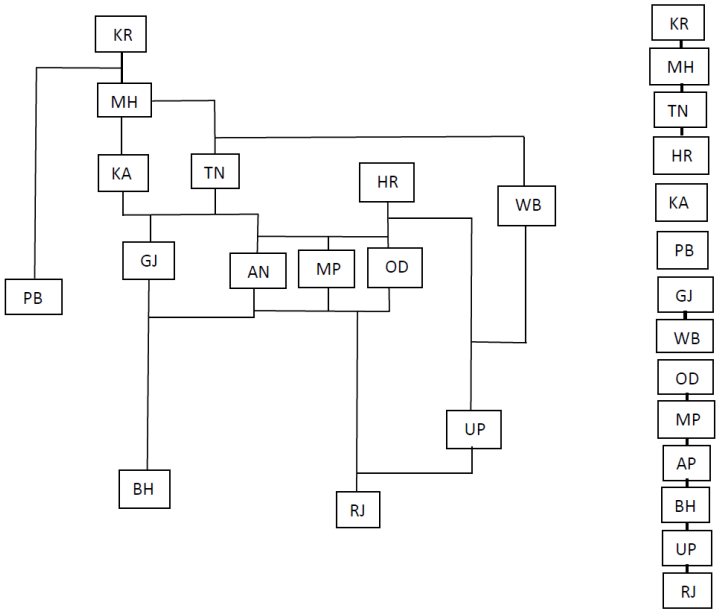

Figure 4: Gender-based QOEZ ranking of allocations of education opportunities in the Indian states.

Figure 3 reveals that QOEZ dominance succeeds in conclusively ranking 43 of the 91 possible pairs of distinct states. As can be seen, the conclusive ranking of the states so obtained is somewhat correlated with the ranking that would result from the mere comparison of the average distribution of educational achievements in those states. Hence the relatively well-educated state of Maharashtra dominates all other states except Kerala, while the relatively low-educated state of Bihar is dominated by all other states except Uttar Pradesh, Rajasthan and Odisha. However, the comparisons reveal some interesting and, possibly, surprising features. One of them is precisely the absence of dominance of Bihar by Odisha despite the fact that the latter state has a better average distribution of education opportunities than the former (Odisha is ranked above Bihar by the index of Expression (10)). It is the significant caste-based inequality of educational opportunity in Odisha that prevents it from dominating Bihar. A similar, somewhat counterintuitive - at least when one neglects equality of educational opportunity - absence of dominance is also observed between Kerala - a state often portrayed as exemplary in terms of educational achievement - and states such as Madhya Pradesh, Andhra Pradesh or Karnataka. Here again,

it is the significant caste-based educational inequality of opportunity in Kerala that prevents this state from dominating the others. Haryana and Punjab stand out as relatively isolated states insofar as their comparability with others is concerned. Punjab is only comparable - unfavorably - with Kerala and Maharashtra and cannot be compared with any other state. Haryana is only comparable - favorably - with Bihar and Uttar Pradesh and is not comparable with any other state. This relative isolation of Punjab and Haryana comes from the fact that they both stand in the upper range insofar as their average distribution of education is concerned, but yet perform somewhat poorly in allocating this average among the caste groups.

Gender is another important driver of educational inequality in India as well as in many other developing areas of the world. Reducing the gender gap is one of the last decade's eight United Nations Millennium Development goals. Several policies have been implemented in India to increase girls' attendance in both primary and secondary school to increase women's education levels (see, e.g., Dhaliwal, Duflo, Glennester, and Tulloch (2013) or Muralidharan and Prakash (2017)), with a varying degree of success. A quick look at the allocations of educational opportunities among gender groups in the 14 states in Figure 7 of Appendix A.4 shows indeed a clear inequality of educational opportunities across genders that take the form of a first-order stochastic dominance of the male distribution of educational outcomes over the female one in all the states. Because of this, one can again use Proposition 2 and apply the sequential test, by first looking for first-order dominance of the worst-off gender - which is female - of the two compared states, and then for the first-order dominance of the symmetric average - over the two genders - of the distributions of educational outcomes. The result of these comparisons using the statistical inference method mentioned above along with the complete ranking of the states as per the index described in Expression (10) is depicted in the Hasse diagram of Figure 4.

The ability of QOEZ dominance to conclusively rank the states is slightly greater (47 conclusive comparisons) for genders than for castes. Kerala, which dominates

all other states, is the absolute best performer in terms of allocation of educational opportunities between genders. Hence, this state does a better job at allocating its very good average distribution of educational outcomes among genders than among castes, even though this increased performance is not recorded by the index-based complete ranking. Another state that sees an improvement in its relative standing when gender is substituted to caste as a source of group differentiation is Haryana, which now dominates six states (instead of two in the preceding case). On the other hand, the position of Maharashtra, that is now dominated by Kerala and ceases to dominate Punjab, is negatively affected by the replacement of caste by gender as the source of group differentiation. Punjab remains isolated in terms of its comparability with other states, as it now compares - unfavorably - only with Kerala.

It is of interest to examine how gender inequality in educational opportunities evolves across castes or, symmetrically, how caste inequality evolves across genders. In fact, very few studies have analyzed educational inequality in the caste-gender nexus. Deshpande (2007) finds that gender inequality in education is more prominent for the deprived castes. Saha (2013) on the other hand observes that the within-caste distribution of educational expenditure is male-skewed for the non-SC/ST. For the case of Mumbai (Maharashtra), Munshi and Rosenzweig (2006) find, however, that low caste girls obtain better schooling than boys, since boys are often withdrawn from school by their parents and sent to work in traditional low-paid jobs exploiting the within-caste mutual networks. The salience of the gender gap in educational opportunities is mediated by the caste belonging, which can contribute to the widening of the differences between boys and girls.

Combining caste and gender leads to the formation of four groups: low caste females (LCF), low caste males (LCM), high caste females (HCF) and high caste males (HCM). This leads to a change in the ubiquitous ordering of the newly formed groups, as shown in Figure 8 of Appendix A.4. While LCF and HCF are always dominated by LCM and HCM respectively, the caste-gender disentanglement marks

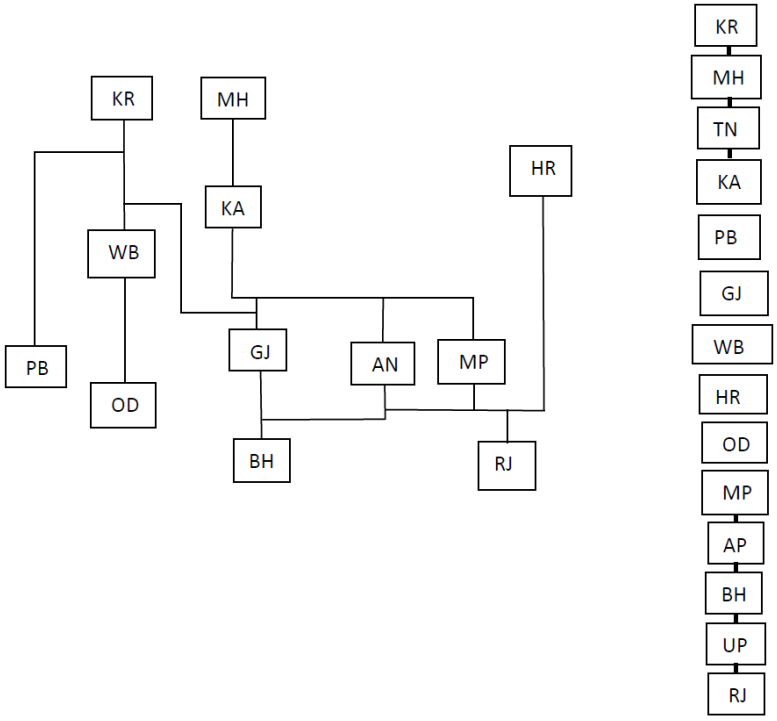

Figure 5: Caste-gender based QOEZ ranking of allocations of education opportunities in the Indian states.

the interesting positions of LCM *vis-à-vis* HCF in different states. In Maharashtra, Andhra and Rajasthan, HCF is dominated by LCM, indicating an acute gender discrimination in these states where educational opportunity is impaired even for HC if they are female. The opposite is true for West Bengal and Odisha, where the distribution of education is more unfavorable for LC irrespective of their gender. Interesting is the case of Kerala, the only state (in our analysis) that shows female dominance among the high caste, although the male-female gap in Kerala is much narrower even for the low castes, despite the prominent caste gap there. However, while all four groups can be ordered by first-order dominance in most states (with some rank reversals), there are two states, Tamil Nadu (TN) and Uttar Pradesh (UP), in which such a complete ordering of the four groups is *not* achieved. In those two states, HCM dominates both LCM and HCF and LCM and HCF both dominate LCF. However, the distributions of educational achievements in LCM and HCF cannot be compared by first-order dominance. Because of this, we cannot apply

to comparisons involving those two states the simple procedure of Proposition 2 and its associated statistical inference test. We have therefore excluded TN and UP from the Hasse diagram of Figure 5, which depicts the statistically significant dominance relationship among the remaining 12 states. We have however shown in this figure the complete ranking of the 14 states based on the index of Expression (10).

The Hasse diagram of Figure 5 happens to be a sort of intersection of the dominance charts of Figures 4 and 3. The ranking of the states obtained is therefore less complete than those of these figures since only 26 pairs of states (on a total of 60) can be conclusively ranked. It should be observed, after Deshpande (2000) some time ago using different tools, that the relatively good performance of Kerala in allocating educational opportunities between genders revealed by Figure 5 is not sufficient to offset the poor performance of this state in terms of caste-based inequality.

We complement the ranking of Figure 5 by looking for sufficient and necessary conditions for the dominance relationships involving UP and TN based on Propositions 3 and 4, which are amenable to Davidson and Duclos (2000) statistical inference tests. The results of those tests for the necessary and sufficient conditions are provided in Table 4 of Appendix A.4. As it turns out, the sufficient and necessary conditions lead to a decision as to whether or not a QOEZ dominance is observed in a majority of cases (15 out of 26 comparisons) involving UP and TN. For the other comparisons, marked with a "?" in Table 4, we resort to the more abstract general procedure of Lemma 4 at the cost of disregarding any concern for the statistical significance of the drawn conclusion.[14] We use LP solvers in Python to identify possible solutions to the relevant systems of inequalities (7) for these comparisons. When the solvers fail to identify solutions, we confirm the presumption of non-dominance by exhibiting solutions to the alternative to (7) implied by Farkas' Lemma. The procedure is detailed in Appendix A.5. It is found that HR, MP and WB QOEZ dominate

[14]We are not aware indeed of a statistical inference procedure that could be used to appraise the statistical significancy of solutions to either of the two alternatives in which the relevant version of Farkas' Lemma is formulated.

UP and KR QOEZ dominates TN, whereas all other pairs of states marked with a "?" in Table 4 are non-comparable.

5 Conclusion

This paper provides a robust and operational definition of what it means for one inter-group allocation of probabilities of achieving outcomes of interest - what we call "opportunities" - to be better than another for a large spectrum of ethical points of view. Our operational definition, which is shown to coincide with the unanimity of all rankings emanating from inequality-of-opportunity-averse UEVEU ethical observers, is the test of extended zonotope inclusion. According to this criterion, undisputable improvements in allocations of opportunities are associated with a shrinking - in the sense of set inclusion - of the associated extended zonotope set. We also show how this extended Zonotope inclusion test, which can be performed in polynomial time and amounts to finding solutions to a finite system of linear inequalities, can be made extremely simple in many cases of interest. Among the cases considered are those where there are only two groups between which opportunities are allocated, and those where the number of groups is arbitrary, but where the groups can all be ordered in terms of the expected utility associated with the probability distribution faced by their members. We have also provided easy-to-check sufficient conditions for extended zonotope inclusion that applies to any two allocations as well as necessary conditions that applies to any two allocations of opportunities when outcomes are completely ordered. We also identify elementary transformations of the allocation of opportunities that are considered worth doing by the extended zonotope inclusion criterion and, in the specific case of two groups facing the same given symmetric average distribution of outcomes, we identify them exactly. Last, but not least, the paper illustrates the usefulness of the criterion for comparing gender and caste-based allocations of educational opportunities in the 14 largest Indian states. The

analysis reveals, among other things, that the good average educational performance of Kerala hides major between-caste inequalities of opportunity that prevent this state from dominating many others. In the same vein, the analysis also points to significant gender inequalities of educational opportunity in Maharashtra that also prevent this wealthy and well-educated state from dominating others.

