Infinite Dimensional Analysis, Quantum Probability and Related Topics


# GAUSSIAN ESTIMATES ON NETWORKS WITH DYNAMIC STOCHASTIC BOUNDARY CONDITIONS

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In the present paper we prove the existence and uniqueness for the solution to a stochastic reaction–diffusion equation, defined on a network, and subjected to non–local dynamic stochastic boundary conditions. The result is obtained by deriving a Gaussian-type estimate for the related leading semigroup, under rather mild regularity assumptions on the coefficients. An application of the latter to a stochastic optimal control problem on graphs, is also provided.

Keywords: Stochastic differential equations, semigroup theory, infinite dimensional setting, stochastic optimal control, evolution equations on networks, non-local boundary conditions, stochastic dynamic conditions, Gaussian estimates.

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## 1. Introduction

Starting with the introductory work [21], where elliptic operators acting on suitable functions spaces on network have been first introduced, several works related to a wide set of physical phenomena whose dynamics are carried out on graphs, have appeared, e.g., concerning the study of heat diffusion, see, e.g. [23], applications to quantum mechanics, see, e.g., [27], the stochastic modelisation of neurobiological activities, particularly with respect to the analysis of the *FitzHugh-Nagumo equation*, see, e.g., [1, 4, 5, 9], and references therein, the quest for invariant measures, see, e.g., [2], the problem of suitable types of estimates, as in the case of the Gaussian one, see, e.g., [17], and references therein, etc.

A powerful technique often used to address aforementioned problems, consists in introducing a suitable infinite dimensional product space and then study the diffusion problem exploiting a semigroup theory approach, see [24] and references therein, for a detailed analysis of the latter subject. Moreover, to what concerns standard problems of existence and uniqueness for the solution of a diffusion problem, as well as the spectral properties of related the leading semigroup, the attention has often been put on the determination of proper boundary condition for the particular diffusion problem one is interested in.

When the focus is on diffusion problems governed by a second order differential operator, then typical boundary condition are the so-call *generalized Kirchhoff conditions*, see, e.g., [23]. Nevertheless, during recent years, also different type of rather general boundary conditions has been proposed. The latter is the case, e.g., of non-local boundary conditions, allowing for non-local interaction of non-adjacent vertex of the graph, see, e.g., [9, 17], dynamic boundary conditions, see, e.g., [5, 25], etc.

The main goal of the present work is to generalize previously mentioned approaches in order to achieve a unified perspective. We will start from a completely general non-local diffusion problem, endowed with non-local boundary conditions which will be both dynamic and static. In such a setting, we state our main result, namely we prove a Gaussian upper bound for the semigroup generated by a proper infinitesimal generator acting on a suitable Hilbert space. We would like to underline that latter type of bound turns out to be extremely powerful when one wants to prove existence and uniqueness of a solution to a stochastic partial differential equation (SPDE), since this immediately leads the operator to be *Hilbert-Schmidt*, allowing to relax regularity assumptions on the coefficients of the SPDE.

The general approach that can be used to show the *Hilbert-Schmidt property* of the leading semigroup, typically relies on the study of its spectral properties. However it is not always possible to give a precise characterization of the semigroup eigenvalues, particularly whit respect to diffusive problems on a graph. In such a case a complete characterization of the spectrum can be obtained by considering the topological structure of the graph. Alternatively, one can try to derive a *heat kernel* which leads to prove a Gaussian upper bound for the semigroup. The latter approach will be the one we will pursue in the present paper.

The work is so structured, in Section 2, exploiting the theory of sesquilinear form, we will introduce a suitable infinite dimensional space, showing that our equation can be rewritten as an infinite dimensional problem where the differential operator generates a strongly continuous analytic semigroup, hence obtaining the well-posedness of the abstract Cauchy problem. Then, in Section 2.2, we will prove a Gaussian estimates for the operator, while in Section 3 a suitable stochastic multiplicative perturbation will be introduced in order to show both the existence and the uniqueness of a *mild solution*, in a suitable sense, under rather mild assumptions on the coefficients. Eventually, in Section 4, a stochastic optimal control application will be proposed.

## 2. General framework

Let us consider a finite connected network identified with a graph $\mathbb{G}$ composed by a finite number $n \in \mathbb{N}$ of vertices, indicated by $\mathrm{v}_1, \dots, \mathrm{v}_n$ and linked by a finite number $m \in \mathbb{N}$ of edges, indicated by $e_1, \dots, e_m$ and assumed to be of unitary length. For the sake of readability, let us also introduce the following notations: we use Latin letters $i, j, k = 1, \dots, m$, to denote quantities related to edges, so that $u_i$ will stand for a function on the edge $e_i$, for $i = 1, \dots, m$; while we use Greek letters $\alpha, \beta, \gamma = 1, \dots, n$, to denote quantities related to vertices, so that $d_\alpha$, $\alpha = 1, \dots, n$, will be the values of the unknown function evaluated at the vertex $\mathrm{v}_\alpha$, with $\alpha = 1, \dots, n$.

In order to describe the structure of the graph $\mathbb{G}$ we will exploit the *incidence matrix* $\Phi := (\phi_{\alpha,i})_{n\times m}$, see, e.g., [24], which is defined as follows: $\Phi := \Phi^+ - \Phi^-$, where the sum is intended componentwise, with $\Phi^+ = \left(\phi^+_{\alpha,i}\right)_{n\times m}$, resp. $\Phi^- = \left(\phi^-_{\alpha,i}\right)_{n\times m}$, is the *incoming incidence matrix*, resp. the *outgoing incidence matrix*. In particular, both of them have value 1, whenever the vertex $v_\alpha$ is the initial point, resp. the terminal point, of the edge $e_i$, and 0 otherwise. The latter implies that

$$\phi^+_{\alpha,i} = \begin{cases} 1 & \mathrm{v}_\alpha = e_i(0)\,, \\ 0 & \text{otherwise} \end{cases}, \quad \phi^-_{\alpha,i} = \begin{cases} 1 & \mathrm{v}_\alpha = e_i(1)\,, \\ 0 & \text{otherwise}\ . \end{cases}$$

Aforementioned definition is consistent with the idea that if $|\phi_{\alpha,i}| = 1$, then we the edge $e_i$ is called *incident* to the vertex $\mathrm{v}_\alpha$, and it remains defined the set

$$\Gamma(\mathrm{v}_\alpha) = \{i \in \{1, \dots, m\} \,:\, |\phi_{\alpha i}| = 1\}\,,$$

of all the *incident edges* to the vertex $\mathrm{v}_\alpha$.

In order to consider the most general framework, we allow the dynamic of the unknown function $u$, defined on the network, to depend non-locally on the underlying graph $\mathbb{G}$, which implies to take into account non-local interactions, namely the process taking place on the edge $e_i$ can be affected by the process that takes place on the edge $e_j$, $i, j = 1, \dots, m$, even if the edge $e_j$ is not directly connected with the edge $e_i$.

We also introduce, see [11], the *ephaptic incidence tensor*, which is defined as follows

$$\mathcal{I} := \mathcal{I}^+ - \mathcal{I}^-\,, \quad \mathcal{I}^+ := \Phi^+ \otimes \Phi^+\,, \quad \mathcal{I}^- := \Phi^- \otimes \Phi^-\,,$$

being $\otimes$ the Kronecker product of two $n \times m$ matrices, defined as

$$(A \otimes B)^{\alpha i}_{\beta j} := a_{\alpha i} b_{\beta j}\,,$$

in particular $(A \otimes B)$ is a $n^2 \times m^2$ matrix and, in our case, it is worth to mention that the matrix $(A \otimes B)$ is symmetric.

Using previous notation, in what follows we will denote by $\iota^{\alpha i}_{\beta j}$, resp. ${}^+\iota^{\alpha,i}_{\beta,j}$, resp. ${}^-\iota^{\alpha i}_{\beta j}$, the entries of the matrix $\mathcal{I}$, resp. of the matrix $\mathcal{I}^+$, resp. of the matrix $\mathcal{I}^-$.

*Remark* 2.1. We underline that the entry $\iota^{\alpha i}_{\beta j}$ represents the influence that the vertex $\mathrm{v}_\beta$, as an endpoint of the edge $e_j$, plays on the vertex $\mathrm{v}_\alpha$ which is an endpoint of the edge $e_i$.

