A Discrete Trinomial Model for the Birth and Death of Stock Financial Bubbles

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Abstract. The present work proposes a novel way to model the dynamic of financial bubbles. In particular we exploit the so called *trinomial tree* technique, which is mainly inspired by the typical market order book (MOB) structure. According to the typical MOB rules, we exploit a bottom-up approach to derive the relevant generator process for the financial quantities characterizing the market we are considering. Our proposal pays attention in considering the real world changes in probability levels characterizing the bid-ask preferences, focusing the attention on the market movements. In particular, we show that financial bubbles are originated by these movements which also act amplify their growth.

Introduction

In this paper we analyse the relation between bubbles birth, their development and and how stocks are exchanged through the so called *Market Order Books*. Inspired by the typical MOB structure, observed in real markets, and its functioning rules, we derive a model for stock prices. In particular we exploit a methodology based on the *trinomial tree* approach, see, e.g., [5]. Studying the typical MOB structure, it can be derived that a trader who wants to buy or sell a certain security can place his orders following two different way, namely the *limit order*, resp. the *market order*. At each time the MOB consists of a queue of buyers and a queue of sellers. When there is an overlap between these two queues, the intersection of all orders will be transacted, consequently a new price is formed. In this paper we focus our attention on the point of view of theories that have the great investor as the triggering factor for the birth of the bubble. In order to model this phenomenon we consider the probability changes for the bid-ask preferences as a consequence of an order settled by a great investor.

The setup

We consider a market consisting of a single stock which is traded through a *Limit Order Book* (LOB). LOBs are typically characterized by different bid-ask levels, hence the resulting structure could easily become rather nested. In order to simplify this structure, we will consider just two order book levels, namely the first bid level, resp. the first ask level. Concerning the first ask level, we set its position equal to 0, with quantity $q_{n,0}$ which represents the maximum amount that can be traded at time n. Furthermore, the first bid level has position 1, with related quantity $q_{n,1}$. The bid-ask spread is fixed at level 2ε, where $\varepsilon > 0$. The dynamic of the order book will be determined by orders arrival. Each trading day traders put their orders in the order book, and we have $0 \leq n \leq N$ trading moments during the day. Orders arrivals are randomly distributed over time, moreover their magnitude is random too and each order can be of bid, or ask type. In our treatment we model orders arrivals supposing that at each time $0 \leq n \leq N$, we have an order with given probabilities. The latter is not the only source of randomness in the order book. In particular, we can identify three sources of randomness: orders arrivals, bid or ask level of the order and magnitude of each single order. The first two sources of randomness are binary since an order can arrive or not, and the order can be of bid or ask type, only. Concerning the quantity of each order, we have to consider a particular probability density function. Furthermore, traders can make limit orders or market orders. While the first type of orders can only decrease the chosen level, because a market order is traded at the best bid or ask level, limit orders can also add quantity. We will model this property of orders by assuming a symmetric probability distribution and considering the sign of the quantity as a characterizing factor. If the quantity is positive we will add, alternatively we will subtract.

For the sake of clarity, let us consider a simple example of the considered order book dynamic:

- $n = 0,\ q_{n,0} = 50,\ q_{n,1} = 38$. We have an order. The order is in the bid level (position 1). The quantity of the order is -20.
- $n = 1,\ q_{n,0} = 50,\ q_{n,1} = 18$. We have no order.
- $n = 2,\ q_{n,0} = 50,\ q_{n,1} = 18$. We have an order. The order is in the ask level (position 0). The quantity of the order is +50.
- $n = 3,\ q_{n,0} = 100,\ q_{n,1} = 18$. We have an order. The order is in the bid level (position 1). The quantity of the order is -30.
- $n = 4,\ q_{n,0} = 100,\ q_{n,1} = 12$. We have no order.

In the last two steps of the above example we can see a change in the price. Indeed, the last order has filled completely the bid quantity, hence a new price is formed. The remaining quantity $12 = 30 - 18$ has to be considered as the new quantity for the new bid level. The latter agrees with the rule that prescribes the price determination whenever the ask or the bid level is completely filled by an order, namely when there is an overlap between the bid queue and the ask queue. In particular, at time $n + 1$ the price X_{n+1} can be $X_n \pm \varepsilon$, or still X_n if there is no overlap. For example if a trader buy $q_{n,0}$ price shares at $X_n + \varepsilon$, then we obtain a new price $X_{n+1} = X_n + \varepsilon$. Equivalently, if a trader sell $q_{n,1}$ price shares at $X_n - \varepsilon$, this implies the new price $X_{n+1} = X_n - \varepsilon$. Otherwise, we remain with the *old* price $X_{n+1} = X_n$.

