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A Semi-Markov Dynamic Capital Injection Problem for Distressed Banks

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Abstract: Our study investigates the optimal dividend strategy for a bank, taking into account the potential for government capital injections. We explore different types of government interventions, such as liberal, transparent, or uncertain strategies, and consider both single and multiple types of interventions. Our approach differs from others as it focuses on interventions that aim to maintain the overall stability of the financial system, rather than just addressing banks that have already sought government assistance or are in dire need of it. Specifically, we focus on situations where the government is more likely to assist banks that have not requested its intervention or that are not too difficult to save. To accomplish this, we conduct a comprehensive examination of all possible scenarios involving a single, one-time capital injection and derive explicit solutions for the associated optimal control problem. Furthermore, we expand the model to include semi-Markov dynamic capital injection processes and show that the optimal control is the unique viscosity solution of a Hamilton–Jacobi–Bellman equation. The government's strategy also takes into account the bank's solvency and any past government interventions.

Keywords: semi-Markov process; capital injection; dividend payment; stochastic control; jump-diffusion; HJB equation

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1. Introduction

The collapse of Lehman Brothers in 2008 sparked extensive debate about the appropriateness of government intervention to save struggling banks. In the event of a bank's bankruptcy, the losses are typically borne by stakeholders such as creditors and shareholders. However, if a default is avoided through intervention by an external entity, such as a government, supranational organization, or central bank, the losses are spread among a wider population, such as taxpayers. Despite the Dodd–Frank reforms and Basel regulatory standards, which have helped to reduce risk in the financial sector, this topic remains highly relevant today.

Our paper presents a stochastic model that describes the dynamics of banks and is based on real-world financial assumptions. From a regulatory perspective, banks are required to assess their exposure to risk and maintain enough capital to prevent large losses. This capital serves as a buffer to absorb unexpected losses and maintain the ability to continue lending during times of stress, in order to maintain a stable financial system. The Federal Reserve and the Basel Committee on Banking Supervision have established guidelines and proposals for assessing financial stability, even in adverse scenarios. These requirements are outlined in the Comprehensive Capital Analysis Review (CCAR) for American bank holding companies or foreign bank holdings operating in the United States and in the Fundamental Review of the Trading Book (Basel 3.1). The latter, initially published in January 2016 and revised in January 2019, introduced new proposals for market risk-related capital requirements for banks. In this paper, we investigate the implications of these regulations from the perspective of the government or central bank when a bank does not comply with these capital requirements. We refer readers to the work of Bonollo et al. in Bonollo et al. (2018) for a comprehensive examination of the mathematical background of minimum capital requirements and the implementation of the Default risk charge, as well as other references Bertagna et al. (2020); Cardone-Riportella et al. (2013); Srivastava and Dashottar (2020) for further information. To simplify matters, in the following sections, we will refer to the external entity that is expected to support distressed banks as "government".

From a mathematical perspective, we establish a probabilistic framework based on a standard filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is the "real-world" probability measure corresponding to the financial scenario we are analyzing, and the time variable $t \in [0, T]$ for a given horizon time $T < +\infty$, following the approach proposed by Hugonnier and Morellec in Hugonnier and Morellec (2017), see also Cordoni et al. (2020) and references therein, in the context of impulsive stochastic control. We assume that banks' financial assets consist of capital invested in risky assets, liquid reserves S (such as cash, cash equivalents, marketable securities, and accounts receivable), liabilities L, and deposits D. In other words, tangible assets (such as inventories, property, plant, and equipment) and intangible assets (such as goodwill and patents) are not considered in this analysis as the focus is on the financial structure and solvency of the bank. For the sake of simplicity, all risky assets are summarized by a global one, A, whose dynamics are determined by a drifted Brownian motion and a compound Poisson process

$$\mathrm{d}A_t = \bar{\mu}\,\mathrm{d}t + \sigma\,\mathrm{d}W_t - \mathrm{d}Y_t\,,$$

with initial value $A_0 = x \in \mathbb{R}^+$, where $\bar{\mu} := \mu\,(1-\vartheta)$, the constants μ and σ are, respectively, the drift and the volatility of the bank's asset value, $\vartheta \in [0,1)$ is the taxation rate paid continuously, and $W = \{W_t\}_{t\geq 0}$ is a Brownian motion (BM), while $Y = \{Y_t\}_{t\geq 0}$ denotes the compound Poisson process, adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, such that

$$Y_t = Y_0 + \sum_{k=0}^{\infty} \zeta_k\, \mathbb{1}_{t\leq \theta_k}\,, \tag{1}$$

$\{\theta_k\}_k$ representing the sequence of random times at which the asset price process A_t experiences (negative) jumps, hence breaking the continuity of its path.

These *shocks*, see, e.g., Nakagawa (2007), have exponentially distributed inter-arrival times, the parameters $\lambda > 0$, ζ_k being the random magnitude of the negative jumps, assumed to be independently, exponentially distributed, with mean $1/\beta$. Alternatively, denoting by N_t the counting process for jumps that have occurred up to time t, we may rewrite the dynamics of Y as $\mathrm{d}Y_t = \zeta_{N_t}\,\mathrm{d}N_t$.[1]

We denote by E the cumulative earnings, whose dynamics are given by the following stochastic differential Equation (SDE):

$$\mathrm{d}E^\alpha_t = \mathrm{d}A_t - (c^\alpha_D(t) + c_L)\,\mathrm{d}t\,,$$

where c_D and c_L are the bank's payments, due to its depositors and creditors, respectively. Both the control α, and the parameters c_D, c_L, will be better specified after having defined the liquidity value.