We will thus define the *weighted incidence tensor* $\mathcal{D} = \left(\delta^{\alpha,i}_{\beta,j}\right)$, $\alpha$, $\beta = 1, \dots, n$, $i$, $j = 1, \dots, m$, as follows

$$\delta^{\alpha i}_{\beta j} = c_{ij}(\mathrm{v}_\beta)\iota^{\alpha i}_{\beta j}\,, \tag{2.1}$$

where the function $c$ is a smooth enough function that we will specify later on.

Eventually, we consider two different type of boundary conditions. In particular we will assume that the vertices $\mathrm{v}_\alpha$, $\alpha = 1, \dots, n_0$, $1 \le n_0 \le n$, have some non-local static generalized *Kirchhoff type* conditions, whereas we equip the remaining nodes $\mathrm{v}_\alpha$, $\alpha = n_0 + 1, \dots, n$, with some non-local dynamic boundary conditions.

Let us thus consider the following diffusion problem on a finite and connected graph $\mathbb{G}$,

$$\begin{cases} \dot{u}_j(t,x) = \sum_{i=1}^m \left(c_{ij}u_i'\right)'(t,x) + \sum_{i=1}^m p_{ij}u_i(t,x)\,, \quad t \ge 0\,,\, x \in (0,1)\,,\, j = 1,\dots,m\,,\\ u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d^u_\alpha(t)\,, \quad t \ge 0\,,\, l,\, j \in \Gamma(\mathrm{v}_\alpha)\,,\, j = 1,\dots,m\,,\\ \sum_{\beta=1}^n b_{\alpha\beta}d^u_\beta(t) = \sum_{i,j=1}^m \sum_{\beta=1}^n \delta^{\alpha i}_{\beta j}u_j'(t,\mathrm{v}_\alpha)\,, \quad t \ge 0\,,\, \alpha = n_0+1,\dots,n\,,\\ \dot{d}^u_\alpha(t) = -\sum_{i,j=1}^m \sum_{\beta=1}^n \delta^{\alpha i}_{\beta j}u_j'(t,\mathrm{v}_\beta) + \sum_{\beta=1}^n b_{\alpha\beta}d^u_\beta(t)\,, \quad t \ge 0\,,\, \alpha = 1,\dots,n_0\,,\\ u_j(0,x) = u^0_j(x)\,, \quad x \in (0,1)\,,\, j = 1,\dots,m\,,\\ d^u_i(0) = d^0_i\,, \quad i = 1,\dots,n_0\,, \end{cases} \tag{2.2}$$

where we have denoted by $\dot{u}(t,x)$ the time derivative of the unknown function $u$, whereas $u'(t,x)$ denotes its space-derivative.

Moreover, for $x \in [0,1]$, $t \in [0,T]$, we defined the unknown functions $u(t,x)$ and $d^u(t)$, by

$$u(t,x) = \left(u_1(t,x),\dots,u_m(t,x)\right)^T\,, \quad d^u(t) = \left(d^u_1(t),\dots,d^u_{n_0}(t), d^u_{n_0+1}(t),\dots,d^u_n(t)\right)^T\,,$$

and we consider the $n \times n$ matrix $B = \left(b_{\alpha,\beta}\right)_{\alpha,\beta=1,\dots,n}$, defined as $B := B_1 + B_2$, $B_1$ being the $n \times n$ matrix defined as

$$B_1 := \begin{pmatrix} b_{1,1} & \dots & b_{1,n}\\ \vdots & \ddots & \vdots\\ b_{n_0,1} & \dots & b_{n_0,n}\\ 0 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & 0 \end{pmatrix}\,, \tag{2.3}$$

while $B_2$ is the $n \times n$ matrix defined as

$$B_2 := \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ b_{n_0+1,1} & \dots & b_{n_0+1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \dots & b_{n,n} \end{pmatrix} .$$

If not stated otherwise, we use $\langle \cdot, \cdot \rangle_m$, resp. $|\cdot|_m$, to denote the standard scalar product, resp. the related norm, in $\mathbb{R}^m$.

Throughout the paper we will assume the following assumptions to hold:

*Assumptions* 2.2. **(i)** for any $i,\ j = 1, \dots, m$, we have that $c_{ij}(x) \in C^1(0,1)$, also assuming that the matrix $C := (c_{ij})_{i,j=1,\dots,m}$ is positive definite, uniformly in $[0,1]$, namely for any $x \in [0,1]$, $\bar{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$, there exists $\lambda^C > 0$ such that

$$\langle C(x)\bar{y}, \bar{y} \rangle_m = \sum_{i,j=1}^{m} c_{ij}(x) y_j y_i \geq \lambda^C |\bar{y}|_m^2 \, ; \tag{2.4}$$

**(ii)** for any $i,\ j = 1, \dots, m$ we have that $p_{ij}(x) \in L^\infty(0,1)$, also assuming that the matrix $P := (p_{ij})_{i,j=1,\dots,m}$ is negative semi-definite, uniformly in $[0,1]$, namely for any $x \in [0,1]$, $\bar{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$, there exists $\lambda^P \geq 0$ such that

$$\langle P(x)\bar{y}, \bar{y} \rangle_m = \sum_{i,j=1}^{m} p_{ij}(x) y_j y_i \leq -\lambda^P |\bar{y}|_m^2 \, ; \tag{2.5}$$

### 2.1. *The abstract setting*

In what follows we introduce the abstract setting which allows us to rewrite equation (2.2) as an abstract Cauchy problem. In particular, let us first consider the following spaces

$$X^2 := \left(L^2([0,1])\right)^m , \qquad \text{resp. } \mathbb{R}^n ,$$

equipped with the standard inner products, denoted by $\langle \cdot, \cdot \rangle_2$, resp. $\langle \cdot \rangle_n$, and norms denoted by $|\cdot|_2$, resp. $|\cdot|_n$. Then, we define the product Hilbert space $\mathcal{X}^2 := X^2 \times \mathbb{R}^n$, equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ d^u \end{pmatrix}, \begin{pmatrix} v \\ d^v \end{pmatrix} \right\rangle_{\mathcal{X}^2} := \sum_{j=1}^{m} \int_0^1 u_j(x) v_j(x) dx + \sum_{\alpha=1}^{n} d_\alpha^u d_\alpha^v ,$$

where $u, v \in X^2$, $d^u, d^v \in \mathbb{R}^n$, with associated norm denoted by $|\cdot|_{\mathcal{X}^2}$. Analogously, we define the Banach space

$$X^p := (L^p([0,1]))^m , \quad \mathcal{X}^p := X^p \times \mathbb{R}^n , \quad p \in [1, \infty] ,$$

*Remark* 2.3. In [17, 25] the authors consider a diffusion problem similar to the one represented by eq. (2.2), and where the boundary conditions depend on some phenomenological positive constants $\mu$ and $\nu$. For ease of notation, we have dropped latter constants in the present work without loose of generality. In fact, our results remain valid also when previous constants are explicitly considered, since it is sufficient to consider some weighted spaces of the form

$$X^2_\mu := \prod_{j=1}^m L^2([0,1];\mu_j dx)\,, \quad \mathbb{R}^n_\nu := \prod_{\alpha=1}^n \mathbb{R}\frac{1}{\nu_i}\,.$$

Recalling the definition of *incidence matrix* $\Phi$ given in Sec. 2, we introduce the associated *Kirchhoff operators* $\Phi^+_\delta$, $\Phi^-_\delta : (H^1(0,1))^m \to \mathbb{R}^n$, which are defined as follows

$$\Phi^+_\delta u' := \left(\sum_{i,j=1}^m \sum_{\alpha=1}^n {}^+\delta^{\alpha i}_{1j} u'_i(\mathrm{v}_1), \dots, \sum_{i,j=1}^m \sum_{\alpha=1}^n {}^+\delta^{\alpha i}_{nj} u'_i(\mathrm{v}_n)\right)^T,$$

$$\Phi^-_\delta u' := \left(\sum_{i,j=1}^m \sum_{\alpha=1}^n {}^-\delta^{\alpha i}_{1j} u'_i(\mathrm{v}_1), \dots, \sum_{i,j=1}^m \sum_{\alpha=1}^n {}^-\delta^{\alpha i}_{nj} u'_i(\mathrm{v}_n)\right)^T,$$

where the notation ${}^+\delta$, resp. ${}^-\delta$, means that $\iota$ in equation (2.1) belongs to $\mathcal{I}^+$, resp. $\mathcal{I}^-$, namely

$${}^+\delta^{\alpha i}_{\beta j} = \begin{cases} c_{ij}(\mathrm{v}_\beta)\iota^{\alpha i}_{\beta j} & \text{if } \iota^{\alpha i}_{\beta j} \in \mathcal{I}^+\,, \\ 0 & \text{otherwise}\ , \end{cases} \quad , \quad {}^-\delta^{\alpha i}_{\beta j} = \begin{cases} c_{ij}(\mathrm{v}_\beta)\iota^{\alpha i}_{\beta j} & \text{if } \iota^{\alpha i}_{\beta j} \in \mathcal{I}^-\,, \\ 0 & \text{otherwise}\ . \end{cases}$$