In order to formalize our approach, let us define the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, and let $o_n : \Omega \to \mathbb{R}$ be a random variable on it, that we use to describe the order arrival, within $0 \le n \le N$. We start supposing $o_n = 0$, with probability $0 \le \lambda \le 1$ and $o_n = 1$, with probability $1 - \lambda$, so that

$$o_n(\omega) = \begin{cases} 0 & \mathbb{P}(o_n = 0) = \lambda \\ 1 & \mathbb{P}(o_n = 1) = 1 - \lambda \end{cases} .$$

Let $P_n : \Omega \to \mathbb{R}$ be a random variable describing the bid-ask level of the market order, once this is arrived. According with the aforementioned description, we have

$$P_n(\omega) = \begin{cases} 0 & \mathbb{P}(P_n = 0) = p \\ 1 & \mathbb{P}(P_n = 1) = 1 - p \end{cases} ,$$

where $0 \le p \le 1$. Moreover, we suppose that $q_n^* \sim N(\mu, \sigma^2)$, where q_n^* is the order quantity, whose distribution is assumed to be of Gaussian type. We underline that such a probability density choice is mainly inspired by calibration aims, see below. We remind here that, once an order is arrived and the trader has taken his position, a negative quantity corresponds to a *subtraction*, resp. a positive quantity corresponds to an *addition*.

Since a new price is formed whenever an order arrives and the quantity q_n^* of the market order is less or equal than $-q_{n,0}$, resp. than $-q_{n,1}$ for a bid order, then the price at time $n + 1$ reads as follow

$$X_{n+1} = X_n + \varepsilon\left[\mathbb{1}_{\{o_n=1\}}\left(\mathbb{1}_{\{q_n^*<-q_{n,0}\}}\mathbb{1}_{\{P_n=0\}} - \mathbb{1}_{\{q_n^*<-q_{n,1}\}}\mathbb{1}_{\{P_n=1\}}\right)\right],$$

and we have

$$X_{n+1}(\omega) = \begin{cases} X_n + \varepsilon & \mathbb{P}(X_{n+1} = X_n + \varepsilon) = \lambda \cdot p \cdot N(-q_{n,0}) \\ X_n - \varepsilon & \mathbb{P}(X_{n+1} = X_n - \varepsilon) = \lambda \cdot (1 - p) \cdot N(-q_{n,1}) \\ X_n & \mathbb{P}(X_{n+1} = X_n) = 1 - \lambda(p \cdot N(-q_{n,0}) - (1 - p) \cdot N(-q_{n,1})) \end{cases} .$$

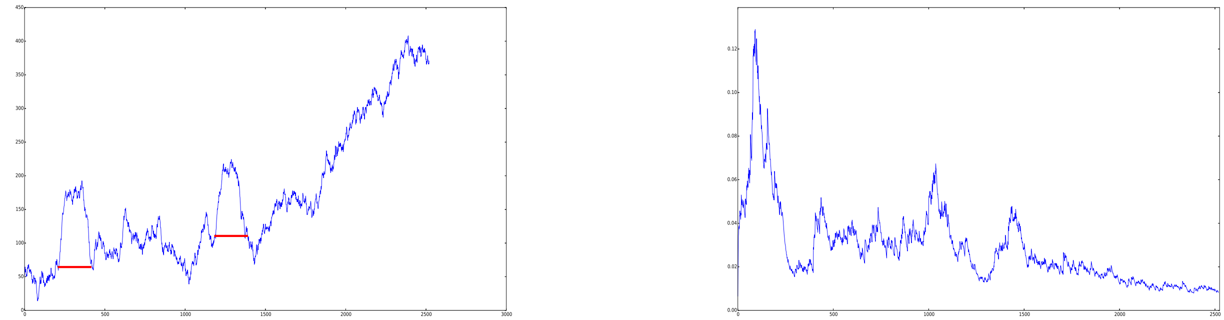

Figure 1. Left: Example of simulated price process. Right: Volatility fitted using GARCH methodologies. Second Scheme.