Alongside the risky asset, the bank can decide to also hold risk-free reserves S, which are liquid and are depleted by the bank's dividend payments. Hence, the time behavior for S is governed by the following differential equation

$$S^\alpha_t = s + E^\alpha_t - Z^\alpha_t + G^\alpha_t\,,$$

where s is the starting value for the liquid reserve, Z is a càdlàg process representing the cumulative dividend payments to the shareholders, and G is the injection of capital made by the government or a financial supervisor aiming to avoid bank default. For a more detailed examination of the role played by the variable G (capital injection), refer to the study by Cordoni et al. in Cordoni et al. (2020). They use a probabilistic constraint approach to examine the government's optimal capital injection problem. A different approach, called mean field game (MFG), can be found in works such as Benazzoli et al. (2020); Huang (2020) and their references. Both this current study and Cordoni et al. (2020) analyze the problem of a government or central bank trying to save struggling banks from default, but with a given tolerance and focus on the too-big-to-fail theory. In other words, Cordoni et al. (2020) suggest that the government is more likely to save banks that are crucial to the entire financial network, while this study argues that the government is more likely to assist banks that have not required interventions in the past. It is also worth mentioning the paper by Capponi and Chen in Capponi and Chen (2013) which examines the role of the lender of last resort in lending capital to systemic important banks in order to maintain an adequate level of wellness for the entire financial network, see also Aguiar and Amador (2020); Carlson and Macchiavelli (2020); Kenny and Turner (2020). The bank has the option to pay dividends using its liquid reserves (while keeping S positive) and, if liquidity is negative, the government can inject the amount of capital G_t to save the bank. However, the government may not guarantee such intervention; see Section 2 for further information on government strategies. In addition to paying dividends, the bank can also increase its liquid reserve by retaining earnings.

Let us define by $\{\tau_n\}_{n\in\mathbb{N}}$ the sequence of times corresponding to moments when the bank reaches a liquidity level lower than a positive threshold below which the bank is experiencing a critical financial situation, namely

$$\tau_n = \inf\{S^\alpha_{s^-} \leq s_c \,:\, s \in (\tau_{n-1}, \infty]\}\,, \tag{2}$$

for $n = 1, 2, \ldots$ with $s_c \in \mathbb{R}^+_0$ and $\tau_0 = 0^-$; the salvage mechanism can be activated whenever the bank enters the region $(-\infty, s_c]$. We remark the fact that the government or financial institution can also intervene with capital injection for positive liquidity values in the *red region*: $s \in [0, s_c)$. One can represent the bank's control by $\alpha = (Z_t)$, where Z is the aforementioned càdlàg increasing process representing the bank's cumulated dividend policy, which cannot exceed the liquidity value of the bank, i.e., $Z_{t^+} - Z_t \leq (S_t + G_{t^+} - G_t)^+$. Once the bank is facing negative liquidity and the government decides not to intervene, the bank will go bankrupt. This occurrence is defined as

$$T^\alpha = \inf\{t \geq 0 \,:\, S^\alpha_{t^+} \leq 0\}\,.$$

In this work, we do not focus on the problem of liabilities, assuming that both the liability value L and c_L are determined internally. Instead, we assume that payments to depositors depend on the total value of deposits D and whether the bank has received external funding from the government. Indeed, the interest rate paid to depositors increases every time the bank experiences a critical situation, as defined by $S^\alpha_{t^-} < s_c$. Additionally, we assume that the government is available to save the bank an indeterminate number of times, and we will further examine this assumption by considering factors such as the number of past salvages and the criticality of the bank's liquidity problem. The goal of the bank manager is to maximize the expected value of dividends until bankruptcy by choosing from a set of admissible control strategies, i.e., aiming at maximizing the following value function

$$v(s) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^{T^\alpha} e^{-\rho t} \mathrm{d}Z^\alpha_t + e^{-\rho T^\alpha}\, l(S^\alpha_{T^\alpha})\right], \tag{3}$$

where $\mathcal{A}$ is the set of all admissible strategies satisfying $Z_{t^+} - Z_t \leq (S_t + G_{t^+} - G_t)^+$, $l(s)$ is the liquidation payment to shareholders if the bank hold $s \leq 0$ in liquid reserves and given by

$$l(s) = (s + \phi\, V^+ - L - D)^+\,, \tag{4}$$

where ϕ is the default recovery rate, while V represents the present value of the infinite stream of cash flows generated by the risky assets, namely

$$V = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\, \mathrm{d}A_t\right] = \frac{1}{\rho}\left(\bar{\mu} - \frac{\lambda}{\beta}\right). \tag{5}$$

2. Government Strategies

In this section, we focus on the main idea of the paper, namely: compare the following three possible government strategies:

- **Liberalism**: The government does not intervene to affect the shortage of liquidity in the banking system. In this case, lack of liquid reserves for a bank implies its liquidation.
- **Transparency**: Banks have perfect knowledge about the strategy of the government, i.e., they know under which conditions they will be rescued.
- **Uncertainty**: Banks do not know exactly what is the government rescuing strategy, but they can make estimations on its savage plans.

Let us first consider the case where the government is available to save a bank only once, so that the interest rate paid to the depositors c_D reads as follow

$$c_D(t) = \left(c_D(0) + \iota\, \mathbb{1}_{\tau_1 < t}\right) D\,. \tag{6}$$

After the bank enters in the critical zone $(-\infty, s_c]$, the government announces the value of $\Delta G_{\tau_1} \in [0, +\infty)$ where

$$\Delta G_t = G_{t^+} - G_t\,. \tag{7}$$

Let us note that $\Delta G_{\tau_1} = 0$ corresponds to the case in which the government decides not to save the bank. Moreover, for any $\Delta G_{\tau_1} > 0$, to avoid bankruptcy, the condition $\Delta G_{\tau_1} > -S^\alpha_{\tau_1^-}$ has to be satisfied. Therefore, different government strategies can be expressed in terms of different measures assigned to G_t. In what follows, we will examine two common government strategies, known as *liberalism* and *transparency*, see, e.g., Chen et al. (2022); Daures-Lescourret and Fulop (2022) and references therein, for insights. It is important to note that the goal of our study is to examine government strategies that involve multiple interventions, and take into account various factors such as the number of past capital injections and the current liquidity level.