Let us then introduce the differential operator $(A, D(A))$ as

$$Au = \begin{pmatrix} (c_{1,1}u'_1)' + p_{1,1}u_1 & \dots & (c_{1,m}u'_1)' + p_{1,m}u_m \\ \vdots & \ddots & \vdots \\ (c_{m,1}u'_1)' + p_{m,1}u_1 & \dots & (c_{m,m}u'_m)' + p_{m,m}u_m \end{pmatrix},$$

which has domain defined as

$$D(A) = \Big\{ u \in \left(H^2(0,1)\right)^m \,:\, \exists\, d^u(t) \in \mathbb{R}^n \ \text{s.t.}\ \left(\Phi^+\right)^T d^u(t) = u(0)\ ,$$
$$\left(\Phi^-\right)^T d^u(t) = u(1)\,,\ \Phi^+_\delta u'(0) - \Phi^-_\delta u'(1) = B_2 d^u(t) \Big\}\ .$$

Then, we define the operator matrix

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ C & B_1 \end{pmatrix}, \tag{2.6}$$

where $C$ represents the *feedback operator* acting from $D(C) := D(A)$ to $\mathbb{R}^n$ and defined as follows

$$Cu := \left(-\sum_{i,j=1}^m \sum_{\beta=1}^n \delta^{1i}_{\beta j} u'_j(\mathrm{v}_1), \dots, -\sum_{i,j=1}^m \sum_{\beta=1}^n \delta^{n_0 i}_{\beta j} u'_j(\mathrm{v}_{n_0}), 0, \dots, 0\right)^T,$$

and

$$D(\mathcal{A})=\left\{\begin{pmatrix}u\\ d^u\end{pmatrix}\in D(A)\times\mathbb{R}^n\ :\ u_i(\mathbf{v}_\alpha)=d^u_\alpha\,,\quad\forall\, i\in\Gamma(\mathbf{v}_\alpha),\,\alpha=1,\dots,n\right\}.$$

Exploiting previous definitions, we can rewrite equation (2.2) as the following abstract infinite dimensional equation stated on the Hilbert space $\mathcal{X}^2$

$$\begin{cases}\dot{\mathbf{u}}(t)=\mathcal{A}\mathbf{u}(t)\,, & t\ge 0\,,\\ \mathbf{u}(0)=\mathbf{u}_0\,,\end{cases}\tag{2.7}$$

where

$$\mathbf{u}:=(u,d^u)^T=\left(u_1,\dots,u_m,d^u_1,\dots,d^u_{n_0},d^u_{n_0+1},\dots,d^u_n\right)^T\in\mathcal{X}^2\,,$$

and

$$\mathbf{u}_0:=\left(u_1(0,x),\dots,u_m(0,x),d^u_1(0),\dots,d^u_{n_0}(0),0,\dots,0\right)^T\ \in\mathcal{X}^2\,.$$

Then we introduce the sesquilinear form $\mathfrak{a}:\mathcal{V}\times\mathcal{V}\to\mathbb{R}$, where the space $\mathcal{V}$ is a suitable subspace of $\mathcal{X}^2$, see below, defined as

$$\begin{aligned}\mathfrak{a}(\mathbf{u},\mathbf{v}):=&\ \langle Cu',v'\rangle_2-\langle Pu,v\rangle_2-\langle B_1d^u,d^v\rangle_n-\langle B_2d^u,d^v\rangle_n=\\ =&\sum_{i,j=1}^m\int_0^1\left(c_{i,j}(x)u_j'(x)v_i'(x)-p_{i,j}(x)u_j(x)v_i(x)\right)dx-\sum_{\alpha,\beta=1}^n b_{\alpha\beta}d^u_\alpha d^v_\beta\,,\end{aligned}\tag{2.8}$$

for any $\mathbf{u}$, $\mathbf{v}\in\mathcal{X}^2$.

In particular, the subspace $\mathcal{V}$, domain of the form $\mathfrak{a}$, is defined by the following lemma

**Lemma 2.4.** *Let us consider the linear subspace*

$$\mathcal{V}:=\left\{\begin{pmatrix}u\\ d^u\end{pmatrix}\in\left(H^1(0,1)\right)^m\times\mathbb{R}^n\ :\ u_i(\mathbf{v}_\alpha)=d^u_\alpha\,,\ \forall\, i\in\Gamma(\mathbf{v}_\alpha),\,\alpha=1,\dots,n\right\},$$

*then $\mathcal{V}$ is densely and compactly embedded in $\mathcal{X}^2$. In particular $\mathcal{V}$ is a Hilbert space equipped with the scalar product*

$$\langle\mathbf{u},\mathbf{v}\rangle_{\mathcal{V}}:=\sum_{j=1}^m\int_0^1\left(u_j'(x)v_j'(x)+u_j(x)v_j(x)\right)dx+\sum_{\alpha=1}^n d^u_\alpha d^v_\alpha\,.\tag{2.9}$$

*The corresponding norm will be denoted by $|\cdot|_{\mathcal{V}}$.*

*Proof.* See, e.g., [11, Lemma 3.1] or [25, Lemma 3.1]. □

*Remark* 2.5. One of the main advantages in using the theory of sesquilinear form is that, under suitable assumptions, a sesquilinear form $\mathfrak{a}$ can be uniquely associated to an infinitesimal generator of an analytic strongly continuous semigroup. In particular, if we prove that the form $\mathfrak{a}$ satisfies some regularity conditions, then we also have a corresponding regularity for the associated semigroup. In the next proposition we gather several properties satisfied by the form $\mathfrak{a}$ defined in (2.8). We would like to underline that such results have already been proved separately, and under a different setting, in different works, see, e.g., [5, 17, 23, 25] and reference therein. Nevertheless, for the sake of completeness, we will provide for the latter a sketch of their proofs.

**Proposition 2.6.**

**(i)** *If Assumptions 2.2 hold, then the form* $\mathfrak{a} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ *defined in* (2.8) *is:*

- *continuous, i.e. it exists* $M > 0$*, such that*

$$|\mathfrak{a}(\mathbf{u}, \mathbf{v})| \leq M|\mathbf{u}|_{\mathcal{V}}|\mathbf{v}|_{\mathcal{V}}\,; \tag{2.10}$$

- $\mathcal{X}^2-$*elliptic, i.e. there exist* $\lambda > 0$ *and* $\omega \in \mathbb{R}$*, such that*

$$\mathfrak{a}(\mathbf{u}, \mathbf{u}) \geq \lambda|\mathbf{u}|^2_{\mathcal{V}} - \omega|\mathbf{u}|^2_{\mathcal{X}^2}\,; \tag{2.11}$$

- *closed, i.e.* $\mathcal{V}$ *is complete with respect to the following norm*

$$\|\mathbf{u}\|^2_{\mathfrak{a}} := \mathfrak{a}(\mathbf{u}, \mathbf{u}) + \|\mathbf{u}\|_{\mathcal{X}^2}\,; \tag{2.12}$$

**(ii)** *If Assumptions 2.2 hold and the matrix* $B$ *is negative defined, i.e. there exists* $\mu > 0$ *such that*

$$\langle B\bar{y}, \bar{y}\rangle_n \leq -\mu|\bar{y}|^2_n\,, \forall\, \bar{y} \in \mathbb{R}^n\,,$$

*then* $\mathfrak{a}$ *is coercive, namely it is* $\mathcal{X}^2-$*elliptic with* $\omega = 0$*, hence*

$$\mathfrak{a}(\mathbf{u}, \mathbf{u}) \geq \lambda|\mathbf{u}|^2_{\mathcal{V}}\,; \tag{2.13}$$

**(iii)** *If Assumptions 2.2 hold and the matrices* $C$*,* $P$ *and* $B$ *are all symmetric, then the form* $\mathfrak{a}$ *is symmetric as well.*