The birth of a bubble

Empirical evidence, see, e.g., [1, 2, 3, 11] and references therein, or [10] about Nasdaq behaviour during the 1990s or, concerning energy markets peculiarities, [4], or [6, 7] for the analysis of specific assets behavior, also in incomplete markets, see [8], show how great order arrivals push traders to change their opinions about future investments. In our model, this fact reflects in a change of the probability level p. From our perspective, a huge order arrives whenever q_n^* is under a certain threshold. Since, we have supposed $q_n^* \sim N(\mu, \sigma^2)$, then we will chose a threshold implying a small probability for huge order to happen. If a huge order arrives, we consider the realistic case of many small traders that start to follow such a possible *new trend*. This is the idea behind our model of bubble bursting. In particular, small traders memorize the great order time arrival, changing the bid/ask level choice probabilities, accordingly, then giving rise to the financial bubble formation by imitation.

Let us consider bubbles with fixed duration t_{vol}. Concerning their *decay time*, let us outline that whenever we are in a bubble regime, we can estimate from empirical data the point in time from which the bubble starts to deflate. Once we have fixed the duration of our bubble, we will model this property considering the changes in probability until we arrive at $\frac{t_{vol}}{2}$ and after that we use reversed probabilities. As an example, let assume that $p = 0.5$, and suppose that at time n a huge order ($o_n = 1, P_n = k, q_n^* < -5.6$) arrives, where k is fixed and can be 0 or 1. When $n \in [n+1, N]$ and in the next $t_{vol}/2$ days, the probabilities of the P_n process changes, hence

$$P_n^{\leq t_{vol}/2}(\omega) = \begin{cases} k & p_1 = 0.6 \\ 1-k & 1-p_1 = 0.4 \end{cases} .$$

From $t_{vol}/2$ to t_{vol}, we invert probabilities so that we can have the deflation of the bubble. In particular

$$P_n^{> t_{vol}/2}(\omega) = \begin{cases} k & p_1 = 0.4 \\ 1-k & 1-p_1 = 0.6 \end{cases} .$$

We define latter scheme as the *first scheme*. Let us underline that in such a scenario we have considered instantaneous jumps in probability, as to realize a more immediate description. However probabilities can increase, resp. decrease, following more complex dynamic, see, e.g., [9]. Indeed, in real world setting financial bubbles are characterized by three phases: their birth, their death and a third period between the latter, characterized by a relatively smooth volatility behavior. Such a middle time interval has a typical duration much more longer than the bubble rise and fall periods. Therefore, to improve our model, we divide the bubble life time interval in four parts, then definingg what we call the "*second scheme*".

Simulation results

In this section we show by simulations the results obtained using our model. In particular, we consider 10 financial years, hence $n^* = 252 \cdot 10$, and we simulate the dynamic of our order book in $N = 10000$ financial intra-day moments, having also fixed $\varepsilon = 0.2$, and $X_0 = 100$. We suppose that the initial order book is normalized, being characterized by $\{q_{0,0} = 1, q_{0,1} = 1\}_{n\geq 0}$. Moreover the orders quantities are distributed as $q_n^* \sim N(\mu = -1, \sigma^2 = 1)$. It is worth to mention that we have chosen $\mu = -1$ because if we consider $\mu = 0$, then we have 50% of having an order adding, resp. subtracting, some quantity to the bid/ask chosen level. The latter implies a non realistic dynamic for our price process. Therefore, our model works if we assign a greater probability of having subtracting, instead of addition, operations on bid/ask level. Regarding probabilities, we set $\lambda = \frac{1}{2}$, $p = \frac{1}{2}$. Great orders arrive within the threshold $q_n^* < -5.6$, and we fix $t_{vol} = 200$ and $p_1 = 0.515$. Figure 1 shows the daily price process, namely the last price of each simulation of the intra-day price process $\{X_n\}_{0\leq n\leq N}$. In particular, we have considered and plotted the new sequence $(X_{N_1}, X_{N_2}, X_{N_3}, ..., X_{N_{n^*}})$, where X_{N_i} is the last price of the i-th day.

Conclusion

In this paper we have derived a discrete approach to model financial bubbles with particular attention to the microstructure of the market in terms of MOBs. In particular, we have described, exploiting a constructive methodology that endogenously determines price processes, how microeconomics trading interactions can determine a shift regarding the probabilistic traders' perspectives. We have also provided a model that, starting from the changes in the bid-ask preferences, determines the formation of a bubble. Numerical simulations based on our model have been also reported, showing that our setting is rather flexible, being not linked to particular probability distribution for the considered random quantities. Furthermore, it can be used to capture very different types of financial and socio-economics factors. Its generalization to consider its continuos counterpart constitutes an ongoing research by the authors, particularly with respect to the general local martingale approach.

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