- **Liberalism:** This can be summarized by equation $\Delta G_{\tau_1} = 0$, which implies that the government will not intervene, regardless of the particular circumstances.
- **Transparency:** $\Delta G_{\tau_1} = \left(s_u - S^\alpha_{\tau_1^-}\right)\mathbb{1}_{\{S^\alpha_{\tau_1^-} > s_l\}}$; therefore, with certainty, the government will inject capital trying to align the liquid reserve back to some fixed level $s_u > s_c \geq 0$, whenever the current reserve level is greater than $s_l < 0$. In this sense, the government's strategy is transparent to the bank and the bank will make decisions based on this *perfect knowledge*.
- **Uncertainty:** Any government strategy that deals with uncertainty about a bank's financial status falls under this category of strategies. In this specific scenario, multiple government injection strategies could be defined. For example, a straightforward strategy could be

$$\Delta G_{\tau_1} = \left(s_u - S^\alpha_{\tau_1^-}\right)\mathbb{1}_{\{S^\alpha_{\tau_1^-} > s_l\}}\mathbb{1}_{\{x=1\}}\,, \tag{8}$$

  where x is an independent Bernoulli distributed random variable with $Prob(x = 1) = p$. In this case, even if the bank's liquid reserve satisfies $S^\alpha_{\tau_1^-} > s_l$, the government will intervene to avoid bankruptcy *just* with probability $0 < p < 1$.

A more interesting and realistic setting can be defined by

$$\Delta G_{\tau_1} = \left(R - S^{\alpha}_{\tau_1^-}\right) \mathbb{1}_{\{S^{\alpha}_{\tau_1^-} > s_l\}}, \tag{9}$$

having assumed that the government has a set limit, represented by $\bar{R}$, on the amount of capital it is willing to inject into a struggling bank. The amount of capital that the government ultimately injects, represented by R, is determined by a random variable with a positive compact support of $[s_c, \bar{R}]$. The distribution of this variable depends on the bank's current liquidity level, represented by $S_{\tau_1^-}$. If multiple government interventions are permitted, the distribution of R may also take into account the number of past interventions and the total amount of financial support already provided.

To provide a comprehensive overview, we will begin by discussing the known results about the liberalism case. Subsequently, we will focus on the uncertainty framework and demonstrate how it can be used to achieve the certainty framework as a specific instance of the former.

2.1. No Government Intervention: The Liberalism Framework

For the liberalism case, there will be no intervention by the government, hence the bank will default as soon as it runs out of liquid reserves. According to the results stated by Hugonnier et al. in (Hugonnier and Morellec 2017, eq. 21), see also results proved in (Hugonnier and Morellec 2017, sec. 3.2) according to the no-government, no-refinancing, intervention scenario, the optimal strategy depends on the constants $a \doteq l(0) = (\phi V^+ - L - D)^+$ and $v_0^*(c_D)$ defined by

$$v_0^*(c_D) \doteq \frac{\rho}{\rho+\lambda}\left(\frac{\bar{\mu} - c_D - c_L}{\rho}\right) + \frac{\lambda}{\rho+\lambda}\mathbb{E}\left[(v_0^* - \zeta_1)^+\right].$$

The reason why we make the reference explicit to c_D, while omitting other constants, is that later on, when we shall consider the intervention case, c_D's value changes in time, according to (6).

Let us consider the problem with respect to the possible relationships of a and $v_0^*(c_D)$:

Case 1. $a > v_0^*(c_D)$. It is optimal to immediately deplete the liquid reserve, meaning liquidating the auxiliary bank immediately, see Lemma A.1 in (Hugonnier and Morellec 2017, eq. 21).

Case 2. $a \leq v_0^*(c_D)$. Recall that E_t^{α} is the cumulative earning at time t. Using the same notation in Hugonnier and Morellec (2017), we first define the auxiliary function

$$\psi(s; a, c_D) \doteq \mathbb{E}_s\left[e^{-\rho\,\delta_0}\,(a + E^{\alpha}_{\delta_0})^+\right], \qquad H(b; a, c_D) = \frac{1 - \psi'(b; a, c_D)}{W'(b; c_D)},$$

where δ_0 is the first time E_t^{α} becomes negative and $W(x; c_D)$ is the ρ-scale function of the uncontrolled liquid reserves process, which is defined by

$$W(x; c_D) = \sum_{i=1}^{3} \mathbb{1}_{x \geq 0}\, \frac{2\,(\beta + B_i)}{\sigma^2 \prod_{k \neq i}(B_i - B_k)}\, e^{B_i x}, \tag{10}$$

where $B_1 < -\beta < B_2 < 0 < B_3$ denote the three real roots of the cubic equation

$$B_i\left(\bar{\mu} - c_D - c_L + B_i\frac{\sigma^2}{2} - \frac{\lambda}{\beta + B_i}\right) - \rho = 0\,.$$

Intuitively , the ρ-scale function W is a solution of the characteristic function

$$\mathcal{L}_s W(s) = \rho\, W(s)\,,$$

where $\mathcal{L}_S$ is the generator of the uncontrolled process of reserve S_t.

Then the optimal strategy is of a barrier type with the optimal barrier $b^*(a) > 0$ satisfying

$$b^*(a) = \text{argmax}_b H(b; a, c_D)\,.$$

Denote by α_b the barrier strategy with barrier b, i.e., $Z_t^{\alpha_b} := \mathbb{1}_{\{t>0\}} \max_{0\le u<t}(X_u - b)^+$ and $X_t = s + E_t$. Then the expected value of the auxiliary problem

$$w(s; a, b, c_D) = \mathbb{E}_{s,c_D}\left[\int_0^{T^{\alpha_b}} e^{-\rho\, t}\, \mathrm{d}Z_t^{\alpha_b} + e^{-\rho\, T^{\alpha_b}}\,(a + S_{T^{\alpha_b}}^{\alpha_b})^+\right],$$

has the form

$$w(s; a, b, c_D) = \begin{cases} (a+s)^+ & \text{for } s \le 0 \\ \psi(s;a) + W(s; c_D)\, H\big(b; a, c_D\big) & \text{for } 0 < s \le b \\ s - b + w\big(b; a, b, c_D\big) & \text{for } s > b \end{cases}. \tag{11}$$

The optimal barrier $b^*(a)$ corresponds to the maximum value

$$w(s; a, b^*(a), c_D) = \max_{b\ge 0}\; w(s; a, b, c_D)\,.$$

Remark 1. *The* liberalism case *provides the cornerstone for possible extensions. Indeed, the value function in* (11) *depends exclusively on the parameter c_D, through the form of functions ψ, W, and v_0^*.*

After the financial crisis of 2008, the Financial Stability Board (FSB) and the G20 introduced a new tool called bail-in, see, e.g., Berger et al. (2022); Lambrecht and Tse (2023) and references therein, to address the failure of financial institutions and reduce the risk of financial contagion. The goal was to create a framework that would shift the cost of failure from taxpayers to shareholders and creditors, which is known as a "liberalism" strategy, see, e.g., Cayla (2022) for a discussion about it within the *Digital Economy* scenario. However, it's important to note that it is not possible to guarantee that this goal will always be achieved, as regulatory tools that were developed in response to past events may not be effective in different future situations. Even today, bail-in is still a relevant and ongoing topic.