*Proof.* **(i)** To simplify notations, let us define the following quantities

$$\begin{aligned}
&\bar{c} := \min_{x\in[0,1]} \sum_{i,j=1}^{m} c_{i,j}(x)\,, \quad \bar{C} := \max_{1\leq j\leq m} \sum_{i,j=1}^{m} c_{i,j}(x)\,,\\
&\bar{p} := \min_{1\leq j\leq m} \sum_{i,j=1}^{m} (1 - p_{i,j}(x))\,, \quad \bar{P} := \max_{1\leq j\leq m} \sum_{i,j=1}^{m} (1 - p_{i,j}(x))\,,\\
&\bar{b} := \min_{i,l} b_{i,l}\,, \quad \bar{B} := \sum_{\alpha,\beta=1}^{n} b_{\alpha,\beta}\,.
\end{aligned}$$

Proceeding as in [25, Lemma 3.2], we have that $\mathcal{V}$, equipped with the inner product defined in equation (2.9), is a Hilbert space, moreover it is a closed subspace of $(H^1(0,1))^m \times \mathbb{R}^n$. From the continuous embedding of $H^1(0,1)$ into $C(0,1)$, see, e.g., [25, Lemma 3.2], we obtain

$$|d^u_i| \leq \max_{1\leq j\leq m} \max_{x\in[0,1]} |u_j(x)| \leq \max_{1\leq j\leq m} |u_j|_{H^1(0,1)} \leq \sum_{j=1}^{m} |u_j|_{H^1(0,1)}\,,$$

hence the norm defined in eq. (2.9) is equivalent to

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{V}} := \sum_{j=1}^{m} (u_j'(x) v_j'(x) + u_j(x) v_j(x)) \mu_j dx \,, \tag{2.14}$$

and, again by [25, Lemma 3.2], it also follows that there exists $K > 0$ such that

$$|d_i^{\mathbf{u}}| \leq K |\mathbf{u}|_{\mathcal{V}} \,, \quad i = 1, \dots, n \,,$$

then, defining

$$\tilde{c} := \min\{\bar{c}, \bar{p}\} \,, \quad \tilde{C} := \max\{\bar{C}, (1 - \bar{B}) K^2, \bar{P}\} \,,$$

we have that the norm generated by $\mathcal{V}$ is equivalent to the one generated by $\mathfrak{a}$, which, from the completeness of $\mathcal{V}$, implies the closure of $\mathfrak{a}$. In what follows the Hilbert space $\mathcal{V}$ will be equipped with the inner product (2.14) and the corresponding norm.
Concerning the continuity of $\mathfrak{a}$, from assumptions 2.2, we have

$$\begin{aligned}
|\mathfrak{a}(\mathbf{u}, \mathbf{v})| \leq & \sum_{i,j=1}^{m} \int_0^1 \left( |c_{i,j}(x) u_i'(x) v_j'(x)| + |p_{i,j}(x) u_i(x) v_j(x)| \right) dx + \\
& - \sum_{\alpha,\beta=1}^{n} b_{\alpha,\beta} |d_\alpha^u| |d_\beta^v| \leq \\
& \leq 2L \sum_{i,j=1}^{m} \langle u_i, v_j \rangle_{H^1((0,1);\mu_j dx)} - K^2 \bar{B} |\mathbf{u}|_{\mathcal{V}} |\mathbf{v}|_{\mathcal{V}} \leq \\
& \leq 2L \left( \sum_{j=1}^{m} |u_j|^2_{H^1((0,1);\mu_j dx)} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{m} |v_j|^2_{H^1((0,1);\mu_j dx)} \right)^{\frac{1}{2}} + \\
& - K^2 \bar{B} |\mathbf{u}|_{\mathcal{V}} |\mathbf{v}|_{\mathcal{V}} = \\
& = \left( 2L - \bar{B} K^2 \right) |\mathbf{u}|_{\mathcal{V}} |\mathbf{v}|_{\mathcal{V}} \leq M |\mathbf{u}|_{\mathcal{V}} |\mathbf{v}|_{\mathcal{V}} \,.
\end{aligned}$$

where $L$, resp. $M$, is defined by $L := \max\{\bar{C}, \bar{P}\}$, resp. by $M := (2L - K^2 \bar{B})$. Moreover assumptions 2.2 also implies that the form

$$\mathfrak{a}_1 := \langle C u', v' \rangle_2 - \langle P u, v \rangle_2 \,,$$

is $\mathcal{X}^2-$elliptic. In fact, by [8, Cor. 4.11], see also [11], we have, for some constant $K > 0$, that the following inequality holds

$$\max_{x \in [0,1]} u(x) \leq K \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}} \,,$$

hence, introducing $\mathfrak{a}_2 := -\langle B d^u, d^u \rangle$, we can decompose $\mathfrak{a}$ as $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$, so that the claim follows from [22, Lemma 2.1], see also [11, Th. 2.3] and [10, Lemma 2.1, Cor. 2.2].

**(ii)** from assumptions 2.2 and denoting by $\mu^C$, resp. $\mu^P$, the constant in equation (2.4), resp. equation (2.5), we have that

$$\begin{aligned}\mathfrak{a}(\mathbf{u},\mathbf{u}) &= \int_0^1 \left(\langle C(x)u'(x),u'(x)\rangle_m - \langle P(x)u(x),u(x)\rangle_m\right)dx - \langle Bd^u,d^v\rangle \\ &\geq \int_0^1 \left(\mu^C|u(x)|_m^2 + \mu^P|u(x)|_m^2\right)dx + \mu^B|d^u|^2 \geq \lambda\|\mathbf{u}\|_{\mathcal{V}}\,.\end{aligned} \tag{2.15}$$

**(iii)** it immediately follows from the very definition of $\mathfrak{a}$, see eq. (2.8).

□

In force of Proposition 2.6, we recall the following result, see [25, Lemma 3.3].

**Proposition 2.7.** *The operator associated with the form* $\mathfrak{a}$ *defined in* (2.8) *is the operator* $(\mathcal{A},D(\mathcal{A}))$ *defined in equation* (2.6).

*Proof.* See [25, Lemma 3.3] or [26, Prop. 1.51, Th. 1.52], □

We end the present subsection characterizing the semigroup generated by the operator $(\mathcal{A},D(\mathcal{A}))$ defined in equation (2.6). Such result will be used later on to prove the *Gaussian bound*, see Sec. 2.2 below.

**Proposition 2.8.** *If assumptions 2.2 hold, then the operator associated with the form* $\mathfrak{a}$ *defined in equation* (2.8)*, is densely defined, sectorial and resolvent compact, hence it generates an analytic and compact* $C_0-$*semigroup* $\mathcal{T}(t)$*. We also have the following properties for the semigroup*

**(i)** *if the matrix* $B$ *is negative definite, then the semigroup is uniformly exponentially stable;*

**(ii)** *if the matrices* $C$*,* $P$ *and* $B$ *are symmetric, then the semigroup is self-adjoint;*

**(iii)** *if the matrices* $C$ *and* $P$ *are diagonal, and the matrix* $B$ *has entries that are positive off-diagonal and it also satisfies*

$$b_{\alpha\alpha} + \sum_{\beta\neq\alpha} b_{\alpha\beta} \leq 0\,, \quad \textit{for any } \alpha = 1,\dots,n\,,$$

*then the semigroup is positive and* $\mathcal{X}^\infty-$*contractive in the sense of [26, Ch. 2].*

*Proof.* The main claim follows exploiting Lemma 2.4, Proposition 2.6, Proposition 2.7 and [15, Th. 1.2.1]. Concerning $(i)$ the uniformly exponential stability it is enough to see that the shifted form $\lambda - \mathfrak{a}(\cdot,\cdot)$ is accreative, whereas point $(ii)$ follows from the fact that the form $\mathfrak{a}$ is symmetric, while point $(iii)$ follows from [11, Th. 2.3] and [23, Cor. 3.4]. □

### 2.2. *Gaussian bounds*

In what follows we state our main result concerning Gaussian estimates and, in order to achieve the result, we require assumptions stated in (2.2) as well as the following

*Assumptions* 2.9. The matrices $C$ and $P$ are diagonal and $B$ has entries that are positive off-diagonal and it satisfies, for any $\alpha = 1, \dots, n$,

$$b_{\alpha\alpha} + \sum_{\beta \neq \alpha} |b_{\alpha\beta}| < 0 \,,$$

Under the current assumptions we have that the semigroup $\mathcal{T}$ generated by $\mathcal{A}$ is analytic, compact, positive, $\mathcal{X}^\infty-$contractive and uniformly exponential stable on $\mathcal{X}^2$, see Proposition 2.8.