These findings open many future research perspectives. One of them is further empirical analysis. As illustrated in this paper, the extended zonotope inclusion criterion is a test that can be implemented rather straighforwardly. It would therefore be interesting to apply this test to other contexts requiring an inequality-sensitive assessment of allocations of opportunity. Another closely related avenue of research would the axiomatic identification of numerical indices that could supplement the incomplete comparisons of allocation of opportunities provided by the extended zonotope inclusion criterion, while remaining compatible with it. The empirical analysis has used one such index as a benchmark but it would be interesting to obtain axiomatic characterization of specific such indices. From a theoretical view point, it would also be important to identify precisely which elementary transformations of the allocations of opportunities lie behind extended zonotope inclusion. While these elementary transformations have been identified in the simple case of two groups who share the same symmetric average distributions over outcomes, they do not suffice to characterize extended zonotope inclusion when there are more than two groups and/or when average distribution over outcomes differs. Last, it might also be worth going beyond the *ex ante* standpoint, favored in this paper, of appraising opportunities *before* they are realized. When outcomes are totally ordered, the only clear indication of improvement to a group's opportunities is first-order stochastic dominance. Yet this criterion is not sensitive to mean preserving spreads or Pigou-Dalton transfers of outcomes. Defining these mean preserving spreads requires, of course, that outcomes be measured in a cardinally meaningful way. It would be worth exploring an extension of the criterion characterized in this paper that would coincide

with a more restricted unanimity of all opportunity-inequality-averse UEVEU ethical observers who assume that the utility numbers used to evaluate expected utility are both increasing and concave with respect to outcomes.

A Appendix of proofs

A.1 Mathematical Notations

The (possibly) non standard mathematical notations and definitions used in this paper are as follows. The convex hull of a collection of vectors $\{v^1, ..., v^n\}$ in $\mathbb{R}^k$ is defined by

$$Co\{v^1, ..., v^n\} = \left\{ x \in \mathbb{R}^k : \; x = \sum_{i=1}^{n} \lambda_i v^i \text{ for some } (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n_+, \; \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

A matrix $\mathbf{m} \in \mathbb{R}^{nk}_+$ is row-stochastic if it satisfies $\sum_{h=1}^{k} m_{ih} = 1$ for all i, and is bistochastic if it is row-stochastic and satisfies $\sum_{i=1}^{k} m_{ih} = 1$ for every h.

Given two vectors u and $v \in \mathbb{R}^k$, we say that u weakly Lorenz-dominates v if the inequality $\sum_{g=1}^{h} u_{(g)} \geq \sum_{g=1}^{h} v_{(g)}$ holds for all $h = 1, ..., k$, where $(v_{(g)})_g$ is the increasingly ordered permutation of v.

By a binary relation $\succsim$ on a set $\mathbf{\Omega}$, we mean a subset of $\mathbf{\Omega} \times \mathbf{\Omega}$. Following the convention in economics, we write $x \succsim y$ instead of $(x, y) \in \succsim$. Given a binary relation $\succsim$, we define its symmetric factor $\sim$ by $x \sim y \Longleftrightarrow x \succsim y$ and $y \succsim x$ and its asymmetric factor $\succ$ by $x \succ y \Longleftrightarrow x \succsim y$ and not $(y \succsim x)$. A binary relation $\succsim$ on $\mathbf{\Omega}$ is reflexive if the statement $x \succsim x$ holds for every x in $\mathbf{\Omega}$, is transitive if $x \succsim z$ always follows $x \succsim y$ and $y \succsim z$ for any $x, y, z \in \mathbf{\Omega}$, is complete if $x \succsim y$ or $y \succsim x$ holds for every distinct x and y in $\mathbf{\Omega}$ and is linear if $x \succsim y$ and $y \succsim x$ implies $x = y$ for any two x and y in $\mathbf{\Omega}$. A reflexive, transitive and complete binary relation is called an ordering and a reflexive and transitive binary relation is called a quasi-ordering.

A.2 Proofs

A.2.1 Remark 1

Let $\mathbf{C}$ be the vector sub-space of $\mathbb{R}^k$, generated by the vector $(1, ..., 1)$. Observe that $\mathbf{C}$ is a convex cone, and is contained in $\mathcal{U}^{\geq_{QO}}$. Hence, by standard results, $\mathcal{U}_*^{\geq_{QO}} \subseteq \mathbf{C}_* = \left\{(v_1, ..., v_k) \in \mathbb{R}^k : \sum_{j=1}^k v_k = 0\right\}$.

A.2.2 Lemma 1

Let $\theta_1, ..., \theta_n \in [0,1]^n$, and suppose without loss of generality (thanks to anonymity) that $\theta_1 \leq \theta_2 \leq ... \leq \theta_n$. Then

$$\sum_{i=1}^n \theta_i p_i = \theta_1 \sum_{i=1}^n p_i + \sum_{l=2}^n \left((\theta_l - \theta_{l-1}) \sum_{i=l}^n p_i \right) + (1 - \theta_l)\mathbf{0}.$$

The right-hand side being a convex combination of the set $\left\{\mathbf{0}, \left(\sum_{i=l}^n p_i\right)_{l=1,...,n}\right\}$, we get the result.

A.2.3 Theorem 1

We first show that Statement 2 of Theorem 1 implies Statement 1 of that theorem and, therefore, that $\mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq_{QO}} \subseteq \mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq_{QO}}$. Since $\mathcal{U}_*^{\geq_{QO}}$ is a cone, it amounts to showing that $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq_{QO}}$. By Lemma 1, it is sufficient to show the following:

$$(11) \quad \forall \alpha_1, ..., \alpha_n \in \{0,1\}^n,\ \exists\ v \in \mathcal{U}_*^{\geq_{QO}},\ \theta_1, ..., \theta_n \in [0,1]^n \quad \text{s.t.} \quad \sum_{i=1}^n \alpha_i q_i = \sum_{i=1}^n \theta_i p_i + v.$$

Since $\sum_{j=1}^k v_j = 0$ (by Remark 1) and p_i and q_i both belong to $\mathbf{S}$, we necessarily have $\sum_{i=1}^n \theta_i = m$, where $m = Card\{i : \alpha_i = 1\}$. Hence, by re-indexing the distributions q_i (for $i = 1, ..., n$) in such a way that $\alpha_i = 1$ for $i = 1, ..., m$, expression (11) can be equivalently written as:

$$\exists\ v \in \mathcal{U}_*^{\geq_{QO}},\ \theta_1, ..., \theta_n \in [0,1]^n \text{s.t.} \quad \frac{1}{m} \sum_{i=1}^m q_i - \sum_{i=1}^n \frac{\theta_i}{m} p_i = \frac{1}{m} v.$$

Let the set $\mathbf{D}$ be defined by $\mathbf{D} := \frac{1}{m}\sum_{i=1}^{m} q_i - Co\{p_1, ..., p_n\}$. We thus need to show that $\mathbf{D} \cap \mathcal{U}_*^{\geq QO} \neq \varnothing$. Suppose by contradiction that $\mathbf{D} \cap \mathcal{U}_*^{\geq QO} = \varnothing$. Since $\mathbf{D}$ is a convex polytope[15] and $\mathcal{U}_*^{\geq QO}$ is a closed convex cone, one can conclude from Theorem 2 at p. 80 of Berge (1959) that there are vectors $(d_1^*, ...d_k^*) \in \mathbf{D}$ and $(v_1^*, ...v_k^*) \in \mathcal{U}_*^{\geq QO}$ such that:

$$\left(\sum_{h=1}^{k}(d_h^* - v_h^*)^2\right)^{1/2} = \min_{(d_1,...,d_k)\in\mathbf{D},(v_1,...,v_k)\in\mathcal{U}_*^{\geq QO}} \left(\sum_{h=1}^{k}(d_h - v_h)^2\right)^{1/2}$$

by continuity of the Euclidian norm, and using the fact that the set $\mathbf{D} \times \mathcal{U}_*^{\geq QO}$ on which it is minimized can be made compact by taking a suitable intersection of $\mathcal{U}_*^{\geq QO}$ with some closed ball in $\mathbb{R}^k$. Define the vector $(\widehat{v}_1, ..., \widehat{v}_k)$ by $\widehat{v}_h = v_h^* - d_h^*$ for $h = 1, ..., k$. Then the hyperplane passing through $(v_1^*, ...v_k^*)$ and orthogonal to $(\widehat{v}_1, ..., \widehat{v}_k)$ strongly separates $\mathbf{D}$ and $\mathcal{U}_*^{\geq QO}$ in the sense that:

$$\inf_{(v_1,...,v_k)\in\mathcal{U}_*^{\geq QO}} \sum_{h=1}^{k} v_h\widehat{v}_h \geq \sum_{h=1}^{k} v_h^*\widehat{v}_h > \sup_{(d_1,...,d_k)\in\mathbf{D}} \sum_{h=1}^{k} d_h\widehat{v}_h \tag{12}$$

Since $(0, ..., 0) \in \mathcal{U}_*^{\geq QO}$ one must have that $0 \geq \sum_{h=1}^{k} v_h^*\widehat{v}_h$. Moreover since $(\lambda v_1^*, ..., \lambda v_k^*) \in \mathcal{U}_*^{\geq QO}$ $\lambda > 0$, one must also have that $\sum_{h=1}^{k} v_h^*\widehat{v}_h \geq 0$. Indeed, assuming $\sum_{h=1}^{k} v_h^*\widehat{v}_h < 0$ would be contradictory, after taking a suitably large λ, with the strict inequality (12). These two last inequalities enable therefore one to rewrite inequality (12) more precisely as:

$$\inf_{(v_1,...,v_k)\in\mathcal{U}_*^{\geq QO}} \sum_{h=1}^{k} v_h\widehat{v}_h \geq 0 > \sup_{(d_1,...,d_k)\in\mathbf{D}} \sum_{h=1}^{k} d_h\widehat{v}_h \tag{13}$$

By the first of these two inequalities, we conclude that $(\widehat{v}_1, ..., \widehat{v}_k)$ belongs to the dual cone of the set $\mathcal{U}_*^{\geq QO}$, which is itself the dual cone of the set $\mathcal{U}^{\geq QO}$. By the bipolar theorem for convex cones (see for example Theorem 14.1 in Rockafellar (1970)), it therefore follows that the dual cone of $\mathcal{U}_*^{\geq QO}$ is $\mathcal{U}^{\geq QO}$ so that $(\widehat{v}_1, ..., \widehat{v}_k) \in \mathcal{U}^{\geq QO}$. Now

[15] see Rockafellar (1970), p. 12. A convex polytope is the convex hull of a finite family of points, called the *vertices* or *extreme points* of this set.

since Statement 2 of the theorem holds, we know that the inequality

$$\sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k}q_{ih}u_h\right)\geq\sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k}p_{ih}u_h\right)$$

holds for all concave Φ and all lists of real numbers $(u_1,...,u_k)\in\mathcal{U}^{\geq_{QO}}$. By the Hardy-Littlewood-Polya theorem (see for example Berge (1959), p. 191), this is equivalent to the requirement that the list of n numbers $\left(\sum_{h=1}^{k}q_{1h}u_h,...,\sum_{h=1}^{k}q_{nh}u_h\right)$ Lorenz-dominates the list of n numbers $\left(\sum_{h=1}^{k}p_{1h}u_h,...,\sum_{h=1}^{k}p_{nh}u_h\right)$ for all $(u_1,...,u_k)\in\mathcal{U}^{\geq_{QO}}$. In particular this is true for $(\hat{u}_1,...,\hat{u}_k)$, and thus there exists an indexing $i_1(\widehat{u}),...,i_n(\widehat{u})$ such that [16]

$$\sum_{h=1}^{k}p_{i_1(\widehat{u})h}\widehat{u}_h\leq\sum_{h=1}^{k}p_{i_2(\widehat{u})h}\widehat{u}_h\leq...\leq\sum_{h=1}^{k}p_{i_n(\widehat{u})h}\widehat{u}_h$$

and

$$\sum_{i=1}^{m}\sum_{h=1}^{k}q_{ih}\widehat{u}_h\geq\sum_{j=1}^{m}\sum_{h=1}^{k}p_{i_j(\widehat{u})h}\widehat{u}_h. \tag{14}$$

However, by the second inequality of Expression (13),we have (remembering the definition of **D**):

$$0>\sum_{h=1}^{k}\overline{q}_h\widehat{u}_h-\sum_{h=1}^{k}p_{ih}\widehat{u}_h \tag{15}$$

for all $i=1,...,n$. It follows therefore from Inequalities (14) and (15) that:

$$\sum_{h=1}^{k}p_{ih}\widehat{u}_h>\sum_{h=1}^{k}\overline{q}_h\widehat{u}_h\geq\frac{1}{m}\sum_{j=1}^{m}\sum_{h=1}^{k}p_{i_j(\widehat{u})h}\widehat{u}_h,\text{ for }i=1,...,n,$$

which is not possible. This concludes the proof of the first implication.