The next section examines the case of government intervention, which can either be certain or uncertain. When government intervention is certain, it can be thought of as a deterministic problem, which is not particularly interesting from a mathematical or practical perspective. However, when government intervention is uncertain, it can be modeled as an optimal stochastic control problem that has a closed-end solution. This uncertain intervention case is particularly interesting when combined with multiple injections, as this is closer to the reality of the financial markets, where past government bailouts of financial institutions have created the perception of future implicit guarantees.

2.2. One-Time Injection with Uncertainty

Starting from the case of a government strategy based on liberalism, we proceed to examining the scenario of a single government intervention. Other studies, such as Neuberg et al. (2019), have looked at the effects of government intervention to rescue distressed banks in greater detail. In our analysis, we assume that the government's decision to save the bank is based on the bank's current financial condition, for example, its level of liquid reserves, liabilities, and deposits. Additionally, we assume that the bank has no choice but to accept the government's capital injection, as is typically the case in real-world scenarios. To begin, we introduce various parameters that define the different regions of a bank's reserve level S_t^α:

Definition 1. $s_b \le 0 \le s_c$ *are constants such that:*

- *If $S_t^\alpha \in (-\infty, s_b]$, the government will not save the bank and the bank has to declare bankruptcy.*
- *If $S_t^\alpha \in (s_b, s_c]$, the bank is considered as undergoing critical financial trouble and the government will save the bank's reserve to level R with probability $P(S_{t-}^\alpha, L, D)$ depending on the current reserve level S_{t-}^α, total value of deposits D, and liability L. Notice that the deposit rate $c_D(t)$ only changes once, according to (6).*
- *If $S_t^\alpha \in (s_c, +\infty)$, the bank is considered to be safe and the government will not intervene.*

Remark 2. *The case $s_b = s_c = 0$, resp. $s_b = -\infty, s_c = 0$, corresponds to the liberalism one, resp. to the case in which the government may save the bank even for an indeterminately large negative value. For $s_c > 0$, the sub-interval $[0, s_c]$ can be seen as a red zone where the bank will not face default provided the government does not intervene, nevertheless being a critical economic situation.*

Assuming that the government will only rescue the bank once, if the bank is indeed salvaged, the optimal expected profit after the first rescue will be the same as that in the case of liberalism. If we call T_c the first time the reserve goes under s_c, namely

$$T_c = \min\{t : S_{t^-}^\alpha \leq s_c\}\,,$$

the optimal expected profit starting from T_c becomes

$$g(s) \doteq \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\{S_{T_c^+}^\alpha = s\}} \left[\int_{T_c}^{T_b} e^{-\rho\, t} \mathrm{d}Z_t^\alpha + e^{-\rho\,(T_b - T_c)}\, l(S_{T_b}^\alpha) \right],$$

where T_b is the bankruptcy time of the bank, i.e., $T_b := \min\{t \,:\, S_{t^+}^\alpha \leq s_b\}$. Notice that if $S_{T_c^+} < s_b$, then $T_c = T_b$.

Having assumed that the salvage event follows a Bernoulli distribution with probability $P(S_{T_c^-}^\alpha, L, D)$, $g(s)$, we have

$$g(s) = P(s, L, D)\, w(R; a, b^*(a), c_D + \iota) + (1 - P(s, L, D))\, w(s; a, b^*(a), c_D + \iota)\,. \quad (12)$$

Indeed, (12) simply represents the optimal expected profit of the bank after the government's salvage. More explicitly, with probability $P(S_{T_c^-}^\alpha, L, D)$, the government decides to inject capital, i.e., $T_b > T_c$ and the optimal expected profit of the bank after the salvage time is $w(R; a, b^*(a), c_D + \iota)$. On the other hand, if the government decides not to inject capital, the bank will adopt the optimal strategy under liberalism and has the optimal profit $w\left(S_{T_c^-}^\alpha; a, b^*(a), c_D + \iota\right)$.

Back to the original optimal control problem (3) for the bank, since X^α is a Markov process, by dynamic programming, the value function (3) can be reformulated as

$$v(s) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{T_c} e^{-\rho\, t}\, \mathrm{d}Z_t^\alpha + e^{-\rho\, T_c}\, g(S_{T_c}^\alpha) \right], \quad (13)$$

where $g(s)$ is defined as (12). Therefore, after the salvage, the optimal strategy for the bank will be the optimal dividend distribution strategy in the liberalism case.

Remark 3. *In the following section, we will make use of the function $g(s)$ defined in (12). Specifically, it will be used in the derivation of the optimal policy and its uniqueness. For now, we will assume that the probability is a Bernoulli distribution with probability $P(S_{T_c^-}^\alpha, L, D)$ and the recovery level R is fixed. However, it is worth noting that the recovery level R can also be modeled as a random distribution whose probability distribution depends on the current liquidity s. In general, the government's salvage strategy is determined by the conditional probability*

$$\mathbb{P}\left(S_{T_c^+}^\alpha \in \mathrm{d}y \,\middle|\, S_{T_c^-}^\alpha = s, L, D\right) = \hat{\rho}(\mathrm{d}y; s, L, D), \quad y \geq s\,,$$

for some function $\hat{\rho}$. Then the function form of g defined in (12) becomes

$$g(s) \doteq \int_s^\infty w(y; a, b^*(a), c_D + \iota)\, \hat{\rho}(y; s, L, D)\, \mathrm{d}y. \tag{14}$$

Notice that (12) is just a special case of (14).