Let us also recall, see [25, Lemma 5.2]. the following lemma,

**Lemma 2.10.** *Let us consider a set of functions* $u_j : [0,1] \to \mathbb{R}$*,* $j = 1, \dots, m$*, and let us then define the map* $Uu : [0,m] \to \mathbb{R}$ *by*

$$Uu(x) := u_j(x - j + 1) \,, \qquad \text{if } x \in (j-1, j) \,,$$

*then the map* $U$ *is a one-to-one map from* $(L^2(0,1))^m$ *onto* $L^2(0,m)$*. Also* $U$ *is an isometry if we consider* $(L^2(0,1))^m$ *with the norm*

$$|u|_{(L^2(0,1))^m} = \left( \sum_{j=1}^m |u_j|_{L^2(0,1)} \right)^{\frac{1}{2}} .$$

We then consider the product space $\mathcal{X}^2 := (L^2(0,1))^m \times \mathbb{R}^n$, hence, in virtue of Lemma 2.10, defining $\Omega := (0,m) \times (0,n)$, and

$$\mu := dx \oplus \delta_1 \oplus \cdots \oplus \delta_n \,,$$

where $\delta_{x_0}$ is the Dirac measure centred at $x_0$, then we have that the map $U : \mathcal{X}^2 \to L^2(\Omega, \mu)$ is an isomorphism. Since we have required assumptions 2.2 to hold, then we know that the operator associated with the form $\mathfrak{a}$, see eq. (2.8), generates an analytic and compact $C_0-$semigroup, which we have defined as $\mathcal{T}(t)$, moreover we have

**Theorem 2.11.** *The semigroup* $\mathcal{T}(t)$*, acting on the space* $\mathcal{X}^2$ *and associated to* $\mathfrak{a}$*, is ultracontractive, namely there exists a constant* $M > 0$ *such that*

$$\|\mathcal{T}(t)\mathbf{u}\|_{\mathcal{X}^\infty} \leq M t^{-\frac{1}{4}} \|\mathbf{u}\|_{\mathcal{X}^2} \,, \quad t \in [0,T], \, \mathbf{u} \in \mathcal{X}^2 \,. \tag{2.16}$$

*Proof.* By the *Nash-type inequality* for weighted $L^p-$space, we have that there exists a constant $M_1 > 0$ such that

$$\|f\|_{L^2(\Omega,\mu)} \leq M_1 \left( \|f'\|_{L^2(\Omega,\mu)} + \|f\|_{L^2(\Omega,\mu)} \right)^{\frac{1}{3}} \|f\|^{\frac{2}{3}}_{L^2(\Omega,\mu)} \leq M_1 \|f\|^{\frac{1}{3}}_{H^1(\Omega,\mu)} \|f\|^{\frac{2}{3}}_{L^2(\Omega,\mu)} \,,$$

hence, for $\mathbf{u} \in V_0$, we have

$$\begin{aligned}\|\mathbf{u}\|^2_{\mathcal{X}^2} &= \sum_{j=1}^m \|u_j\|_2^2 + \sum_{i=1}^n |d_i^u| \le M_1^2 \sum_{j=1}^m \|u_j\|_{H^1}^{\frac{2}{3}} \|u_j\|_2^{\frac{4}{3}} + \sum_{i=1}^n |d_i^u| \,, \le \\ &\le M_1^2 \left( \sum_{j=1}^m \|u_j\|^2_{H^1} + \sum_{i=1}^n |d_i^u| \right)^{\frac{1}{3}} \left( \sum_{j=1}^m \|u_j\|^2_{L^1} + \sum_{i=1}^n |d_i^u| \right)^{\frac{1}{3}} \le \\ &\le M_2 \|\mathbf{u}\|_{V_0}^{\frac{2}{3}} \|\mathbf{u}\|_{\mathcal{X}^1}^{\frac{4}{3}} \,,\end{aligned}$$

and the claim follows from the equivalence between the norms $\|\cdot\|_{\mathfrak{a}}$ and $\|\cdot\|_{V_0}$, as have been shown in Prop. 2.6 and [26, Lemma 5.2]. □

Moreover, Th. 2.11 implies the following

**Corollary 2.12.** *The semigroup $\mathcal{T}(t)$ on $\mathcal{X}^2$ satisfies*

$$\|\mathcal{T}(t)\mathbf{u}\|_{\mathcal{X}^\infty} \le M \left( \frac{1 - t\omega}{t} \right)^{\frac{1}{4}} e^{1+t\omega} \|\mathbf{u}\|_{\mathcal{X}^2} \,,$$

*where $\omega < 0$ is the spectral bound of the semigroup $\mathcal{T}(t)$.*

*Proof.* The claim follows from Prop. 2.8, Th. 2.11 and [26, Lemma 6.5]. □

Besides the ultracontrattivity of $\mathcal{T}(t)$ together with Cor. 2.12, implies that the semigroup has an integral Kernel, see [15, Lemma 2.1.2.]. More precisely let us denote by $\tilde{\mathcal{T}}(t) := U^{-1}\mathcal{T}(t)U$ the similar semigroup, see, e.g., [18], acting on $L^2(\Omega, \mu)$, being $U$ the isomorphism introduced above. Then, Lemma [15, Lemma 2.1.2] gives us that the action of $\left(\tilde{\mathcal{T}}(t)\right)_{t\ge 0}$, reads as follow

$$\left(\tilde{\mathcal{T}}(t)g\right)(\cdot) = \int_\Omega K_t(\cdot, y) g(y) \mu(dy) \,, \quad g \in L^2(\Omega, \mu) \,,$$

for a suitable kernel $K_t \in L^\infty(\Omega \times \Omega)$. Besides, we can rewrite eq. (2.16) as follows

$$\|\mathcal{T}(t)\mathbf{u}\|_{\mathcal{X}^\infty} \le e^{\kappa(t)} \|\mathbf{u}\|_{\mathcal{X}^2} \,, \quad t \in [0, T], \,, \mathbf{u} \in \mathcal{X}^2 \,,$$

where

$$\kappa(t) := \log M - \frac{1}{4} \log t \,.$$

Then, applying [15, Th. 2.2.3], we can derive the following logarithmic Sobolev inequality

$$\int_\Omega \tilde{\mathbf{u}} \log \tilde{\mathbf{u}} dx \le \epsilon \mathfrak{a}(\mathbf{u}, \mathbf{u}) + \kappa(\epsilon) \|\mathbf{u}\|^2_{\mathcal{X}^2} + \|\mathbf{u}\|^2_{\mathcal{X}^2} \log \|\mathbf{u}\|_{\mathcal{X}^2} \,, \tag{2.17}$$

for any $\mathbf{u} \ge 0$, $\mathbf{u} \in V_0$ and $\epsilon > 0$, and where $\tilde{\mathbf{u}} \in L^2(\Omega, \mu)$ denotes the function isometric to $\mathbf{u}$ under the isomorphism $U$. Evenually, inequality (2.17) implies the next result

**Theorem 2.13.** *The Gaussian upper bound*

$$0 \leq K_t(x,y) \leq c_\delta t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{\sigma t}}, \tag{2.18}$$

*holds for the heat kernel $K_t$ introduced above, such that it holds*

$$[\mathcal{T}(t)g](x) = \int_\Omega K_t(x,y)g(y)\mu(dy), \quad y \in L^2(\Omega,\mu).$$

*Proof.* The claim follows from [15, Th. 3.2.7], taking into account the logarithmic Sobolev inequality (2.17), see, e.g., [23, Th. 4.8] and [17]. □

Exploiting Th. 2.13 it is also possible to prove the existence of a *mild solution*, in a suitable sense, to equation (2.2) perturbed by a multiplicative Gaussian noise. Before state latter result, let us denote by $\mathcal{L}_2(\mathcal{X}^2)$ the class of *Hilbert-Schmidt* operator from $\mathcal{X}^2$ to $\mathcal{X}^2$, while $|\cdot|_{HS}$ denotes the standard *Hilbert-Schmidt* norm. We refer the reader to, e.g., [13, Appendix. C], for a dense *rÃľsumÃľ* of the main properties of *Hilbert-Schmidt* operators.