Let us now prove the reverse implication. Suppose that Statement 1 of the Theorem holds and pick any $(u_1,...,u_k)\in\mathcal{U}^{\geq_{QO}}$. We must show, using again the Hardy-

[16]Which depends of course upon the k-tuple $(\widehat{u}_1,...,\widehat{u}_k)$.

Littlewood-Polya theorem, that the list of n numbers $\left(\sum_{h=1}^{k} q_{1h}u_h, ..., \sum_{h=1}^{k} q_{nh}u_h\right)$ Lorenz-dominates $\left(\sum_{h=1}^{k} p_{1h}u_h, ..., \sum_{h=1}^{k} p_{nh}u_h\right)$. Without loss of generality (since the ranking of allocations is anonymous), we can write the indices of the rows of the two matrices $\mathbf{q}$ and $\mathbf{p}$ in such a way that the two lists are increasingly ordered so that:

$$\sum_{h=1}^{k} q_{1h}u_h \leq ... \leq \sum_{h=1}^{k} q_{nh}u_h \text{ and } \sum_{h=1}^{k} p_{1h}u_h \leq \sum_{h=1}^{k} p_{nh}u_h.$$

Hence, we need to show that for any $n_0 \leq n-1$,

$$\sum_{i=1}^{n_0}\sum_{h=1}^{k} p_{1h}u_h \leq \sum_{i=1}^{n_0}\sum_{h=1}^{k} q_{1h}u_h$$

Since statement 1 of the theorem holds, we know that there exists $v \in \mathcal{U}_*^{\geq QO}$ and $\theta_1, ..., \theta_n \in [0,1]$ such that $\sum_{i=1}^{n}\theta_i = n_0 \leq n$ and $\sum_{i=1}^{n_0} q_i = \sum_{l=1}^{n}\theta_l p_l + v$. It thus follows that:

$$\begin{aligned}
\sum_{i=1}^{n_0}\sum_{h=1}^{k} q_{1h}u_h &= \sum_{j=1}^{n}\theta_h\sum_{h=1}^{k} p_{jh}u_h + \sum_{h=1}^{k} v_h u_h \geq \sum_{j=1}^{n}\theta_h\sum_{h=1}^{k} p_{jh}u_h \text{ (since } v \in \mathcal{U}_*^{\geq QO}) \\
&\geq \sum_{j=1}^{n_0}\theta_j\sum_{h=1}^{k} p_{jh}u_h + \sum_{g=n_0+1}^{n}\theta_g\sum_{h=1}^{k} p_{n_0 h}u_h \text{ (rows are ordered)} \\
&= \sum_{j=1}^{n_0}\theta_j\sum_{h=1}^{k} p_{jh}u_h + \sum_{j=1}^{n_0}[1-\theta_j]\sum_{h=1}^{k} p_{n_0 h}u_h \text{ (since } \sum_{j=1}^{n}\theta_i = n_0) \\
&\geq \sum_{j=1}^{n_0}\theta_j\sum_{h=1}^{k} p_{jh}u_h
\end{aligned}$$

as required.

A.2.4 Remark 2

Suppose that $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ and, as a result, that $\mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq QO} \subseteq \mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq QO}$. Since in particular $\sum_{i=1}^{n} q_i \in \mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq QO}$, there is a collection of n numbers $\theta_1, ..., \theta_n$ in the

$[0,1]$ interval and a vector $v \in \mathcal{U}_*^{\geq QO}$such that:

$$\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} \theta_i p_i + v.$$

or, writing this equality for outcome j :

$$\sum_{i=1}^{n} q_{ij} = \sum_{i=1}^{n} \theta_i p_{ij} + v_j.$$

Summing over all outcomes, and exploiting the fact that $\sum_{j=1}^{k} v_j = 0$ and $\sum_{j=1}^{k} p_{ij} = \sum_{j=1}^{k} q_{ij} = 1$, one has:

$$\sum_{j=1}^{n}\sum_{i=1}^{n} q_{ij} = n = \sum_{i=1}^{n} \theta_i \sum_{j=1}^{n} p_{ij} + \sum_{j=1}^{n} v_j = \sum_{i=1}^{n} \theta_i,$$

which implies that $\theta_i = 1$ for all i. Hence:

$$\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} p_i + v$$

and

$$\sum_{i=1}^{n} q_i - \sum_{i=1}^{n} p_i = v \in \mathcal{U}_*^{\geq QO},$$

as required.

A.2.5 Remark 3

Observing that the inequality:

$$\sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} q_{ih} u_h\right) \geq \sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} p_{ih} u_h\right)$$

holds for all concave Φ and all lists of real numbers $u_1, ..., u_k$ is equivalent, thanks to the Hardy-Littlewood-Polya theorem, to the requirement that the matrix $\mathbf{q}$ price majorizes (using Kolm (1977) terminology) the matrix $\mathbf{p}$ for all price vectors $(u_1, ..., u_k)$. Koshevoy (1995) (Theorem 1) proves that the fact for a matrix $\mathbf{q} \in \mathbb{R}^{nd}$ to price ma-

jorize a matrix $\mathbf{p} \in \mathbb{R}^{nd}$ is equivalent to observing:

$$\begin{aligned}\overline{\mathbf{Z}}(\mathbf{q}) &= \left\{ \mathbf{z} \in \mathbb{R}^{k+1} : \ \mathbf{z} = \sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, q_{i1}, ..., q_{ik} \right), \ \theta_i \in [0,1] \ \forall i = 1, \ldots, n \right\} \\ &\subseteq \left\{ \mathbf{z} \in \mathbb{R}^{k+1} : \ \mathbf{z} = \sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, p_{i1}, ..., p_{ik} \right), \ \theta_i \in [0,1] \ \forall i = 1, \ldots, n \right\} \\ &= \overline{\mathbf{Z}}(\mathbf{p})\end{aligned}$$

Observe that the set $\overline{\mathbf{Z}}(\mathbf{a})$ (for any matrix $\mathbf{a} \in \mathbb{R}^{nd}$) defined in Koshevoy (1995) is somewhat similar to the set defined in Equation 5 above, with the exception that it takes the Minkowski sums over the population share extended vectors $(\frac{1}{n}, p_{i1}, ..., p_{ik})$ rather than over the vectors $(p_{i1}, ..., p_{ik})$ themselves. Hence we only need to prove that $\overline{\mathbf{Z}}(\mathbf{q}) \subseteq \overline{\mathbf{Z}}(\mathbf{p})$ is equivalent to $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p}) \Longleftrightarrow (\mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq^{\varnothing}}) \subseteq (\mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq^{\varnothing}})$ to complete the argument. The fact that $\overline{\mathbf{Z}}(\mathbf{q}) \subseteq \overline{\mathbf{Z}}(\mathbf{p})$ implies $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p})$ is obvious. To establish the other direction, assume that $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p})$. This means that for any list of numbers $\theta_1, ..., \theta_n$ in the $[0,1]$ interval, one can find a list of numbers $\theta'_1, ..., \theta'_n$ in the $[0,1]$ interval such that:

$$\sum_{i=1}^{n} \theta_i q_i = \sum_{i=1}^{n} \theta'_i p_i$$

Observe that this equality implies that for any $j = 1, ..., k$ one has:

$$\sum_{i=1}^{n} \theta_i q_{ij} = \sum_{i=1}^{n} \theta'_i p_{ij}$$

Summing these equalities over all j yields (exploiting the fact that the probability distributions lie in $\mathbf{S}$):

$$\sum_{i=1}^{n} \theta_i \sum_{j=1}^{k} q_{ij} = \sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} \theta'_i \sum_{j=1}^{k} q_{ij} = \sum_{i=1}^{n} \theta'_i$$

But this implies that for any for any list of numbers $\theta_1, ..., \theta_n$ in the $[0, 1]$ interval, one can find a list of numbers $\theta'_1, ..., \theta'_n$ in that same interval such that;

$$\sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, q_{i1}, ..., q_{ik} \right) = \sum_{i=1}^{n} \theta'_i \left(\frac{1}{n}, p_{i1}, ..., p_{ik} \right)$$

That is, this implies that $\overline{\mathbf{Z}}(\mathbf{q}) \subseteq \overline{\mathbf{Z}}(\mathbf{p})$ holds, as required. The fact that $\overline{p} = \overline{q}$ is an immediate consequence of Remark 2 and the fact that $\mathcal{U}_*^{\geq^{\varnothing}} = \{\mathbf{0}^k\}$.

A.2.6 Lemma 2

For uniform averaging, we simply observe that the function $\Psi : \mathbf{S} \to \mathbb{R}$ defined, for every $(s_1, ..., s_k) \in \mathbf{S}$ by:

$$\Psi(s_1, ..., s_k) = \Phi \left(\sum_{h=1}^{k} s_h u_h \right)$$

is concave if Φ is concave irrespective of what the real numbers $(u_1, ..., u_k)$ are. Hence, by virtue of Theorem 3 in Kolm (1977),

$$\sum_{i=1}^{n} \Phi \left(\sum_{h=1}^{k} q_{ih} u_h \right) \geq \sum_{i=1}^{n} \Phi \left(\sum_{h=1}^{k} p_{ih} u_h \right)$$

if there exists a bistochastic matrix $n \times n$ bistochastic matrix $\mathbf{b}$ such that $\mathbf{q} = \mathbf{b}.\mathbf{p}$.

Assume now that $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$ and that $\mathbf{q}$ results from from $\mathbf{p}$ through an equalizing transfer as per Definition 6. We must show that:

$$\sum_{i=1}^{n} \Phi \left(\sum_{h=1}^{k} q_{ih} u_h \right) \geq \sum_{i=1}^{n} \Phi \left(\sum_{h=1}^{k} p_{ih} u_h \right).$$

Since all rows others than i_1 and i_2 in the matrix $\mathbf{p}$ and others than i'_1 and i'_2 in the

matrix $\mathbf{q}$ are unaffected by the change, we have:

$$\sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k}q_{ih}u_h\right)\geq\sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k}p_{ih}u_h\right)$$

$$\Longleftrightarrow$$

$$\Phi\left(\sum_{h=1}^{k}q_{i'_1h}u_h\right)+\Phi\left(\sum_{h=1}^{k}q_{i'_2h}u_h\right)\geq\Phi\left(\sum_{h=1}^{k}p_{i_1h}u_h\right)+\Phi\left(\sum_{h=1}^{k}p_{i_2h}u_h\right) \tag{16}$$

We now observe that

$$\left(\sum_{h=1}^{k}q_{i'_1h}u_h,\sum_{h=1}^{k}q_{i'_2h}u_h\right)\text{ Lorenz-dominates }\left(\sum_{h=1}^{k}p_{i_1h}u_h,\sum_{h=1}^{k}p_{i_2h}u_h\right).$$

Indeed, one has:

$$\sum_{h=1}^{k}p_{i_1h}u_h\leq\sum_{h=1}^{k}p_{i_2h}u_h-\sum_{h=1}^{k}v_hu_h=\sum_{h=1}^{k}q_{i'_2h}u_h\leq\sum_{h=1}^{k}p_{i_2h}u_h$$

and:

$$\sum_{h=1}^{k}p_{i_1h}u_h\leq\sum_{h=1}^{k}p_{i_1h}u_h+\sum_{h=1}^{k}v_hu_h=\sum_{h=1}^{k}q_{i'_1h}u_h\leq\sum_{h=1}^{k}p_{i_2h}u_h$$

Inequality (16) then follows from the Hardy-Littlewood-Polya Theorem.