2.2.1. Viscosity Approach

Particular Case: Certain Government Intervention

As a starting step we consider the simplest case, hence taking $s_b = -\infty$, $s_c = 0$ and the salvage probability equal 1. In other words we remove the uncertainty, making the government rescue sure for the first time the bank faces negative liquidity. Therefore, for this particular case, we are fixing to one the probability of intervention, i.e., $P(S^{\alpha}_{T_c}, L, D) = 1$.

Consequently (12) simplifies to

$$g_{\{(-\infty,0],P=1\}}(s) \equiv w(R; a, b^*(a), c_D + \iota)\,, \tag{15}$$

which is independent of the bank's liquidity value and therefore the problem reduces to an optimal dividend distribution problem.

Let us denote by $\mathcal{L}_S v$ the second order integro-differential infinitesimal generator associated with the liquidity process S with no dividends and government intervention

$$\mathcal{L}_S v(s) = (\bar{\mu} - c_D - c_L)\,\partial_s v(s) + \frac{\sigma^2}{2}\,\partial^2_{ss} v(s) + \lambda \int_0^s v(s-\zeta) - v(s)\,\nu(\mathrm{d}\zeta)\,.$$

where ν is the exponential distribution associated with the wide shocks.

Hence the HJB is

$$\begin{cases} \min\{\rho\, v(s) - \mathcal{L}_S v(s),\ \partial_s v(s) - 1\} = 0 & \text{for } s > 0\,, \\ v(s) = w(R; a, b^*(a), c_D + \iota) & \text{for } s \le 0\,. \end{cases} \tag{16}$$

For the formal proof one can see, e.g., Avanzi et al. (2013); Yin and Yuen (2015).

Extension to the General Uncertainty Framework

By modifying the assumption beyond the dividend strategy as

$$Z_{t^+} - Z_t \le S_t + G_{t^+} - G_t - s_c\,,$$

the bank is not allowed to enter the red zone voluntarily and the HJB (16) can be extended to the general one time injection case by considering a liquidity dependent starting value function $g(s)$ given by (12) instead of the constant function $g_{\{(-\infty,0],P=1\}}(s)$. In what follows we provide an analogous result and determine the optimal barrier strategy. Furthermore, we provide an explicit solution of the problem.

Remark 4. *The HJB satisfied by the value function associated with the uncertainty framework will be crucial in the case of multiple injections, see next section.*

Proposition 1. *The value function $v(s)$ is the unique classical C^2 solution to*

$$\min\{\rho\, v(s) - \mathcal{L}_S v(s);\ \partial_s v(s) - 1\} = 0\,, \qquad s > 0\,,$$

subjected to the boundary condition

$$v(0^+) = \lim_{s\to 0} v(s) = g(s)\,,$$

and g given by (12).

Proof. The proof is a direct result of (Yin and Yuen 2015, sec. 3). □

2.2.2. Barrier Strategy Approach

We aim at explicitly computing the control strategy and the value function exploiting the dividend-penalty identity in Gerber et al. (2006). Let $b > 0$ and consider the barrier strategy α_b maintaining the reserve level at or below b. The corresponding value is denoted by

$$w(s;b) = \mathbb{E}_s\left[\int_0^{T_c} e^{-\rho t}\, \mathrm{d}Z_t^{\alpha_b} + e^{-\rho T_c}\, g(S_{T_c}^{\alpha_b})\right].$$

Here we omit the c_D reference in the definition of $\rho-$scale function $W(x;c_D)$ in (10) and simply denote it by $W(x)$. The present value of the terminal cost for the uncontrolled system is given by

$$\psi(s) = \mathbb{E}_s[e^{-\rho T_c} g(S_{T_c})]\,. \tag{17}$$

By the dividend-penalty identity in Gerber et al. (2006), we have

$$w(s;b) = \begin{cases} g(s) & \text{for } s \le 0 \\ \psi(s) + \frac{W(s)}{W'(b)}(1-\psi'(b)) & \text{for } 0 < s \le b \\ s - b + w(b;b) & \text{for } s > b \end{cases},$$

hence, the optimal barrier is determined by the minimizer b^* of the function

$$H(b) = \frac{1-\psi'(b)}{W'(b)}\,, \tag{18}$$

and, by verification theorem, the barrier strategy α_{b^*} is the optimal control to the problem. We are left to show that the minimizer b^* is unique. By (17) we have

$$\psi(s) = \mathbb{E}_s\left[e^{-\rho\, T_c}\, g(S_{T_c})\, \mathbb{1}_{\triangle S_{T_c}=0}\right] + \mathbb{E}_s\left[e^{-\rho\, T_c}\, g(S_{T_c})\, \mathbb{1}_{\triangle S_{T_c}>0}\right]. \tag{19}$$

The first term in (19) can be easily computed as

$$\mathbb{E}_s\left[e^{-\rho\, T_c}\, g(S_{T_c})\, \mathbb{1}_{\triangle S_{T_c}=0}\right] = \frac{g(s_c)\,\sigma^2}{2}\big(W'(s-s_c) - B_3\, W(s-s_c)\big)\,,$$

moreover, the potential measure of the uncontrolled liquid reserves process terminated at T_c evolves as follows

$$U(s,\mathrm{d}y) = \mathbb{E}_s\left[\int_0^{+\infty} e^{-\rho\, t}\, \mathbb{1}_{\{T_c>t\}\cap\{S_t\in\,\mathrm{d}y\}}\,\mathrm{d}t\right] = \left(e^{-B_3(y-s_c)}W(s-s_c) - W(s-y)\right)\mathrm{d}y\,,$$

and the second term in (19) can be computed as

$$\begin{aligned}
\mathbb{E}_s\left[e^{-\rho\, T_c}\, g(S_{T_c})\, \mathbb{1}_{\triangle S_{T_c}>0}\right] &= \int_{s_c}^{+\infty} \lambda\, U(s,\mathrm{d}y) \int_{y-s_c}^{+\infty} \beta\, e^{-\beta\, u}\, g(y-u)\,\mathrm{d}u \\
&= \int_{s_c}^{+\infty} \lambda\, U(s,\mathrm{d}y) \int_{-\infty}^{s_c} \beta\, e^{-\beta\,(y-u)}\, g(u)\,\mathrm{d}u \\
&= \int_{s_c}^{+\infty} e^{-\beta\, y}\, U(s,\mathrm{d}y) \int_{-\infty}^{s_c} \lambda\, \beta\, e^{\beta\, u} g(u)\,\mathrm{d}u\,.
\end{aligned}$$