**Proposition 2.14.** *Let assumptions 2.2-2.9 hold, then, for any $t > 0$, the semigroup $\mathcal{T}(t) \in \mathcal{L}_2(\mathcal{X}^2)$, moreover there exists $M > 0$ such that*

$$|\mathcal{T}(t)|_{HS} \leq M t^{-\frac{1}{4}}.$$

*Proof.* Since

$$|\mathcal{T}(t)|_{HS} = |\tilde{\mathcal{T}}(t)|_{HS} = |K_t|_{L^2(\Omega\times\Omega)},$$

where $K_t$ is the kernel defined in equation (2.18), then, by Th. 2.13, eq. (2.18), see also [17, Cor.2], we obtain the existence of a constant $C > 0$ such that, $\forall\ t \in [0,T]$, it holds

$$|\mathcal{T}(t)|^2_{HS} = \int_{\Omega\times\Omega} |K_t(x,y)|^2 dxdy \leq C\sqrt{2\pi\sigma t^{-1}},$$

which implies the existence of a positive constant $M$ such that, $\forall\ t \in [0,T]$, the following hold

$$|\mathcal{T}(t)|_{HS} \leq M t^{-\frac{1}{4}}.$$

□

## 3. The perturbed stochastic problem

In the present section we focus our attention on the problem (2.2) by perturbing it with multiplicative Gaussian noise. Let us first consider the following complete, filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, with respect to which, we state the following system

$$\begin{cases} \dot{u}_j(t,x) = \left(c_j u_i'\right)'(t,x) + p_i u_i(t,x) + g_j(t,x,u_j(t,x))\dot{W}_j^1(t,x)\,, \\ \qquad \text{for } t \geq 0\,,\, x \in (0,1)\,,\, j = 1,\dots,m\,, \\ u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d_\alpha^u(t)\,, \quad t \geq 0\,,\, l,\, j \in \Gamma(\mathrm{v}_\alpha)\,,\, j = 1,\dots,m\,, \\ \sum_{\beta=1}^n b_{\alpha\beta} d_\beta^u(t) = \sum_{i,j=1}^m \sum_{\beta=1}^n \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\alpha)\,, \quad t \geq 0\,,\, \alpha = n_0+1,\dots,n\,, \\ \dot{d}_\alpha^u(t) = -\sum_{i,j=1}^m \sum_{\beta=1}^n \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\beta) + \sum_{\beta=1}^n b_{\alpha\beta} d_\beta^u(t) + \tilde{g}_\alpha(t, d_\alpha^u(t))\dot{W}_\alpha^2(t,\mathrm{v}_\alpha)\,, \\ \qquad \text{for } t \geq 0\,,\, \alpha = 1,\dots,n_0\,, \\ u_j(0,x) = u_j^0(x)\,, \quad x \in (0,1)\,,\, j = 1,\dots,m\,, \\ d_i^u(0) = d_i^0\,, \quad i = 1,\dots,n_0\,, \end{cases} \tag{3.1}$$

where, for every $(j,\alpha) \in \{1,\dots,m\} \times \{1,\dots,n_0\}$, $W_j^1$ and $W_\alpha^2$ are independent Wiener processes adapted to $\mathcal{F}_t-$, while $\dot{W}$ is the formal time derivative. In particular, for every $j = 1,\dots,m$, $W_j^1$, is a space time Wiener process with values in $L^2(0,1)$. Then, we denote by $W^1 := (W_1^1,\dots,W_m^1)$, a space time Wiener process with values in the product space $X^2 := \left(L^2(0,1)\right)^m$. Analogously, for every $\alpha = 1,\dots,n$, $W_\alpha^2$ is a space time Wiener process taking values in $\mathbb{R}$, hence we denote by $W^2 := (W_1^2,\dots,W_n^2)$ the standard Wiener process with values in $\mathbb{R}^n$. Consequently, $W := (W^1,W^2)$ indicates the standard space time Wiener process with values in $\mathcal{X}^2 := X^2 \times \mathbb{R}^n$, being $(\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration generated by $W$, augmented by all $\mathbb{P}-$null sets of $\mathcal{F}_T$.

Besides assumptions 2.2 and 2.9 we will also assume the following to hold.

*Assumptions* 3.1.

**(i)** For every $j = 1,\dots,m$, the functions $g_j : [0,T]\times[0,1]\times\mathbb{R} \to \mathbb{R}$, are measurable, bounded and uniformly Lipschitz in the third component, namely there exist constants $C_j > 0$ and $K_j$ such that, for any $(t,x,y_1) \in [0,T]\times[0,1]\times\mathbb{R}$ and $(t,x,y_2) \in [0,T]\times[0,1]\times\mathbb{R}$, the following holds

$$|g_j(t,x,y_1)| \leq C_j\,, \quad |g_j(t,x,y_1) - g_j(t,x,y_2)| \leq K_j|y_1 - y_2|\,;$$

**(ii)** For every $\alpha = 1,\dots,n_0$, the functions $\tilde{g}_\alpha : [0,T]\times\mathbb{R} \to \mathbb{R}$, are measurable, bounded and uniformly Lipschitz with respect to the second component, namely there exist constants $C_\alpha > 0$ and $K_\alpha$ such that, for any $(t,y_1) \in [0,T]\times\mathbb{R}$ and $(t,y_2) \in [0,T]\times\mathbb{R}$, the following holds

$$|\tilde{g}_\alpha(t,y_1)| \leq C_\alpha\,, \quad |\tilde{g}_\alpha(t,y_1) - \tilde{g}_\alpha(t,y_2)| \leq K_\alpha|y_1 - y_2|\,.$$

With the help of the notations just introduced, see also Sec. 2.1, the problem (3.1) can be rewritten as an abstract infinite dimensional Cauchy problem of the form

$$\begin{cases} d\mathbf{u}(t) = \mathcal{A}\mathbf{u}(t)dt + G(t,\mathbf{u}(t))dW(t)\,, \quad t \geq 0\,, \\ \mathbf{u}(0) = \mathbf{u}_0 \in \mathcal{X}^2\,, \end{cases} \tag{3.2}$$

where $\mathcal{A}$ is the operator introduced in (2.6), while $G : [0,T]\times\mathcal{X}^2 \to \mathcal{L}(\mathcal{X}^2)$, $\mathcal{L}(\mathcal{X}^2)$ being the space of linear and bounded operator from $\mathcal{X}^2$ to $\mathcal{X}^2$ equipped with standard operator norm $|\cdot|_{\mathcal{L}}$, is defined as

$$G(t,\mathbf{u})\mathbf{v} = (\sigma_1(t,u)v, \sigma_2(t,y)z)^T\ , \quad \mathbf{u}=(u,y),\, \mathbf{v}=(v,z) \in \mathcal{X}^2, \tag{3.3}$$

with

$$(\sigma_1(t,u)v)\,(x) = (g_1(t,x,u_1(t,x)),\dots,g_m(t,x,u_m(t,x)))^T\ ,$$
$$\sigma_2(t,y)z = (\tilde g_1(t,y_1)z_1,\dots,\tilde g_{n_0}(t,y_{n_0})z_{n_0},0,\dots,0)^T\ .$$

It is worth to mention that, in order to guarantee the existence and uniqueness of a *mild solution* to equation (3.2), in a suitable sense to be introduced in a while, we have to require the stronger property that $G:[0,T]\times\mathcal{X}^2\to\mathcal{L}_2(\mathcal{X}^2)$, where $\mathcal{L}_2(\mathcal{X}^2)$ is the space of *Hilbert-Schmidt* operator from $\mathcal{X}^2$ into itself equipped with standard *Hilbert-Schmidt norm*denoted by $|\cdot|_{HS}$, see, e.g., [13, Appendix C]. Nevertheless, by Prop. 2.14, we can show that the semigroup $\mathcal{T}(t)$ is *Hilbert-Schmidt*, and that to have a unique solution in a *mild sense* we can weaken the condition on $G$ requiring it to take values in $\mathcal{L}(\mathcal{X}^2)$.

The aforementioned *mild solution* to equation (3.2), is intended in the following sense

**Definition 3.1.1.** We will say that $\mathbf{u}$ is a *mild solution* to equation (3.2), if it is a mean square continuous $\mathcal{X}^2-$valued process adapted to the filtration generated by $W$, such that for any $t\geq 0$ we have that $\mathbf{u}\in L^2\left(\Omega, C([0,T];\mathcal{X}^2)\right)$, and it holds

$$\mathbf{u}(t) = \mathcal{T}(t)\mathbf{u}_0 + \int_0^t \mathcal{T}(t-s)G(s,\mathbf{u}(s))dW(s)\,, \quad t\geq 0\,. \tag{3.4}$$

We thus have the following.