A.2.7 Lemma 3

The fact that

$$\mathcal{U}_*^{\geq_{QO}}\subseteq\left\{(v_1,...,v_k)\in\mathbb{R}^k:\ \sum_{j=1}^{k}v_ju_j\geq 0\ \ \forall(u_1,...,u_k)\in\mathcal{U}^{\geq_{QO}}\cap\{0,1\}^k\right\}$$

directly follows from the fact that $\mathcal{U}^{\geq_{QO}}\cap\{0,1\}^k\subset\mathcal{U}^{\geq_{QO}}$.

To prove the reverse inclusion, consider any $(v_1,...,v_k)$ satisfying $\sum_{j=1}^{k}v_ju_j\geq 0$ for all $(u_1,...,u_k)\in\mathcal{U}^{\geq_{QO}}\cap\{0,1\}^k$. We must show that it satisfies also $\sum_{j=1}^{k}v_ju_j\geq 0$ for any $(u_1,...,u_k)\in\mathcal{U}^{\geq_{QO}}$. Consider therefore any such $(u_1,...,u_k)\in\mathcal{U}^{\geq_{QO}}$. By continuity of the map $(u_1,...,u_k)\mapsto\sum_{j=1}^{k}v_ju_j$, we may assume without loss of generality that $u_h\neq u_i$ for any two distinct h and i in $\{1,...,k\}$. Let $j:\{1,...,k\}\longrightarrow$

$\{1, ..., k\}$ be a one-to-one function such that such that $u_{j(1)} < u_{j(2)} < ... < u_{j(k)}$. We have:

$$\sum_{j=1}^{k} v_j u_j = \sum_{h=1}^{k} v_{j(h)} u_{j(h)} = \sum_{h=2}^{k} v_{j(h)} (u_{j(h)} - u_{j(1)}) \tag{17}$$

since $\sum_{h=1}^{k} v_h = 0$. Using Abel decomposition formula, one can alternatively write this equality as:

$$\sum_{j=1}^{k} v_j u_j = \sum_{h=2}^{k} (u_{j(h)} - u_{j(h-1)}) \sum_{g=h}^{k} v_{j(g)}$$

Now, for any $h = 2, ..., k$, let $w^h \in \{0, 1\}^k$ be defined by:

$$w^h_{j(g)} = \begin{cases} 0 & \text{if} \quad g < h, \\ 1 & \text{if} \quad g \geq h. \end{cases}$$

We observe that, for any $h \in \{2, ..., k\}$, $(w^h_{j(1)}, ..., w^h_{j(k)}) \in \mathcal{U}^{\geq_{QO}}$. Indeed, if $l >^{QO} g$ for two distinct outcomes g and l in $\{1, ..., k\}$, then $u_l > u_g$ by definition of $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$. Given h, three cases are possible:

(i) $l < h$. In this case, one has $w^h_{j(g)} = 0 = w^h_{j(l)}$ from the definition of w^h;

(ii) $g < h \leq l$. In this case, $w^h_{j(g)} = 0 < 1 = w^h_{j(l)}$ holds from the definition of w^h and the required weak inequality $w^h_{j(g)} \leq w^h_{j(l)}$ is also satisfied;

(iii) $h \leq g < l$. In this case $w^h_{j(g)} = 1 = w^h_{j(l)}$ holds from the definition of w^h.

Hence, in all the three cases, the required weak inequality $w^h_{j(g)} \leq w^h_{j(l)}$ is satisfied. Since $(w^h_{j(1)}, ..., w^h_{j(k)}) \in \mathcal{U}^{\geq_{QO}} \cap \{0, 1\}^k$ for any $h = 2, ..., k$, we have $\sum_{g=1}^{k} v_{j(g)} w^h_{j(g)} = \sum_{g=h}^{k} v_{j(g)} \geq 0$ for any such h. But this implies that

$$\sum_{h=2}^{k} v_{j(h)} (u_{j(h)} - u_{j(1)}) \geq 0$$

for any such h which, thanks to Equality (17), establishes the result.

A.2.8 Lemma 4

As noted in the proof of Theorem 1 (before Expression (11)), $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ if and only if, for all $\alpha_1, ..., \alpha_n \in \{0,1\}^n$, there exist $v \in \mathcal{U}_*^{\geq QO}$ and $\theta_1, ..., \theta_n \in [0,1]^n$ such that

$$\sum_{i=1}^{n} \alpha_i q_i = \sum_{i=1}^{n} \theta_i p_i + v.$$

This is in turn equivalent to having that, for all $l = 1, \ldots, n$ and all $m = 1, \ldots, m(l)$, there exist $v \in \mathcal{U}_*^{\geq QO}$ and $\theta_1, ..., \theta_n \in [0,1]^n$ such that $\sum_{i=1}^{n} \theta_i = l$ and

$$\sum_{i \in J_l^m} q_i = \sum_{i=1}^{n} \theta_i p_i + v. \tag{18}$$

Proving the lemma therefore amounts to showing that, for any $l = 1, ..., n$, we have

$$\left\{ \sum_{i=1}^{n} \theta_i p_i : \theta_1, ..., \theta_n \in [0,1]^n, \ \sum_{i=1}^{n} \theta_i = l \right\} = Co \left\{ \sum_{i \in J_l^1} p_i, \ldots, \sum_{i \in J_l^{m(l)}} p_i \right\} \tag{19}$$

The set on the left-hand side of the equality is the intersection of a convex polytope, $\mathbf{Z}(\mathbf{p})$, and an affine subspace of $\mathbb{R}^k$. Hence it is also a convex polytope. Since it contains the points $\sum_{i \in J_l^1} p_i, \ldots, \sum_{i \in J_l^{m(l)}} p_i$, we have:

$$Co \left\{ \sum_{i \in J_l^1} p_i, \ldots, \sum_{i \in J_l^{m(l)}} p_i \right\} \subseteq \left\{ \sum_{i=1}^{n} \theta_i p_i : \theta_1, ..., \theta_n \in [0,1]^n, \ \sum_{i=1}^{n} \theta_i = l \right\}.$$

Let $x^1, ..., x^P$ be the extreme points of $\{\sum_{i=1}^{n} \theta_i p_i : \theta_1, ..., \theta_n \in [0,1]^n, \ \sum_{i=1}^{n} \theta_i = l\}$. Let us now show that

$$x^p \in \left\{ \sum_{i \in J_l^1} p_i, \ldots, \sum_{i \in J_l^{m(l)}} p_i \right\}, \ \forall p = 1, ..., P.$$

Suppose by contradiction that this is not the case. Then $x^p = \sum_{i=1}^{n} \theta_i p_i$, where we can assume without loss of generality that $\theta_1 \in]0,1[$. Since $\sum_{i=1}^{n} \theta_i = l$, there must

exist another real number θ_2 such that $\theta_2 \in]0,1[$. We then have

$$x^p = \frac{1}{2}(x_+ + x_-),$$

where $x_+ := (\theta_1+\varepsilon)p_1+(\theta_2-\epsilon)p_2+...+\theta_n p_n$ and $x_- := (\theta_1-\epsilon)p_1+(\theta_2+\epsilon)p_2+...+\theta_n p_n$ both belong to $\{\sum_{i=1}^n \theta_i p_i : \theta_1,...,\theta_n \in [0,1]^n,\ \sum_{i=1}^n \theta_i = l\}$, provided that ε is small enough. Hence x^p is not an extreme point of $\{\sum_{i=1}^n \theta_i p_i : \theta_1,...,\theta_n \in [0,1]^n,\ \sum_{i=1}^n \theta_i = l\}$, a contradiction. Hence (19) holds, and this concludes the proof.

A.2.9 Proposition 2

If $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ then there exists some $\tilde{p} \in Co\left\{\sum_{i\in J_l^1} p_i, \ldots, \sum_{i\in J_l^{m(l)}} p_i\right\}$ such that

$$\frac{1}{l}\sum_{i=1}^{l} q_i - \frac{1}{l}\tilde{p} \in \mathcal{U}_*^{\geq_{QO}},$$

by Lemma 4. Since $\frac{1}{l}\tilde{p} - \frac{1}{l}\sum_{i=1}^{l} p_i \in \mathcal{U}_*^{\geq_{QO}}$ we get Expression (8) of Proposition 2.

Suppose now that Expression (8) holds. Since $\frac{1}{l}\sum_{i\in J_l^m} q_i - \frac{1}{l}\sum_{i=1}^{l} q_i \in \mathcal{U}_*^{\geq_{QO}}$ for every $m = 1,...,m(l)$ we have:

$$\frac{1}{l}\sum_{i\in J_l^m} q_i - \frac{1}{l}\sum_{i=1}^{l} p_i = \frac{1}{l}\sum_{i\in J_l^m} q_i - \frac{1}{l}\sum_{i=1}^{l} q_i + \frac{1}{l}\sum_{i=1}^{l} q_i - \frac{1}{l}\sum_{i=1}^{l} p_i \in \mathcal{U}_*^{\geq_{QO}}$$

for all such m and this concludes the proof.

A.2.10 Theorem 2

Using the reasoning in the proof of Remark 2, one can observe that if $n = 2$, the statement $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ is equivalent to the requirement that $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq_{QO}}$ and that there exist θ_1 and $\theta_2 \in [0,1]$ such that $q_1 - (\theta_1 p_1 + (1-\theta_1)p_2) \in \mathcal{U}_*^{\geq_{QO}}$ and $q_2 - (\theta_2 p_1 + (1-\theta_2)p_2) \in \mathcal{U}_*^{\geq_{QO}}$. Since these θ_1 and θ_2 may belong respectively to $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$, this establishes one direction of the implication.

For the other direction, it is sufficient to prove that the statement $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ implies the existence of $\lambda_1 \in \mathbf{\Lambda}_1$ such that $q_1 - (\lambda_1 p_1 + (1-\lambda_1)p_2) \in \mathcal{U}_*^{\geq_{QO}}$ (the argument being similar for λ_2). If $q_1 - p_1 \in \mathcal{U}_*^{\geq_{QO}}$, then one selects $\lambda_1 = 1 \in \mathbf{\Lambda}_1$ and the proof

is over. If $q_1 - p_1 \notin \mathcal{U}_*^{\geq_{QO}}$, then we know that since $q_1 - (\theta_1 p_1 + (1-\theta_1)p_2) \in \mathcal{U}_*^{\geq_{QO}}$ for some $\theta_1 \in [0,1]$, there exists some $v_1 \in \mathcal{U}_*^{\geq_{QO}}$ such that $q_1 = \theta_1 p_1 + (1-\theta_1)p_2 + v_1$. Let $\mathbf{D}(q_1)$ denote the (compact) set of distributions that are weakly dominated by q_1, with respect to the quasi-ordering, defined by:

$$\mathbf{D}(q_1) = \{x \in \mathbf{S} : q_1 - x \in \mathcal{U}_*^{\geq_{QO}}\}$$

Consider the continuous map $x : [0,1] \to [0,1]$ defined by $x(t) = tp_1 + (1-t)p_2$. Since $q_1 - p_1 \notin \mathcal{U}_*^{\geq_{QO}}$ one has that $x(1) \notin \mathbf{D}(q_1)$ while $x(\theta_1) \in \mathbf{D}(q_1)$. Let $\overline{\theta}_1$be defined by:

$$\overline{\theta}_1 = \max\{t \geq \theta_1 : x(t) \in \mathbf{D}(q_1)\} \tag{20}$$

We then have $\overline{\theta}_1 \in [\theta_1, 1[$ and $x(\overline{\theta}_1) \in \mathbf{D}(q_1)$. We therefore have:

$$q_1 = \overline{\theta}_1 p_1 + (1 - \overline{\theta}_1)p_2 + \overline{v}_1$$

for some $\overline{v}_1 \in \mathcal{U}_*^{\geq_{QO}}$. Also observe that $\overline{v}_1$ must be such that $\sum_{j\in J} \overline{v}_{1j} = 0$ for some $J \in \mathcal{F}^{\geq_{QO}}$. Indeed, using Expression (6), assuming that $\sum_{j\in J} \overline{v}_{1j} > 0$ for all $J \in \mathcal{F}^{\geq_{QO}}$ would imply the possibility of increasing a bit the t above $\overline{\theta}_1$ while maintaining $x(t)$ in the set $\mathbf{D}(q_1)$ in the maximization described by Expression (6), and will therefore be contradictory. Hence for the set J where $\sum_{j\in J} \overline{v}_{1j} = 0$, one has $q_1(J) = \overline{\theta}_1 p_1(J) + (1 - \overline{\theta}_1 p_2(J)$ and this completes the proof.