By introducing the function

$$F(s_c) = \lambda\, e^{\beta\, s_c}\, \mathbb{E}[g(s_c - Y_1)] = \int_{-\infty}^{s_c} \lambda\, \beta\, e^{\beta u}\, g(u)\,\mathrm{d}u\,,$$

we have the following claim:

Claim 1. *Assume that*

$$s_c \leq \frac{1}{\beta} \log\left(\frac{\beta}{\beta + B_2}\right), \tag{20}$$

then we have

$$\psi(s) = \sum_{i=1}^{2} \alpha_i(s_c)\, e^{B_i(s-s_c)}, \quad \text{for } s \geq s_c\,, \tag{21}$$

with coefficients defined by

$$\alpha_i(s_c) = (-1)^i \left[\frac{2F(s_c) + g(s_c)(\beta + B_i)(\beta + B_3)\sigma^2}{(B_2 - B_1)(\beta + B_3)\sigma^2}\right] \; > \; 0\,.$$

Proof. Due to the non-negativity of $F(s_c)$ and $g(s_c)$, it is obvious that $\alpha_2 > 0$. On the other hand, using the fact that

$$\prod_{i=1}^{3}(\beta + B_i) + \frac{2\,\lambda\,\beta}{\sigma^2} = 0\,,$$

along with

$$F(s_c) = \lambda\, e^{\beta\, s_c}\, \mathbb{E}[g(s_c - Y_1)] < \lambda\, e^{\beta\, s_c}\, \mathbb{E}[g(s_c)] = \lambda\, \frac{\beta\, g(s_c)}{\beta + B_2}\,,$$

and the assumption (20), we have

$$\alpha_1(s_c) = \frac{2}{(B_1 - B_2)(\beta + B_3)} \left(F(s_c) - \lambda\, \frac{\beta\, g(s_c)}{\beta + B_2}\right) > 0\,.$$

□

Remark 5. *We can follow a similar process as outlined in Hugonnier and Morellec (2017) with the exception that our terminal condition is $g(s)$ instead of $(s+\alpha)^+$ as previously mentioned in Hugonnier and Morellec (2017). It is important to note that a key aspect that we use in this claim is the monotonicity of $g(x)$ in the range $(-\infty, s_c]$. By utilizing the information provided in Hugonnier and Morellec (2017), we can conclude that there is a unique optimal barrier, $b^*(s_c)$, for any value of $s_c > 0$ and it also serves as the optimal solution to Equation (13).*

3. Semi-Markov Multiple Injection

The previous section thoroughly examined the case of a one-time government injection. However, in reality, government assistance is not limited to just one instance. Usually, a "red zone" is established, represented by the range $[s_l, s_u]$. If a bank's situation falls within this red zone, the government will provide aid with varying intensity. In the one-time injection case, the government can save the bank only at the first time in which S_t^α falls below the threshold s_c. In contrast, in the case of multiple injections, aid is given with some probability density whenever $S_t^\alpha \in [s_l, s_u]$. In both cases, the bank can continue operations as long as $s_l < S_t < s_u$, but must declare bankruptcy if $S_t \leq s_l$. It is worth noting that a specific scenario of the setting being considered can be obtained by setting $s_l = 0, s_u = s_c$. Another important feature of the multiple injection case is that the salvage rate $\theta(t)$ for the government is no longer a Markov process, namely, it is not of the form $\theta(S_t)$. Instead, we assume that the salvage rate $\theta(t)$ also depends on time passed since the last salvage event, denoted by h. In other words, the salvage rate $\theta(t)$ is of the form $\theta(S_t, h(t))$. Let us also note that, since both L and D are fixed, we drop them from the parameters list.

If we denote the salvage time sequence by $T_0 = 0 < T_1 < T_2 < \ldots < T_n < \ldots$, then h is defined as

$$h(t) = \sum_{i=0}^{+\infty} (t - T_i)\, \mathbb{1}_{t \in [T_i, T_{i+1})}\,.$$

Notice that T_n can be $+\infty$ and in this case $T_i = +\infty$ for all $i \geq n$. In other terms, we have that the distribution of the next jump T_i at time $t \in [T_{i-1}, T_i]$ is given by the following formula

$$\mathbb{P}\big(T_i > r \,\big|\, S_t = s,\, T_{i-1} = t - h\big) = \exp\left(-\int_h^{h+r-t} \theta(s,k)\,\mathrm{d}k\right), \quad \text{for } r > t\,, \tag{22}$$

that is, we consider the strong Markov process $\{(S_t, h(t)),\, t \geq 0\}$ to be equal to (s, h) at time t. From now on we will denote the conditional probability in Equation (22) by $\mathbb{P}_t^{s,h}$, that is the probability conditioned by $S_t = s$ and $h = t - T_{i-1}$.