**Proposition 3.2.** *Let assumptions 2.2-2.9-3.1 hold, then the map* $G:[0,T]\times\mathcal{X}^2\to\mathcal{L}(\mathcal{X}^2)$ *defined in eq.* (3.3) *satisfies:*

**(i)** *for any* $\mathbf{u}\in\mathcal{X}^2$, *the map* $G(\cdot,\cdot)\mathbf{u}:[0,T]\times\mathcal{X}^2\to\mathcal{X}^2$ *is measurable;*

**(ii)** $\mathcal{T}(t)G(s,\mathbf{u})\in\mathcal{L}_2(\mathcal{X}^2)$, *for any* $t>0$, $s\in[0,T]$ *and* $\mathbf{u}\in\mathcal{X}^2$*;*

**(iii)** *for any* $t>0$, $s\in[0,T]$, $\mathbf{u}$, $\mathbf{v}\in\mathcal{X}^2$, *and for some constant* $M>0$, *it holds*

$$|\mathcal{T}(t)G(s,\mathbf{u})|_{HS} \leq Mt^{-\frac14}(1+|\mathbf{u}|_{\mathcal{X}^2})\,, \tag{3.5}$$
$$|\mathcal{T}(t)G(s,\mathbf{u})-\mathcal{T}(t)G(s,\mathbf{v})|_{HS} \leq Mt^{-\frac14}|\mathbf{u}-\mathbf{v}|_{\mathcal{X}^2}\,, \tag{3.6}$$
$$|G(s,\mathbf{u})|_{\mathcal{L}} \leq M(1+|\mathbf{u}|_{\mathcal{X}^2})\,. \tag{3.7}$$

*Proof.* Point **(i)** immediately follows from assumptions 3.1, whereas **(ii)** follows from equation (3.5). Concerning point **(iii)**, we have that eq. (3.5) immediately follows from assumptions 3.1. To derive eq. (3.6), we first denote by $\{\phi_k\}_{k\in\mathbb{N}}$ an orthonormal basis in $\mathcal{X}^2$. Then, denoting in what follows by $M>0$ several different constants, and exploiting assumptions 3.1, we have

$$\begin{aligned}|\mathcal{T}(t)G(s,\mathbf{u})|_{HS}^2 &= \sum_{j,k\in\mathbb{N}} \langle \mathcal{T}(t)G(s,\mathbf{u})\phi_j,\phi_k\rangle_{\mathcal{X}^2}^2 = \\ &= \sum_{j,k\in\mathbb{N}} \langle G(s,\mathbf{u})\phi_j,\mathcal{T}(t)\phi_k\rangle_{\mathcal{X}^2}^2 \leq |G(s,\mathbf{u})|_{\mathcal{L}}^2 |\mathcal{T}(t)|_{HS}^2 \leq \\ &\leq M(1+|\mathbf{u}|_{\mathcal{X}^2}^2)|\mathcal{T}(t)|_{HS}^2 \leq Mt^{-\frac{1}{4}}(1+|\mathbf{u}|_{\mathcal{X}^2})\,,\end{aligned} \tag{3.8}$$

where the last inequality follows from Prop. 2.14, hence, proceeding as for eq. (3.8), we obtain eq. (3.6). ☐

**Theorem 3.3.** *Let assumptions 2.2-2.9-3.1 hold, then there exists a unique mild solution in the sense of Def. 4.1.1.*

*Proof.* The result can be derived exploiting [14, Th. 5.3.1], together with Prop. 3.2, see also [17]. ☐

### 3.1. *Existence and uniqueness for the non-linear equation*

In what follows we generalize eq. (3.1), and consequently the abstract Cauchy problem (3.2), taking into account a non-linear Lipschitz perturbation. The notation is as in previous sections. In particular we consider the following non-linear stochastic boundary value problem

$$\begin{cases} \dot u_j(t,x) = \left(c_j u_i'\right)'(t,x) + p_i u_i(t,x) + f_j(t,x,u_j(t,x)) + g_j(t,x,u_j(t,x))\dot W_j^1(t,x)\,, \\ \qquad\quad \text{for } t\geq 0\,,\ x\in(0,1)\,,\ j=1,\dots,m\,, \\ u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d_\alpha^u(t)\,, \quad t\geq 0\,,\ l,\,j\in\Gamma(\mathrm{v}_\alpha)\,,\ j=1,\dots,m\,, \\ \sum_{\beta=1}^n b_{\alpha\beta} d_\beta^u(t) = \sum_{i,j=1}^m \sum_{\beta=1}^n \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\alpha)\,, \quad t\geq 0\,,\ \alpha=n_0+1,\dots,n\,, \\ \dot d_\alpha^u(t) = -\sum_{i,j=1}^m \sum_{\beta=1}^n \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\beta) + \sum_{\beta=1}^n b_{\alpha\beta} d_\beta^u(t) + \tilde g_\alpha(t,d_\alpha^u(t))\dot W_\alpha^2(t,\mathrm{v}_\alpha)\,, \\ \qquad \text{for } t\geq 0\,,\ \alpha=1,\dots,n_0\,, \\ u_j(0,x) = u_j^0(x)\,, \quad x\in(0,1)\,,\ j=1,\dots,m\,, \\ d_i^u(0) = d_i^0\,, \quad i=1,\dots,n_0\,. \end{cases} \tag{3.9}$$

Besides the assumptions 2.2-2.9-3.1, we also require that

*Assumptions* 3.4. For every $j=1,\dots,m$, the functions $f_j:[0,T]\times[0,1]\times\mathbb{R}\to\mathbb{R}$, are measurable, bounded and uniformly Lipschitz continuous with respect to the third component, namely there exist constants $C_j>0$ and $K_j$, such that, for any $(t,x,y_1)\in[0,T]\times[0,1]\times\mathbb{R}$ and $(t,x,y_2)\in[0,T]\times[0,1]\times\mathbb{R}$, it holds

$$|f_j(t,x,y_1)|\leq C_j\,, \quad |f_j(t,x,y_1)-f_j(t,x,y_2)|\leq K_j|u-v|\,.$$

Analogously to what has been made in Sec. 3, we reformulate eq. (3.9) as follows

$$\begin{cases} d\mathbf{u}(t) = [\mathcal{A}\mathbf{u}(t)+F(t,\mathbf{u}(t))]\,dt + G(t,\mathbf{u}(t))dW(t)\,, \quad t\geq 0\,, \\ \mathbf{u}(0)=\mathbf{u}_0\in\mathcal{X}^2\,, \end{cases} \tag{3.10}$$

moreover we define $F:[0,T]\times\mathcal{X}^2\to\mathcal{X}^2$, such that

$$F(t,\mathbf{u}) := (f(t,u),0)^T\,, \quad \mathbf{u}=(u,y)\in\mathcal{X}^2 := X^2\times\mathbb{R}^n, \tag{3.11}$$

with

$$(f(t,u))(x) := (f_1(t,x,u_1(t,x)),\dots,f_m(t,x,u_m(t,x)))^T \ .$$

Then, we can state the following result for the existence and uniqueness of a *mild solution* to the eq. (3.10)

**Theorem 3.5.** *Let assumptions 2.2-2.9-3.1-3.4 hold, then there exists a unique mild solution to eq.* (3.10) *in the sense of Def. 4.1.1.*

*Proof.* It is enough to show that the map $F$ defined in eq. (3.11) is Lipschitz continuous on the space $\mathcal{X}^2$. In fact, from assumptions 3.4, it holds

$$|F(t,\mathbf{u}) - F(t,\mathbf{v})|_{\mathcal{X}^2} = |f(t,u) - f(t,v)|_{X^2} \le K|u-v|_{X^2} \ . \tag{3.12}$$

Then, exploiting eq. (3.12) together with Prop.3.2, the existence of a unique mild solution is a direct application of [14, Th. 5.3.1], see also [17]. □

*Remark* 3.6. A result similar to Th.3.5 can be also proved under the assumption of $F$ to be only a function of polynomial growth at infinity, see, e.g., [6].

## 4. Application to stochastic optimal control

In the present section, in the light of previously obtained results, we consider an optimal control problem related to a general nonlinear control system, written in the following form

$$\begin{cases} d\mathbf{u}(t)^z = [\mathcal{A}\mathbf{u}^z(t) + F(t,\mathbf{u}^z) + G(t,\mathbf{u}^u(t))R(t,\mathbf{u}(t),z(t))]\,dt \\ \qquad\qquad\qquad + G(t,\mathbf{u}^z(t))dW(t)\,, \quad t\in[t_0,T]\,, \\ \mathbf{u}^z(t_0) = \mathbf{u}_0 \in \mathcal{X}^2\,, \end{cases} \tag{4.1}$$

where $z$ denotes the control and the subscript $\mathbf{u}^z$ denotes the dependence of the process $\mathbf{u} \in \mathcal{X}^2$ from the control $z$. In particular, we analyse the system (4.1) following the approach given in [20], searching for its weak solutions, see, e.g., [19].