A.2.11 Remark 4

That no equalizing transfers can be performed to go from $\mathbf{p}$ to $\mathbf{q}$ is clear since neither of the two distributions p_1 and p_2 first-order stochastically dominates the other. One can also see that no uniform averaging operation, however small, can be performed. Indeed, for any $\lambda \in [0,1[$, $q_1 - (\lambda p_1 + (1-\lambda)p_2) \notin \mathcal{U}_*^{\geq_C}$. This is so because the probability of achieving the worst outcome for the first group in allocation $\mathbf{q}$ is strictly larger than any mixture of the probabilities of achieving that worst outcome for the two groups in allocation $\mathbf{p}$ ($q_{11} = 16/36 > \lambda 16/36 + (1-\lambda)13/36$ for all $0 \leq \lambda < 1$). We finally show that there is no margin to perform an anonymous

and unanimous utility improvement, however small, on the initial allocation $\mathbf{p}$ in a way that preserves dominance of $\mathbf{q}$ over the transformed $\mathbf{p}$. Let the transformed allocation $\mathbf{p}'$ be defined by $p'_1 = p_1 + w_1$ and $p'_2 = p_2 + w_2$ for some $w_1, w_2 \in \mathcal{U}_*^{\geq C}$. We claim that if $\mathbf{q} \succsim_{EZ}^{C} \mathbf{p}'$ then $w_1 + w_2 = 0$. Suppose indeed that:

$$q_1 - (\theta_1 p'_1 + (1-\theta_1)p'_2) \in \mathcal{U}_*^{\geq C}, \;\; q_2 - (\theta_2 p'_1 + (1-\theta_2)p'_2) \in \mathcal{U}_*^{\geq C}$$

and:

$$q_2 + q_1 - (p'_1 + p'_2) \in \mathcal{U}_*^{\geq C}.$$

Then it follows that $\theta_1 = 1$, as we have seen in the argument that we just made about the impossibility of performing a uniform averaging. This implies that $q_1 - p_1 - w_1 \in \mathcal{U}_*^{\geq C}$, that is $\frac{1}{36}(0,-2,2,0) - w_1 \in \mathcal{U}_*^{\geq C}$. Secondly

$$q_2 - (\theta_2 p_1 + (1-\theta_2)p_2) = \frac{1}{36}\left(-3\theta_2, 2-\theta_2, -3+6\theta_2, 1-2\theta_2\right).$$

This vector belongs to $\mathcal{U}_*^{\geq C}$ if and only if $\theta_2 = 1/2$ and it is then equal to $\frac{1}{72}\left(-3,3,0,0\right)$. To sum up we have:

$$\frac{1}{36}(0,-2,2,0) - w_1 \in \mathcal{U}_*^{\geq C}, \;\; \frac{1}{72}\left(-3,3,0,0\right) - \frac{1}{2}(w_1+w_2) \in \mathcal{U}_*^{\geq C}$$

and:

$$\frac{1}{36}\left(0,0,-1,1\right) - (w_1+w_2) \in \mathcal{U}_*^{\geq C}.$$

Now $w_1 + w_2 = (a,b,c,d)$ is by assumption an element of $\mathcal{U}_*^{\geq C}$. The condition $\frac{1}{36}\left(-3,3,0,0\right) - (w_1+w_2) \in \mathcal{U}_*^{\geq C}$ implies that $c = d = 0$. On the other hand the condition $\frac{1}{36}\left(0,0,-1,1\right) - (w_1+w_2) \in \mathcal{U}_*^{\geq C}$ implies that $a = b = 0$. Thus, $w_1 + w_2 = 0$ and, actually, $w_1 = w_2 = 0$.

A.2.12 Theorem 3

The fact that Statement 1 implies Statement 2 has been proved (for any number of groups) by Lemma 2 while the implication of Statement 3 by Statement 2 has been established by Theorem 1. We therefore only need to prove that Statement 3 implies

Statement 1. Suppose therefore that $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$

• Consider first the case where $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$.[17] Then:

$$\mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq QO} \subseteq \{\theta p_1 + v : \ \theta \in [0,2], \ v \in \mathcal{U}_*^{\geq QO}\}.$$

Since $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ we have $q_1 = \theta_1 p_1 + v_1$; $q_2 = \theta_2 p_1 + v_2$, where $\theta_1, \theta_2 \in [0,2]$ and $v_1, v_2 \in \mathcal{U}_*^{\geq QO}$. Now, q_1 and q_2 being both in $\mathbf{S}$ and v_1 and v_2 having both their components summing to zero, we must have $\theta_1 = \theta_2 = 1$. As a result $q_1 = p_1 + v_1$ and $q_2 = p_1 + v_2$. Since $p_1 + p_2 = q_1 + q_2$, we have $p_2 = p_1 + v_1 + v_2$. Hence $q_1 = p_1 + v_1$, $q_2 = p_2 - v_1$ and $p_2 - p_1 - v_1 = v_2 \in \mathcal{U}_*^{\geq QO}$ which means that $\mathbf{q}$ has been obtained from $\mathbf{p}$ through an equalizing transfer.

• Consider now the case where neither $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$ nor $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$. Since $\mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq QO} \subseteq \mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq QO}$ and both q_1 and $q_2 \in \mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq QO}$, there are numbers θ_1^1, θ_1^1, θ_1^2 and $\theta_2^2 \in [0,1]$ satisfying $\theta_1^1 + \theta_2^1 = \theta_1^2 + \theta_2^2 = 1$ such that $q_1 = \theta_1^1 p_1 + \theta_2^1 p_2 + v_1$ and $q_2 = \theta_1^2 p_1 + \theta_2^2 p_2 + v_2$, for some v_1 and $v_2 \in \mathcal{U}_*^{\geq QO}$. Since $p_1 + p_2 = q_1 + q_2$ we then have:

$$v_1 + v_2 = p_1 + p_2 - \theta_1^1 p_1 - \theta_2^1 p_2 - \theta_1^2 p_1 - \theta_2^2 p_2 = (1 - \theta_1^1 - \theta_1^2)(p_1 - p_2). \tag{21}$$

Now, since neither $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$ nor $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$ while $v_1 + v_2 \in \mathcal{U}_*^{\geq QO}$, the only way by which Equality (21) can hold is if $(1 - \theta_1^1 - \theta_1^2) = 0$ and, as a result, $v_1 + v_2 = 0$. Setting in that case $\theta_1 = \theta_1^1 = \theta_1^2$, we must therefore have:

$$q_1 = \theta_1 p_1 + (1 - \theta_1) p_2; \ \ q_2 = (1 - \theta_1) p_1 + \theta_1 p_2$$

so that:

$$\mathbf{q} = \mathbf{m}.\mathbf{p}, \ \text{ where } \ \mathbf{m} = \begin{bmatrix} \theta_1 & 1 - \theta_1 \\ 1 - \theta_1 & \theta_1 \end{bmatrix} \ \text{ is bistochastic.}$$

A.2.13 Proposition 3

Suppose that Expression (9) holds for any ordering $R \in \mathfrak{R}^q$ and $R' \in \mathfrak{R}^p$ and any

[17] The case where $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$ is similar.

$l = 1, \ldots, n$. Take any $R \in \mathfrak{R}^q$. By definition of $\left(q_{\sigma^R(1)}, ..., q_{\sigma^R(n)}\right)$, $\frac{1}{l}\sum_{i\in J_l^m} q_i - \frac{1}{l}\sum_{i=1}^{l} q_{\sigma^R(i)} \in \mathcal{U}_*^{\geq QO}$ for every $l = 1, ..., n$ and $m = 1, ..., m(l)$. Moreover, for any $l = 1, ..., m$, and $R' \in \mathfrak{R}^p$, we clearly have that $\frac{1}{l}\sum_{i=1}^{l} p_{\sigma^{R'}(i)} \in Co\left\{\sum_{i\in J_l^1} p_i, \ldots, \sum_{i\in J_l^{m(l)}} p_i\right\}$. Hence, we have, for all $m = 1, ..., m(l)$ and $l = 1, ..., n$:

$$\frac{1}{l}\sum_{i\in J_l^m} q_i - \frac{1}{l}\sum_{i=1}^{l} p_{\sigma^{R'}(i)} = \frac{1}{l}\sum_{i\in J_l^m} q_i - \frac{1}{l}\sum_{i=1}^{l} q_{(i)}^R + \frac{1}{l}\sum_{i=1}^{l} q_{\sigma^R(i)} - \frac{1}{l}\sum_{i=1}^{l} p_{\sigma^{R'}(i)} \in \mathcal{U}_*^{\geq QO}$$

so that $\mathbf{q} \succsim_{EZ}^{QO} \mathbf{p}$ holds by Lemma 4.

A.2.14 Proposition 4

Assume that

$$\sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k} q_{ih}u_h\right) \geq \sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k} p_{ih}u_h\right)$$

holds for every increasing and concave function Φ and every list of numbers $u_1, ..., u_k$ satisfying $u_1 \leq ... \leq u_k$. This clearly implies that this inequality holds for those numbers $u_1, ..., u_k$ that take their value in the set $\{0, 1\}$. Hence one has

$$\sum_{i=1}^{n}\Phi\left(\sum_{h=l}^{k} q_{ih}\right) \geq \sum_{i=1}^{n}\Phi\left(\sum_{h=l}^{k} p_{ih}\right) \tag{22}$$

for all $l \in \{1, ..., k\}$ and all increasing and concave function $\Phi : [0, 1] \to R$. This implies in particular that Inequality (22) holds for the concave and increasing function Φ^π defined by: $\Phi^\pi(q) = \min(q - \pi, 0)$, for any $\pi \in [0, 1]$. Hence one must have

$$\sum_{i=1}^{n}\min\left[\sum_{h=l}^{k} q_{ih} - \pi, 0\right] \geq \sum_{i=1}^{n}\min\left[\sum_{h=l}^{k} p_{ih} - \pi, 0\right], \text{ for all } \pi \in [0, 1],$$

and we obtain that $\mathbf{q} \succsim_2^{success} \mathbf{p}$.

A.2.15 Example 3

We can check that $\mathbf{q}$ cannot be obtained from $\mathbf{p}$ through uniform averaging or equalizing transfer. Using Theorem 3, we conclude that $\mathbf{q} \not\succsim_{EZ}^{C} \mathbf{p}$.