Remark 6. *It is worth noting that h represents the amount of time passed since the last salvage event in which the bank has survived. Once a salvage event is triggered at time t, h resets to 0. It is thus reasonable to assume that $\theta(s,h)$ is an increasing function of h. Indeed, the higher h, the healthier the bank and the government is thus more eager to save the bank.*

There are still many options for determining the specifics of the multiple injection model. On the one hand, the relationship between θ and s is not fixed, and it is generally assumed that θ increases as s increases within the range of $s \in [s_l, s_u]$ and is equal to zero outside of this range. On the other hand, the amount of capital injection can also be subject to randomness. Specifically, the jump component in $S_{T_i^+}$ can be a random variable defined in the range of $[s_u - s, +\infty]$ and its dynamics are determined by the following equation.

$$\mathrm{d}G_t = \int_{s_u - S_t}^{+\infty} \gamma\, \mathcal{N}^{S_t, h_t}(\mathrm{d}t, \mathrm{d}\gamma)\,, \tag{23}$$

where its associated Lévy measure is denoted by $A \mapsto \mathcal{N}^{s,h}(t, A)$, or alternatively $\hat{\nu}$: $A \mapsto \hat{\nu}(s, h, A)$, with jump rate function given by $\theta(s,h)$ for $A \subset [s_u - s, \infty]$. Therefore, the conditional probability that the process associated with the government intervention is in A at a jump time, see (22), is given by

$$\mathbb{P}_t^{s,h}\big(G_{T_i} - G_{T_{i-1}} \in A \,\big|\, T_i = t\big) = \hat{\nu}(s, h, A)\,, \tag{24}$$

that is, with (24) we define the conditional probability of the jump process ΔG_t immediately after a jump at time $T_i = t$. The arrival times are exponentially distributed with intensity θ, i.e.,

$$\mathbb{P}^{s,h}(T_1 > r) = \exp\left(-\int_h^r \theta(s,k)\,\mathrm{d}k\right). \tag{25}$$

We denote the number of salvage events happening between time 0 and time t by $N^G = N^G(t)$, defined as

$$N^G(t) = \sum_i \mathbb{1}_{T_i \leq t}\,.$$

We assume that the jump process has finite activity, which ensures that there will be a finite number of jumps within a finite time interval. This assumption can be satisfied by adding an upper barrier $N^{G,\max} > 0$, which represents the maximum number of events. Once this maximum number of events has been reached, no further action by the government or central bank should occur.

Finally, the semi-Markov multiple injection problem can be formulated as

$$v(s,h) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{s,h}\left[\int_0^{T^\alpha} e^{-\rho\, t}\mathrm{d}Z_t^\alpha + e^{-\rho\, T^\alpha}\, l(S_{T^\alpha})\right], \tag{26}$$

subject to

$$S_t = s + E_t - Z_t + G_t\,,$$

with $S_0 = s$ and the previous capital injection happening in $T_{-1} = -h$, possibly $-\infty$, and where G_t is the cumulated value of the pure jump process defined in Equations (22)–(24).

The generator of $v(s,h)$ becomes

$$\tilde{\mathcal{L}}v(s,h) = \partial_h v(s,h) + (\bar{\mu} - c_D - c_L)\,\partial_s v(s,h) + \frac{\sigma^2}{2}\,\partial^2_{ss} v(s,h) + \lambda\int_0^{s-s_l} v(s-\zeta,h) - v(s,h)\,\nu(\mathrm{d}\zeta)\,. \tag{27}$$

It is important to note that the support of the function $\theta(s,h)$ is (s_l, s_u), meaning that the government will not provide aid to the bank for any reserve level s that is not within this range. The choice of $\theta(s,h)$ is flexible, for example one can choose the function $\theta(s,h) = (s - s_l), \mathbb{1}_{s\in(s_l,s_u)}, C_1, (1 - e^{-C_2,h})$, where $C > 0$ is a constant. It is important to note that, when using this form of the function, the salvage rate density increases with respect to h for any fixed $s \in (s_l, s_u)$. An example simulation of the dynamics involved in the determination of the probability of Central Bank intervention can be seen in Figure 1. In this scenario, we assumed some possible fixed values for the parameters in Equation (1) and also set that, if the reserve level drops by more than 20% from the current level, the bank is in a very critical economic condition. This means that, at the starting time, the bank is already in a stressed condition. Of course, other assumptions can be considered but, for the purpose of seeing possible multiple government interventions in the next five years, we chose this starting point. If the bank reserves drop by just 2%, the bank is in the "red region", meaning that the government has the possibility to inject capital in order to contribute to the health of the bank. We also assumed some intensity of the negative jumps that may potentially bring the bank close to collapse despite being very liquid. This situation is similar to what can be seen during the great financial crisis, when some banks were assumed to be very liquid but, after a dramatic drop in their equity value, they were close to bankruptcy. Here, we do not consider the intervention event, which would depend on the intervention rate, in order to see the "natural" evolution of the processes A and θ.

The HJB equation after the introduction of the multiple injection feature becomes

$$\begin{cases} \min\big\{\rho\, v(s,h) - \tilde{\mathcal{L}}v(s,h) - \mathcal{H}\big(s,h,v(\cdot,0) - v(s,h)\big)\ ;\ \partial_s v(s,h) - 1\big\} = 0 & \text{for } s > s_l, \\ v(s,h) = l(s) & \forall\, s \le s_l,\ h \ge 0\,, \end{cases} \tag{28}$$

where $\mathcal{H}$ is given by the following expression

$$\mathcal{H}(s,h,z(\cdot)) = \theta(s,h)\int_{s_u - s}^{+\infty} z(s+\gamma)\,\hat{\nu}(s,h,\mathrm{d}\gamma)\,. \tag{29}$$

Let us also underline that, when $s > s_u$, $\theta(s,h)$ becomes 0 and the last term in (27) disappears.

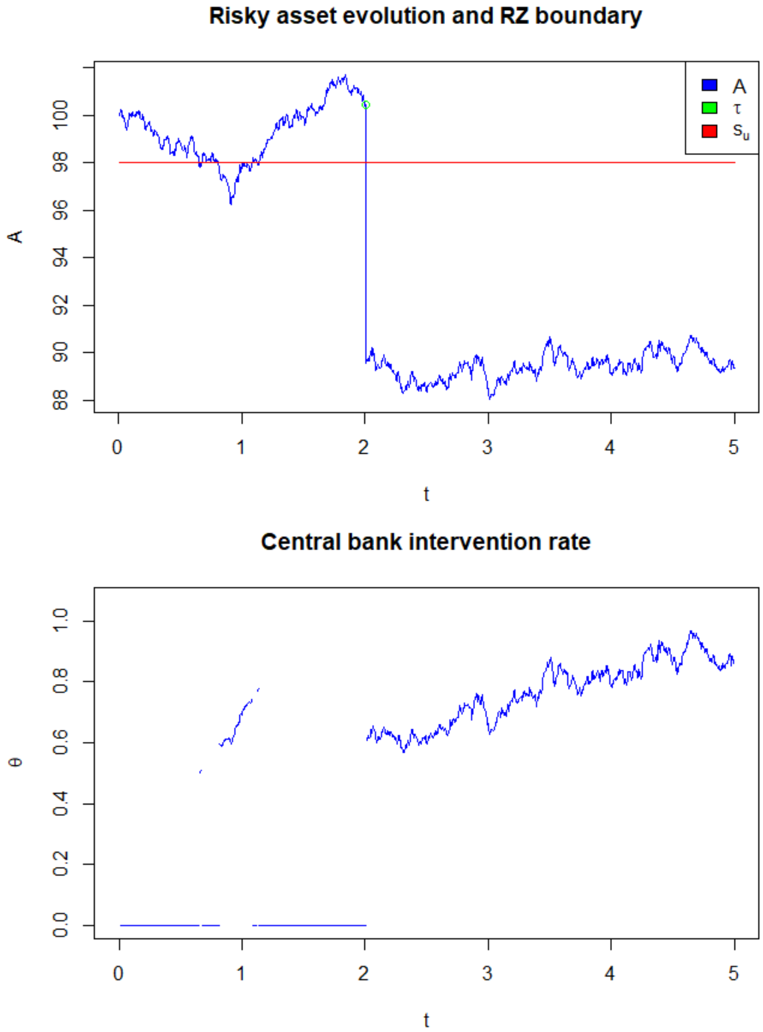

Figure 1. Let us consider a red zone $RZ = [s_l = 80, s_u = 98]$, $s_0 = 100$, $\overline{\mu} = 0.1$, $\sigma = 2$, $T = 5$. We assume that the compound Poisson process described in Equation (1) has an intensity of $\lambda = 1/3$, meaning that one jump is expected every three years. The jump size is assumed to be distributed as a gamma variable with shape parameter 2 and scale parameter 2.5, resulting in a jump size expectation of 5. In the upper figure, the dynamics of the bank's risky asset is shown, with the green circle τ representing the negative jump and the red line representing the upper bound of the red region. The lower figure shows the dynamics of $\theta(\cdot, \cdot)$ assuming no dividends and no government intervention over the 5 years considered, and with the parameters C_1 and C_2 equal to $1/100$ and $1/2$, respectively. For this analysis, it was assumed that $h_0 = 0$. It can be observed that, after one year, the government is eager to help the bank as the bank's value is far away from the lower boundary of the red region. However, as time goes by, the government increases the intensity of intervention as the time since the last salvage event has increased.

Uniqueness of the Solution

It is worth noting that the arguments in the min-operator in (28) for $s > s_l$ can be split into two parts

- The jump part corresponding to the government intervention, that is

$$\mathcal{G}v(s,h) = \partial_h v(s,h) + \theta(s,h) \int_{s_u - s}^{\infty} [v(s+\gamma,0) - v(s,h)]\hat{\nu}(s,h,\mathrm{d}\gamma) \; ; \tag{30}$$

- The residual part which equals the one already considered in Sections 2.1 and 2.2.

We will focus on the component corresponding to the multiple injection of the government intervention, that is, Equation (30).

The main result of this section is to show that (26) is the unique viscosity solution of (28). To prove it, we consider the following standard assumptions on the regularity of θ and $\nu^s(A) \doteq \hat{\nu}(s + A)$:

- $\theta(\cdot,\cdot)$ is measurable on its domain;
- There exists a constant dominating θ on its whole domain, i.e., $\sup \theta(s,h) \leq C$, $\forall\, s,h$;
- $A \mapsto \hat{\nu}(s,h,A)$ is measurable, $\forall\, A \subset [s_u - s, \infty]$ and all s,h;
- $\hat{\nu}^{s,h}(A) := \nu(s,h,A)$ is a probability measure on $[s_u - s, \infty]$.

see (Bandini and Confortola 2017, Hyp. 2.1) for further details. Notice that $\theta(\cdot)$ satisfies the above assumptions. As a result, we consider a setting similar to the one presented in Bandini and Confortola (2017) and we have the following result:

Corollary 1. *Let us assume that the penalization component of the value function, that is $e^{-\rho\, T^{\alpha}}\, l(S_{T^{\alpha}})$, is measurable and bounded. If we further assume* (29) *to be measurable with respect to its sigma-algebra and bounded and to satisfy a growth condition, see (*Bandini and Confortola 2017*, Hyp. 2.3), then we have that Equation* (28) *has a unique solution in the viscosity sense and this solution is given by* (26)*.*

Proof. This corollary is a particular case of Theorem 4.7 in Bandini and Confortola (2017), see also Proposition 1. □

Clearly the liquidation payment function $l(\cdot)$ defined in (4) satisfies the assumption in Corollary 1.

4. Conclusions

In this work, we have discussed multiple government approaches to addressing the problem of banks experiencing financial difficulties from a liquidity and solvency perspective. The first approach, liberalism, and the second approach, transparency, both involve deterministic control functions. The liberalism approach does not include government interventions and banks are allowed to go bankrupt if they cannot meet their liabilities. The transparency approach, on the other hand, is a purely transparent strategy in which the system knows when the government will rescue banks in trouble. However, this strategy is not optimal as banks tend to take on more risk while keeping their capital structure within the "salvage bands". To address this limitation, we considered a third strategy that introduces randomness in the government's optimal control function. The government's injections are determined by solving a Hamilton–Jacobi–Bellman equation. Although this equation does not have a closed-form solution, it can be accurately approximated numerically through finite element methods, which is a step we plan to consider in future projects. The present study has focused on analyzing a realistic framework in which the government's strategy is a compound Poisson process with random intensity. The resulting capital dynamics are a jump-diffusion process and the optimal problem is a semi-Markov value function. We have proven the uniqueness of the solution to the problem and provided a scenario analysis to facilitate understanding of the main players involved.

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Note

1 Notice that, for some time t, the risky asset value A_t could become negative. Negative asset values are never considered in the model since the optimal control problem is considered for times prior to T^α. See later sections for a formal definition of the stopping time T^α.

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