On the other hand, $\mathbf{q} \succsim_2^{success} \mathbf{p}$ because, for $l = 2, 3$, the list $\left(\sum_{h=l}^{3} q_{1h}, \sum_{h=l}^{3} q_{2h}\right)$

Lorenz dominates the list $\left(\sum_{h=l}^{3} p_{1h}, \sum_{h=l}^{3} p_{2h}\right)$: when $l = 2$, $(711/864, 717/864)$ Lorenz-dominates $(708/864, 720/864)$ and, when $l = 3$, $(281/864, 295/864)$ Lorenz-dominates $(264/864, 312/864)$

A.3 Axiomatic characterization of UEVEU orderings

We provide here a description of the axioms that happen to characterize the UEVEU family of orderings of allocations of opportunities. We start by giving additional notations. If $\mathbf{p} \in \mathbf{S}^m$ and $\mathbf{q} \in \mathbf{S}^n$, we denote by $(\mathbf{p}, \mathbf{q})$ the allocation of opportunities in $\mathbf{S}^{m+n}$ where the m first groups have the opportunities associated with $\mathbf{p}$ and the n last groups have the opportunities associated with $\mathbf{q}$. For any distribution ρ in $\mathbf{S}$, we abuse notation by also denoting by ρ the *one group allocation* in which people in the considered group face the distribution ρ.

The first axiom that enters in the characterization of the family of UEVEU family of criteria is the *anonymity* principle according to which the names of the groups don't matter for appraising the opportunities offered to their members. Hence, any between-group permutation of the probability distributions is a matter of indifference.

Axiom A1 *(Anonymity)* $\pi.\mathbf{p} \sim \mathbf{p}$ *must hold for every* $\mathbf{p} \in \mathbb{A}$ *and every* $n(\mathbf{p}) \times n(\mathbf{p})$ *permutation matrix* π*.*

While Anonymity seems plausible when evaluating allocations of opportunities among groups formed on the basis of race, gender, and other (morally arbitrary) qualitative characteristics of that sort, it may not seem so when groups are formed on the basis of a more quantitative attribute like, for example, the income category of the parents. In such a setting, often considered in mobility measurement (see e.g. Atkinson (1981) and Dardanoni (1993)), it has been suggested that it could be better to give the good opportunities to children from low-income families and the bad opportunities to children from high-income family. In such a case, permuting the opportunities offered to children coming from different backgrounds is not be a matter of social indifference.

The second axiom is a continuity condition that concerns the ranking of opportunities faced by one group *vis-à-vis* others. This axiom requires that the strict ranking of any one-group allocation of opportunities *vis-à-vis* any other be robust to small changes in the probabilities of achieving any given outcome. Its formal statement is as follows.

Axiom A2 *(Continuity) For every* $\mathbf{p} \in \mathbb{A}$*, the sets* $B(\mathbf{p}) = \{\rho \in \mathbf{S} : \rho \succsim \mathbf{p}\}$*,* $W(\mathbf{p}) = \{\rho \in \mathbf{S} : \mathbf{p} \succsim \rho\}$ *are both closed in* $\mathbb{R}^k_+$*.*

The third axiom is called *averaging* in Gravel, Marchant, and Sen (2012). It is the only axiom that restricts the ranking of allocations of opportunities among different numbers of groups. Specifically, the axiom evaluates what happens when two allocations of opportunities are merged together. To illustrate this, consider the allocations of educational opportunities in West Bengal and Odisha discussed in the introduction and their states' partition between low- and high-caste adults. Suppose, as suggested earlier, that educational opportunities are considered more equally distributed between low and high caste adults in West Bengal than in Odisha. Assume that the two states merge into a larger jurisdictional entity where there will now be four groups: Odisha low caste, Odisha high caste, West Bengal low caste and West Bengal high caste. The averaging axiom requires the ranking of the opportunities offered to this enlarged four-group jurisdiction to lie between that of the two initial two-groups states (West Bengal and Odisha). That is, opportunities should be better distributed in West Bengal than in the newly enlarged jurisdiction, and should be better distributed in this enlarged jurisdiction than they were in Odisha. The formal statement of this axiom is as follows.

Axiom A3 *(Averaging) For all* $\mathbf{p}, \mathbf{q} \in \mathbb{A}$*, we have* $\mathbf{p} \succsim \mathbf{q} \Leftrightarrow \mathbf{p} \succsim (\mathbf{p}, \mathbf{q}) \succsim \mathbf{q}$*.*

The next axiom requires the ranking of any two allocations of opportunities among the same number of groups to be robust to the addition, to the two allocations, of a common allocation of opportunities. Said differently, the ranking of two allocations of opportunities among the same number of groups should not depend upon any common allocation of opportunities among some of the groups.

Axiom A4 *(Same Number Group Independence) For all allocations* $\mathbf{p}$*,* $\mathbf{p}'$ *and* $\mathbf{p}''$ *in* $\mathbb{A}$ *such that* $n(\mathbf{p}) = n(\mathbf{p}')$*,* $(\mathbf{p}, \mathbf{p}'') \succsim (\mathbf{p}', \mathbf{p}'')$ *if and only if* $\mathbf{p} \succsim \mathbf{p}'$*.*

The last two axioms deal with the ranking of one-group allocations where by definition there is no concern for inequality of opportunities. The first of these axioms requires the ranking of one-group allocations, which is simply the ranking of probability distributions, to obey the well-known Von Neumann and Morgenstern (1947) independence axiom.

Axiom A5 *(VNM for One-Group) For every probability distributions* p*,* p' *and* p" $\in$ $\mathbf{S}$ *and every number* $\lambda \in [0, 1]$*,* $p \succsim p'$ *if and only if* $\lambda p + (1 - \lambda)p'' \succsim \lambda p' + (1 - \lambda)p''$*.*

The second of these two axioms ensures the consistency of the ranking of one-group allocations, at least when the members of the group face no uncertainty at all, with the (possibly incomplete) ranking of outcomes provided by $\geq_{QO}$.

Axiom A6 *(Consistency with* $\geq_{QO}$ *for One-Group) For every two distinct outcomes* h *and* $j \in \{1, ..., k\}$ *such that* $j \geq_{QO} h$*, one should have* $\delta_j \succsim \delta_h$*.*[18]

It can be checked that any ordering as per (2) satisfies Axioms A1 - A6. Using and adapting results in Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2011), the converse implication can be established. Hence, we have:

Proposition A5 *Let* $\succsim$ *be an ordering on* $\mathbb{A}$ *satisfying Axioms A1 - A6. Then, there exists a function* $\Phi : \mathbb{R} \to \mathbb{R}$ *and* $u_1, ..., u_k \in \mathcal{U}^{\geq_{QO}}$ *such that (2) holds for any two allocations* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbb{A}$*. Furthermore, the function* Φ *is unique up to a positive affine transformation, and is continuous and increasing.*

[18] Where, for $j \in \{1, ..., k\}$, $\delta_j \in \mathbf{S}$ is defined by $\delta_j(j) = 1$.

A.4 Additional results from the empirical analysis

State	State abbreviations	State GDP	Male	Upper caste	High edu.	Low edu.	Sample size
Uttar Pradesh	UP	14526.6	0.51	0.80	0.09	0.43	7911
Maharastra	MH	25688.1	0.50	0.80	0.12	0.22	6540
Bihar	BH	4958.45	0.51	0.83	0.05	0.45	4118
West Bengal	WB	10442.5	0.51	0.70	0.09	0.31	4977
Madhya Pradesh	MP	6331.11	0.49	0.69	0.07	0.36	4188
Tamil Nadu	TN	15077.1	0.51	0.82	0.10	0.24	4634
Rajasthan	RJ	8724.13	0.51	0.71	0.08	0.46	3369
Karnataka	KA	12158.4	0.50	0.80	0.10	0.30	3422
Gujrat	GJ	12350.9	0.50	0.79	0.08	0.28	2950
Andra Pradesh	AN	7611.93	0.51	0.80	0.09	0.41	5121
Odisha	OD	4634.29	0.51	0.63	0.06	0.35	3427
Kerala	KR	7303.89	0.55	0.90	0.10	0.07	3114
Haryana	HR	5969.52	0.49	0.78	0.13	0.28	2338
Punjab	PB	5349.36	0.48	0.67	0.09	0.27	2587
India	IN	361134	0.50	0.73	0.08	0.33	298787

Table 1: Sample summary: Major Indian states.

Note: State GDP for 2012 in trillions 2015-USD; High education means graduate and above; Low education means no formal schooling. The columns Males, Upper caste, High educ. and Low educ. report sample shares by state.

Education group	Education level
1	Illiterate
2	Literate without formal schooling
3	Below primary (less than 4 years of schooling)
4	Below secondary (5 to 10 years of schooling)
5	Below graduate (11 to 15 years of schooling)
6	Graduate and above (16 or more years of formal education)

Table 2: Education groups and the corresponding levels of education.

	Gender		Caste		Caste and Gender			
	M	F	H	L	HCM	HCF	LCM	LCF
Overall share in population	0.50	0.50	0.73	0.27	0.37	0.36	0.13	0.13
Share in education categories								
Illiterate	0.22	0.43	0.29	0.43	0.19	0.38	0.31	0.56
Informally literate	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
Below primary level	0.10	0.09	0.09	0.10	0.09	0.09	0.11	0.09
Below secondary level	0.29	0.23	0.26	0.26	0.28	0.24	0.31	0.21
Below graduate level	0.27	0.18	0.25	0.15	0.30	0.20	0.20	0.11
Graduate and above	0.11	0.07	0.11	0.04	0.13	0.08	0.05	0.02

Table 3: Distribution of Indian adult population by education categories and circumstances (2012)

Note: M:Male, F: Female, H: High-caste, L: Low-caste, HCM: High-caste-male, HCF: High-caste-female, LCF: Low-caste-male, LCF: Low-caste-female. Each entry represents percentage share of the respective cohort in the overall Indian adult population during 2012.

	Indian states													
	AP	BH	GJ	HR	KA	KR	MH	MP	OD	PB	RJ	TN	UP	WB
TN	TN	TN	TN	–	?	?	MH	TN	TN	–	TN		?	TN
UP	?	–	–	?	KA	?	MH	?	?	–	UP	TN		?

Table 4: Caste-gender based QOEZ ranking of allocations of education opportunities in the Indian states (necessary and sufficient conditions).

Note: Dominance tests based on the union-intersection approach at 95% confidence level. Sufficient conditions are given in Proposition 3. Necessary conditions are from Definition 8 and are tested at a fixed grid of reference shares $\pi \in \{0, 0.01, 0.02, \ldots, 0.99\}$. Standard errors are bootstrapped (500 samples drawn with replacement and stratified by state). The bounds of the rejection region for the necessary conditions test is 3.64, the available estimates of the 95% quantile of a Studentized Maximum Modulus distribution with infinite degree of freedom and the maximum amount of constraints. Each entry of the table reports either the name of the Indian State that QOEZ-dominates the other (identified by one row and one column of the table, for instance, TN QOEZ-dominates AP because at least the sufficient conditions for dominance are satisfied when the two states are compared), or the "–" sign to indicate that the two states are not QOEZ-comparable (because the necessary condition fails for every dominance comparison involving the two states), or the "?" for cases in which we cannot conclude about whether or not a state dominates the other (because both sufficient and necessary conditions fail).

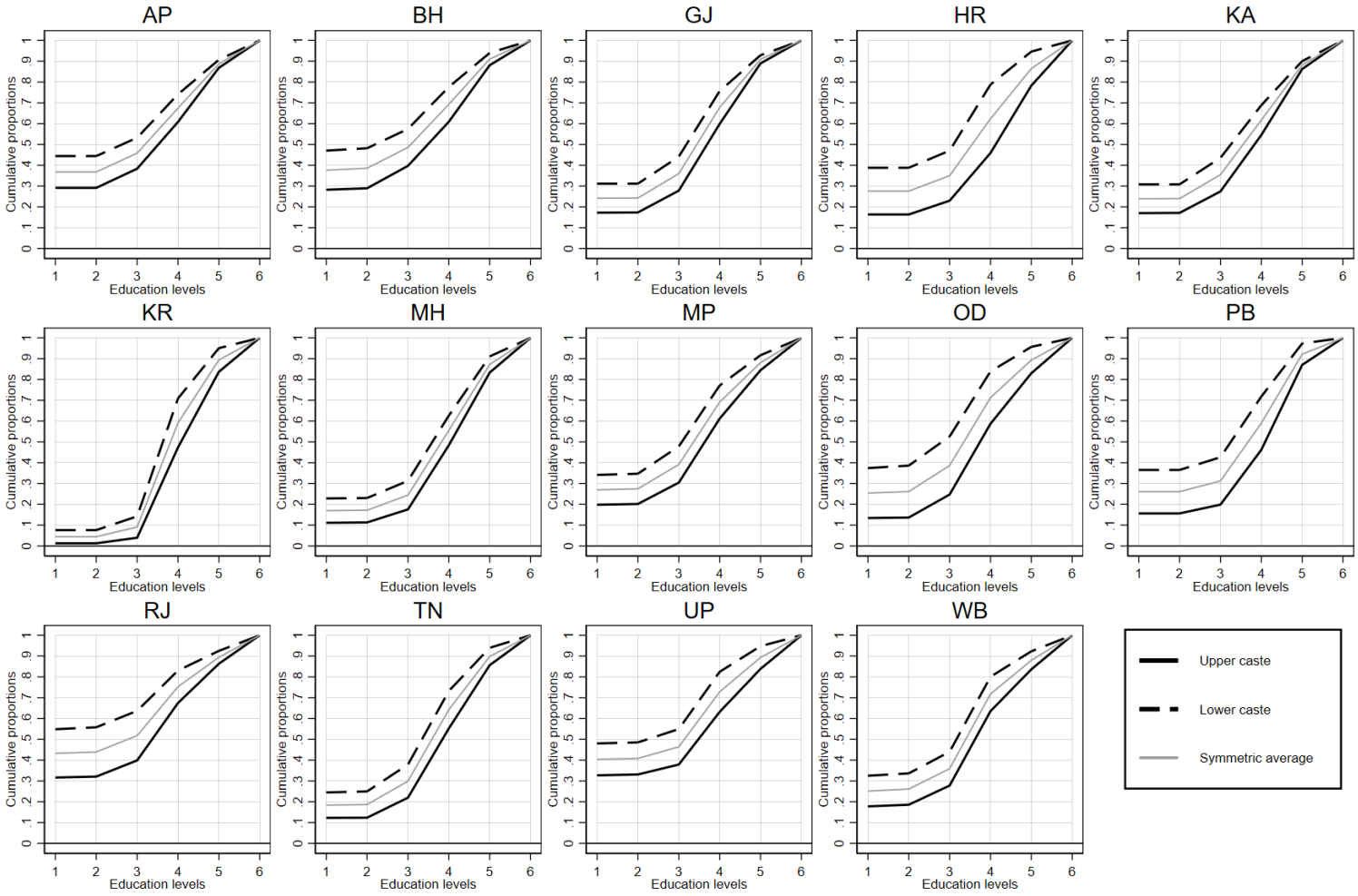

Figure 6: Cumulative distribution of education levels by caste, all states.

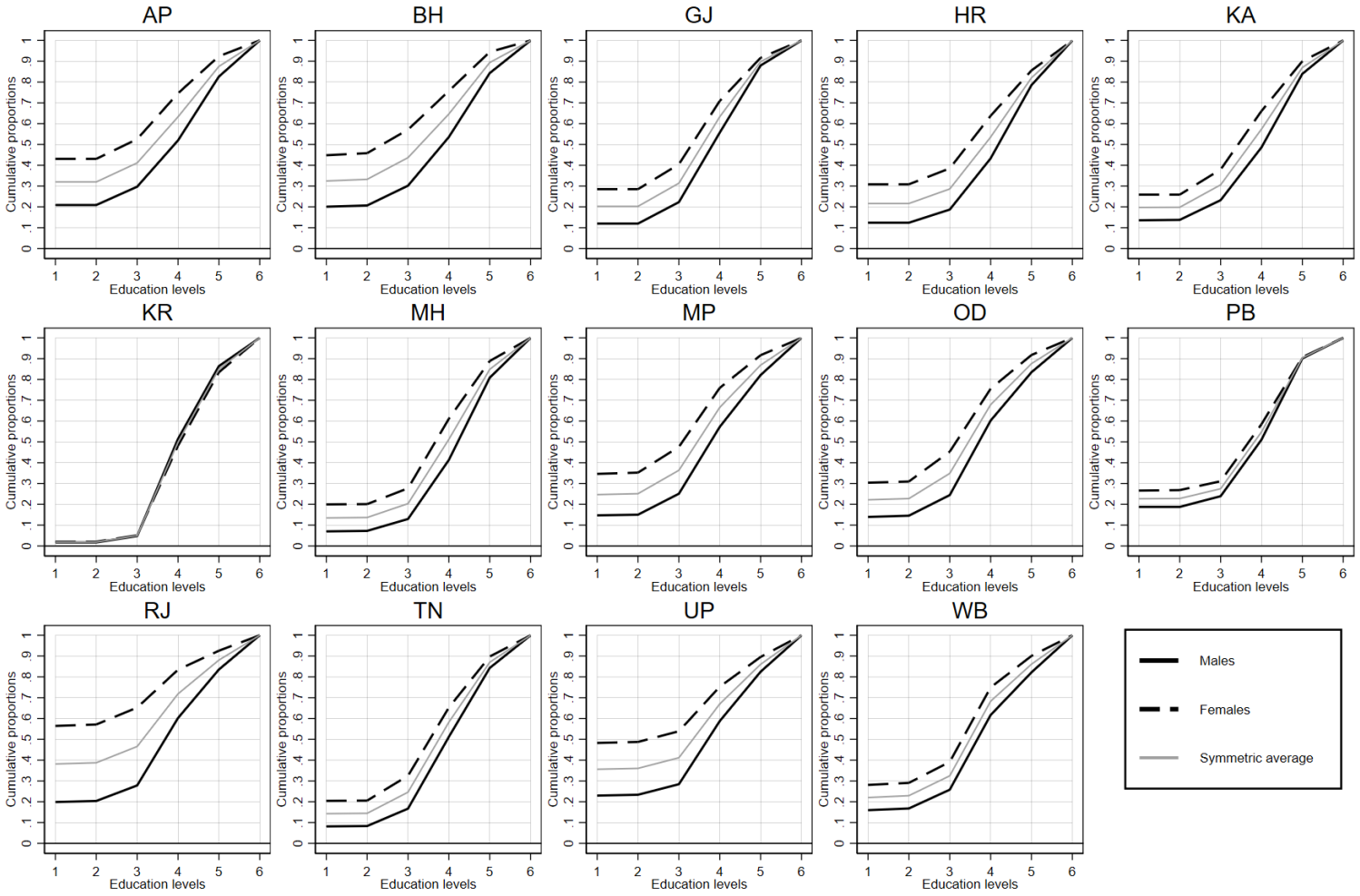

Figure 7: Cumulative distribution of education levels by gender, all states.

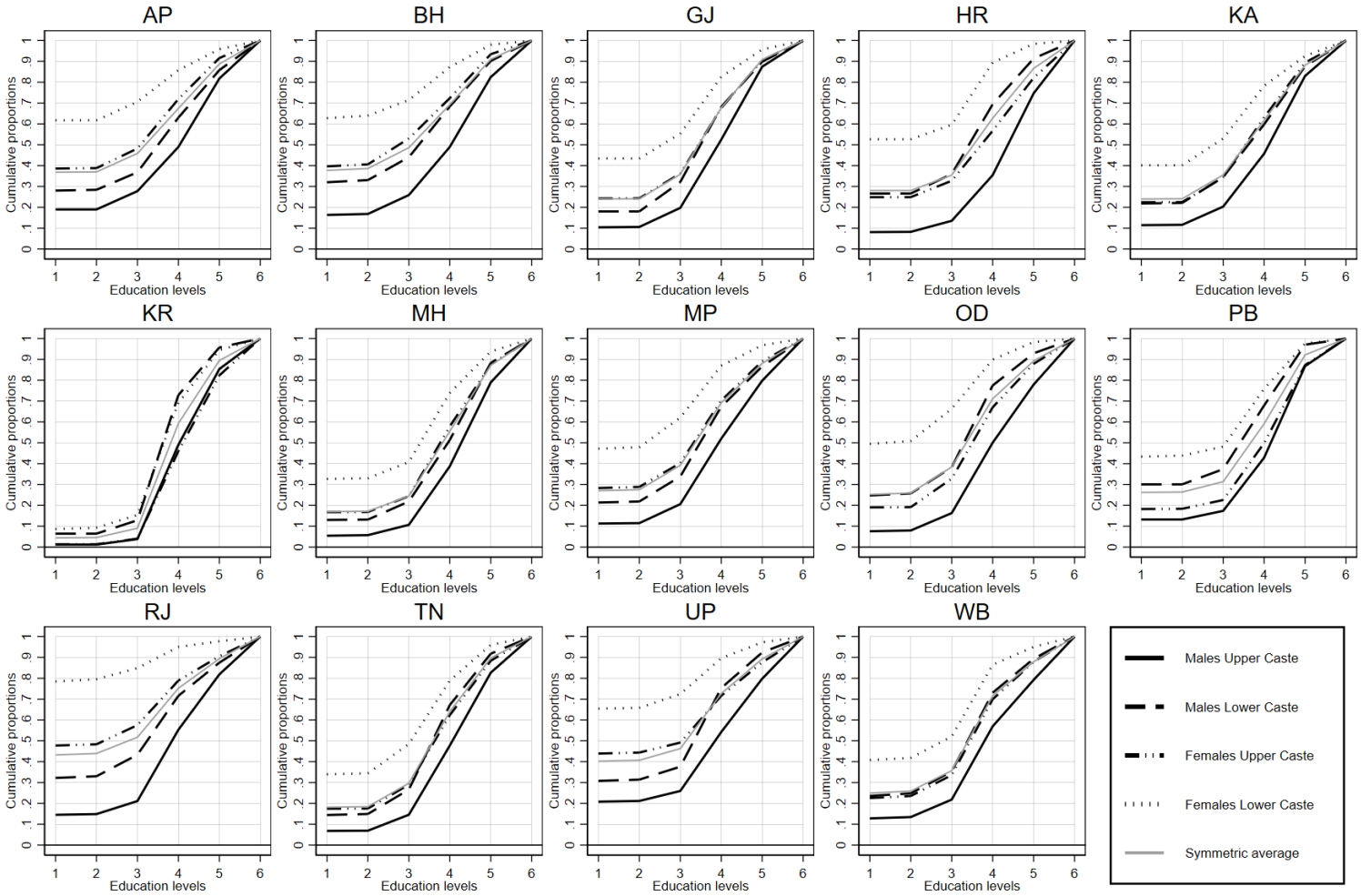

Figure 8: Cumulative distribution of education levels by caste and gender, all states.

A.5 Applications of Lemma 4: Some examples

This appendix illustrates the abstract method of Lemma 4 for verifying if QOEZ dominance holds between two allocations in the case where the outcomes are completely ordered. We first provide a theoretical example with two groups and four outcomes in which a QOEZ dominance relationship is identified by the procedure of Lemma 4. We then make use of data for India on educational opportunity distributions across caste-gender groups in Uttar Pradesh and Andhra Pradesh to show how the procedure of Lemma 4 can lead to the conclusion of non-comparability.

A.5.1 An example of QOEZ dominance

Consider the following two allocations involving three groups and four completely ordered outcomes.

$\mathbf{q} =$

	a	b	c	d
group 1	0.025	0.375	0.35	0.25
group 2	0.15	0.2	0.35	0.3
group 3	0.1	0.35	0.2	0.35

$\mathbf{p} =$

	a	b	c	d
group 1	0	0.3	0.6	0.1
group 2	0	0.6	0	0.4
group 3	0.3	0	0.3	0.4

Let us use the procedure of Lemma 4 to verify if $\mathbf{q}$ QOEZ dominates $\mathbf{p}$. We know from Remark 2 that the reverse dominance cannot hold because the symmetric average distribution in $\mathbf{q}$ first-order stochastically dominate that of $\mathbf{p}$. In this specific example, there are three collections of 1 group, three collections of 2 groups, and one collection of 3 groups. Hence, there are seven systems of linear inequalities of the form (7) to solve. However the system for the unique collection of one group is nothing else than the verification of first order dominance between symmetric average distributions, which has been already established. This leaves one with the

following six systems which can be written, in vector notation as in (7):

$$
\begin{aligned}
&i) \quad \left(\sum_{j=h}^{4} p_{1j},\ \sum_{j=h}^{4} p_{2j},\ \sum_{j=h}^{4} p_{3j}\right)\cdot \boldsymbol{\lambda}^{1^1} \leq \sum_{j=h}^{4} q_{1j} \quad for\ h=2,\ldots,4;\\
&ii) \quad \left(\sum_{j=h}^{4} p_{1j},\ \sum_{j=h}^{4} p_{2j},\ \sum_{j=h}^{4} p_{3j}\right)\cdot \boldsymbol{\lambda}^{2^1} \leq \sum_{j=1}^{h} q_{2j} \quad for\ h=2,\ldots,4;\\
&iii) \quad \left(\sum_{j=h}^{4} p_{1j},\ \sum_{j=h}^{4} p_{2j},\ \sum_{j=h}^{4} p_{3j}\right)\cdot \boldsymbol{\lambda}^{3^1} \leq \sum_{j=1}^{h} q_{3j} \quad for\ h=2,\ldots,4;\\
&iv) \quad \left(\sum_{j=h}^{4} p_{1j}+p_{2j},\ \sum_{j=h}^{4} p_{1j}+p_{3j},\ \sum_{j=h}^{4} p_{2j}+p_{3j}\right)\cdot \boldsymbol{\lambda}^{1^2} \leq \sum_{j=h}^{4} q_{1j}+q_{2j} \quad for\ h=2,\ldots,4;\\
&v) \quad \left(\sum_{j=h}^{4} p_{1j}+p_{2j},\ \sum_{j=h}^{4} p_{1j}+p_{3j},\ \sum_{j=h}^{4} p_{2j}+p_{3j}\right)\cdot \boldsymbol{\lambda}^{2^2} \leq \sum_{j=h}^{4} (q_{1j}+q_{3j}) \quad for\ h=2,\ldots,4;\\
&vi) \quad \left(\sum_{j=1}^{4} p_{1j}+p_{2j},\ \sum_{j=1}^{4} p_{1j}+p_{3j},\ \sum_{j=1}^{4} p_{2j}+p_{3j}\right)\cdot \boldsymbol{\lambda}^{3^2} \leq \sum_{j=h}^{h} q_{2j}+q_{3j} \quad for\ h=2,\ldots,4;
\end{aligned}
$$

A linear programming solver (in Python) determines that the vectors:

$$
\boldsymbol{\lambda}^{1^1}=\begin{pmatrix}1/2\\1/3\\1/6\end{pmatrix},\ \boldsymbol{\lambda}^{2^1}=\begin{pmatrix}1/3\\1/6\\1/2\end{pmatrix},\ \boldsymbol{\lambda}^{3^1}=\begin{pmatrix}1/6\\1/2\\1/3\end{pmatrix},\ \boldsymbol{\lambda}^{1^2}=\begin{pmatrix}1/3\\1/2\\1/6\end{pmatrix},\ \boldsymbol{\lambda}^{2^2}=\begin{pmatrix}1/2\\1/6\\1/3\end{pmatrix},\ \boldsymbol{\lambda}^{3^2}=\begin{pmatrix}1/6\\1/3\\1/2\end{pmatrix}
$$

are solutions to their respective systems. Hence $\mathbf{q}$ QOEZ dominates $\mathbf{p}$. Observe that there are potentially many vectors of solutions to the above systems of inequalities. Take, for instance the first system of inequalities. The vectors $\boldsymbol{\lambda}^{1^1}$ that solve this system are all those vectors that write:

$$
\boldsymbol{\lambda}^{1^1}=\begin{pmatrix}\lambda\\ \mu\\ 1-\lambda-\mu\end{pmatrix}
$$

for some real numbers λ and μ satisfying $\lambda \geq 1/2,\ \mu \geq 1/3,\ \lambda+\mu \leq 11/12$. This set is thus a convex subset of the interior of $Co(p_1, p_2, p_3)$. Yet, verifying QOEZ dominance does not require that we know the whole set of solutions to the relevant systems of inequalities. It only requires to find one of them.

When no solution is found for at least one of the systems, then QOEZ dominance cannot be claimed. Of course, this absence of solution may be due to the inability

of the particular computer program to find it. When this is the case, one may look, again with the help of computer solvers, for existence of solution to the Farkas's alternatives to the system of linear inequalities in (7), for example "Farkas's Alternative 38" in Border (2020). If a solution to this alternative inequality system is found, then we know for sure that no solution exists to the systems (7) and, accordingly, that no dominance holds between the two allocations. In the next section, we illustrate how one obtains this conclusion in the comparison of Uttar Pradesh and Andhra Pradesh. Of course, a theoretical possibility - of serious computer solver incompetence - could emerge if the computer solver was failing to find solutions to systems of inequalities (7) and was also failing to find solutions to the Farkas's alternatives. The recommendation would then be to tune up the computer algorithm.

A.5.2 An example of lack of QOEZ dominance

We consider the possibility of ranking Uttar Pradesh (UP) and Andhra Pradesh (AP) by QOEZ dominance. As discussed in the empirical application, it is not possible to compare these two states using the simple criterion outlined in Proposition 2, nor according to the necessary and sufficient conditions identified in Propositions 3 and 4. We therefore resort on Lemma 4 and show how the test has been applied to the data.

The distributions of educational opportunities across four caste-gender groups in UP (matrix $\mathbf{p}$) and AP (matrix $\mathbf{q}$) are the following:

$AP:\ \mathbf{q} =$

	1	**2**	**3**	**4**	**5**	**6**
LCF	0.617	0.000	0.089	0.152	0.101	0.041
UCF	0.386	0.002	0.094	0.237	0.197	0.084
LCM	0.280	0.004	0.084	0.262	0.228	0.142
UCM	0.190	0.000	0.088	0.212	0.328	0.183

$UP:\ \mathbf{p} =$

	1	**2**	**3**	**4**	**5**	**6**
LCF	0.654	0.004	0.066	0.172	0.076	0.028
UCF	0.439	0.005	0.048	0.224	0.161	0.124
LCM	0.308	0.006	0.062	0.377	0.171	0.076
UCM	0.208	0.003	0.048	0.283	0.256	0.202

There are 4 groups in each allocation. Adopting the notation in Lemma 4, we form the following collections of groups for $m(1) = 4$, $m(2) = 6$, $m(3) = 4$, $m(4) = 1$:

$$\begin{aligned}
\{\mathcal{J}_1^1, \ldots, \mathcal{J}_1^4\} =& \{\{LCF\}, \{UCF\}, \{LCM\}, \{UCM\}\}, \\
\{\mathcal{J}_2^1, \ldots, \mathcal{J}_2^6\} =& \{\{LCF, UCF\}, \{LCF, LCM\}, \{LCF, UCM\}, \{UCF, LCM\}, \{UCF, UCM\}, \{LCM, UCM\}\}, \\
\{\mathcal{J}_3^1, \ldots, \mathcal{J}_3^4\} =& \{\{LCF, UCF, LCM\}, \{LCF, UCF, UCM\}, \{LCF, LCM, UCM\}, \{UCF, LCM, UCM\}\}, \\
\mathcal{J}_4^1 =& \{LCF, UCF, LCM, UCM\}.
\end{aligned}$$

The test proposed in Lemma 4 requires one to solve 14 separate systems of linear inequalities, one for each element of each of the collections $\{\mathcal{J}_1^1, \ldots, \mathcal{J}_1^4\}$, $\{\mathcal{J}_2^1, \ldots, \mathcal{J}_2^6\}$, $\{\mathcal{J}_3^1, \ldots, \mathcal{J}_3^4\}$, and then verify that the inequality holds when the (trivial) collection $\mathcal{J}_4^1$ is involved. Each of the 14 systems features six inequalities (in fact five since the inequality concerning the whole distribution which sums to 1 is trivial). The system of inequalities corresponding to a given pair l, m^l with $l \in \{1, 2, 3\}$ and $m^l \in \{1, ..., m(l)\}$ can be written as follows:

$$\left(\sum_{i \in \mathcal{J}_l^1} \sum_{j=h}^{6} p_{ij}, \ldots, \sum_{i \in \mathcal{J}_l^{m(l)}} \sum_{j=h}^{6} p_{ij} \right) \cdot \boldsymbol{\lambda}_l^{m^l} \leq \sum_{i \in \mathcal{J}_l^{m^l}} \sum_{j=1}^{h} q_{ij} \quad for\ h = 2, \ldots, 6. \tag{23}$$

We can readily verify that the symmetric average distribution of AP first-order dominates that of UP. Hence, if any dominance is observed, AP will be the dominant distribution. Using a Python linear programming solver, we find that almost all systems of inequalities of the form (23) admit at least one solution. The only exception to this is the system associated to $l = 2$ and $m^l = 1^2$. The solutions to the other systems are:

$$\begin{aligned}
\boldsymbol{\lambda}^{1^1} = \boldsymbol{\lambda}^{2^1} = \boldsymbol{\lambda}^{3^1} = \boldsymbol{\lambda}^{4^1} =& (1, 0, 0, 0)' \\
\boldsymbol{\lambda}^{2^2} = \boldsymbol{\lambda}^{3^2} = \boldsymbol{\lambda}^{4^2} = \boldsymbol{\lambda}^{5^2} = \boldsymbol{\lambda}^{6^2} =& (0, 1, 0, 0, 0, 0)' \\
\boldsymbol{\lambda}^{1^3} = \boldsymbol{\lambda}^{2^3} = \boldsymbol{\lambda}^{3^3} = \boldsymbol{\lambda}^{4^3} =& (1, 0, 0, 0)'.
\end{aligned}$$

These rather simple solutions follow from the fact that the worse-off group in UP - that is the LCF- is also worse-off in terms of first-order stochastic dominance than any other group in AP.

By contrast, verifying the existence of a vector $\boldsymbol{\lambda}^{1^2}$ solving the corresponding system of inequalities (23) is less obvious. This verification amounts to finding an element $\mathbf{x}$ in the 5-dimensional simplex, which solves the system $-\mathbf{A}\mathbf{x} \leq -\mathbf{b}$, where

$$\mathbf{b} := \begin{pmatrix} 1.004 \\ 1.005 \\ 1.188 \\ 1.577 \\ 1.874 \\ 2 \end{pmatrix}; \quad \mathbf{A} := \begin{pmatrix} 1.093 & 0.962 & 0.863 & 0.747 & 0.647 & 0.516 \\ 1.102 & 0.972 & 0.870 & 0.758 & 0.655 & 0.526 \\ 1.216 & 1.100 & 0.984 & 0.868 & 0.751 & 0.635 \\ 1.612 & 1.649 & 1.439 & 1.469 & 1.258 & 1.295 \\ 1.849 & 1.896 & 1.770 & 1.800 & 1.674 & 1.722 \\ 2.000 & 2.000 & 2.000 & 2.000 & 2.000 & 2.000 \end{pmatrix}.$$

Widely used linear programming solvers (for instance scipy and cvxpy in Python) are unable to find a solution to this system. However, by virtue of Farkas' Lemma, proving that this system has no solution is equivalent to showing that there exists a row vector $\boldsymbol{\alpha} \in \mathbb{R}^6_+$, as well as a scalar $\beta \in \mathbb{R}$ such that

$$-\boldsymbol{\alpha} \cdot \mathbf{A} + \beta \mathbf{1} \geq \mathbf{0}, \text{ and } -\boldsymbol{\alpha} \cdot \mathbf{b} + \beta < 0. \tag{24}$$

One can verify that the values

$$\boldsymbol{\alpha} = (0, 0, 0.45, 0, 0.55, 0), \quad \beta = 1.565$$

are solutions to (24). Hence, Farkas's alternatives are true, and this implies that AP does not compare with UP in terms of QOEZ dominance.

Data Availability Statement

The data that support the findings of this study are openly available in the ICPSR repository at https://doi.org/10.3886/E237323V1, reference number openicpsr-237323.

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