Let us fix $t_0 \ge 0$ and $\mathbf{u}_0 \in \mathcal{X}^2$, then an *admissible control system* (ACS) is given by $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P},(W(t))_{t\ge0},z\right)$, where

- $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P}\right)$ is a complete probability space;
- $(\mathcal{F}_t)_{t\ge0}$ is a filtration, in the aforementioned probability space, satisfying the usual assumptions;
- $(W(t))_{t\ge0}$ is a $\mathcal{F}_t-$adapted Wiener process with values in $\mathcal{X}^2$;
- $z$ is a process taking values in the space $Z$, predictable with respect to the filtration $(\mathcal{F}_t)_{t\ge0}$, and such that $z(t)\in\mathcal{Z}$ $\mathbb{P}-$a.s. , for almost any $t\in[t_0,T]$, $\mathcal{Z}$ being a suitable domain of $Z$.

To each ACS we associate the mild solution of the abstract equation (4.1) $\mathbf{u}^z \in C([t_0,T];L^2(\Omega;\mathcal{X}^2))$, and we introduce the following cost functional

$$J(t_0,\mathbf{u}_0,z) := \mathbb{E}\int_{t_0}^{T} l\left(t,\mathbf{u}^z(t),z(t)\right)dt + \mathbb{E}\varphi(\mathbf{u}^z(T))\,, \tag{4.2}$$

where the function $l$, resp. the function $\varphi$, denotes the *running cost*, resp. the *terminal cost*. Then, the main goal is to chose a control $z$ belonging to a given set of admissible controls, and such that it minimizes the cost functional (4.2). If such a control $z$ exists, it will be called optimal control.

In what follows, besides the assumptions 2.2-2.9-3.13.4, we will also require the following to hold

*Assumptions* 4.1. **(i)** the map $R:[0,T]\times\mathcal{X}^2\times\mathcal{Z}\to\mathcal{X}^2$ is measurable and, for some $C_R>0$, it satisfies

$$\begin{aligned}&|R(t,\mathbf{u},z)-R(t,\mathbf{u},z)|_{\mathcal{X}^2}\le C_R(1+|\mathbf{u}|_{\mathcal{X}^2}+|\mathbf{v}|_{\mathcal{X}^2})^m|\mathbf{u}-\mathbf{v}|_{\mathcal{X}^2}\,,\\&|R(t,\mathbf{u},z)|_{\mathcal{X}^2}\le C_R\,;\end{aligned}$$

**(ii)** the map $l:[0,T]\times\mathcal{X}^2\times\mathcal{Z}\to\mathbb{R}\cup\{+\infty\}$ is measurable and, for some $C_l>0$ and $C\ge 0$, it satisfies

$$\begin{aligned}&|R(t,\mathbf{u},z)-R(t,\mathbf{u},z)|\le C_l(1+|\mathbf{u}|_{\mathcal{X}^2}+|\mathbf{v}|_{\mathcal{X}^2})^m|\mathbf{u}-\mathbf{v}|_{\mathcal{X}^2}\,,\\&|R(t,0,z)|_{\mathcal{X}^2}\ge -C\,,\\&\inf_{z\in\mathcal{Z}} l(t,0,z)\le C_l\,;\end{aligned}$$

**(iii)** for some $C_\varphi>0$ and $m\ge 0$, the map $\varphi:\mathcal{X}^2\to\mathbb{R}$ satisfies

$$|\varphi(\mathbf{u})-\varphi(\mathbf{v})|\le C_\varphi(1+|\mathbf{u}|_{\mathcal{X}^2}+|\mathbf{v}|_{\mathcal{X}^2})^m|\mathbf{u}-\mathbf{v}|_{\mathcal{X}^2}\,.$$

Following [20], if we let assumptions 2.2-2.9-3.13.4-4.1 to hold, then an ACS can be constructed as follows: first we arbitrarily chose the probability space $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge 0},\mathbb{P}\right)$ and $W$ as above, then we consider the uncontrolled problem

$$\begin{cases}d\mathbf{u}(t)=[\mathcal{A}\mathbf{u}(t)+F(t,\mathbf{u})]\,dt+G(t,\mathbf{u}(t))dW(t)\,, & t\ge 0\,,\\ \mathbf{u}(0)=\mathbf{u}_0\in\mathcal{X}^2\,,\end{cases} \tag{4.3}$$

under above assumptions. Then, by Th. 3.5, we have the existence for a unique mild solution to eq. (4.3). Moreover, by the boundedness of $R$ and applying the *Girsanov theorem*, we obtain that, for any fixed $\zeta\in\mathcal{Z}$, there exists a probability measure $\mathbb{P}^\zeta$ such that

$$W^\zeta(t):=W(t)-\int_{t_0\wedge t}^{t\wedge T}R(s,\mathbf{u}(s),\zeta)ds\,,$$

is a Wiener process, so that, for any $t\in[0,T]$, $\mathbf{u}$, $\mathbf{v}\in\mathcal{X}^2$, we can classically define the Hamiltonian function associated to the problem (4.3), as follows

$$\begin{aligned}&\psi(t,\mathbf{u},\mathbf{v})=-\inf_{z\in\mathcal{Z}}\{l(t,\mathbf{u},z)+\mathbf{v}R(t,\mathbf{u},z)\}\,,\\&\Gamma(t,\mathbf{u},\mathbf{v})=\{z\in\mathcal{Z}\,:\,\psi(t,\mathbf{u},\mathbf{v})+l(t,\mathbf{u},z)+\mathbf{v}R(t,\mathbf{u},z)=0\}\,,\end{aligned} \tag{4.4}$$

where we note that $\Gamma(t,\mathbf{u},w)$ is a (possibly empty) subset of $\mathcal{Z}$, and the function $\psi$ satisfies assumptions 4.1. In the present setting we can apply [20, Th. 5.1] which allows us to write the *Hamilton-Jacobi-Bellman* (HJB) equation for the problem (4.1)-(4.2), as follows

$$\begin{cases} \frac{\partial w(t,\mathbf{u})}{\partial t} + \mathcal{L}_t w(t,\mathbf{u}) = \psi(t,\mathbf{u},\nabla w(t,\mathbf{u})G(t,\mathbf{u}))\,, \\ w(T,\mathbf{u}) = \varphi(\mathbf{u})\,, \end{cases} \tag{4.5}$$

where

$$\mathcal{L}_t w(\mathbf{u}) := \frac{1}{2} Tr\left[G(t,\mathbf{u})G(t,\mathbf{u})^* \nabla^2 w(\mathbf{u})\right] + \langle \mathcal{A}\mathbf{u}, \nabla w(\mathbf{u})\rangle_{\mathcal{X}^2}\ ,$$

is the infinitesimal generator associated to the eq. (4.1), $Tr$ denotes the trace, $G^*$ is the adjoint of $G$ and $\nabla$ is a suitable notion of gradient to be introduced in a while.

In particular, see, e.g., [20, Def. 5.1], $w$ is said to be a *mild solution in the sense of generalized gradient*, or simply *mild solution*, according to the following definition

**Definition 4.1.1.** We say that a function $w : [0,T] \times \mathcal{X}^2 \to \mathbb{R}$ is a mild solution to equation (4.5) if the following hold:

**(i)** there exist $C > 0$ and $m \geq 0$, such that for any $t \in [0,T]$, and for any $\mathbf{u}$, $\mathbf{v} \in \mathcal{X}^2$, it holds

$$\begin{aligned} &|w(t,\mathbf{u}) - w(t,\mathbf{v})| \leq C(1 + |\mathbf{u}|_{\mathcal{X}^2} + |\mathbf{v}|_{\mathcal{X}^2})^m |\mathbf{u}-\mathbf{v}|_{\mathcal{X}^2}\,, \\ &|w(t,0)| \leq C\,; \end{aligned}$$

**(ii)** for any $0 \leq t \leq T$, $\mathbf{u} \in \mathcal{X}^2$, we have that

$$w(t,\mathbf{u}) = P_{t,T}\varphi(\mathbf{u}) - \int_t^T P_{t,s}\psi(s,\cdot,w(s,\cdot),\rho(s,\cdot))(\mathbf{u})ds\,,$$

where $\rho$ is an arbitrary element of the *generalized directional gradient* $\nabla^G w$ defined in [20], while $P_{t,T}$ is the Markov semigroup generated by the forward process (4.1).

In particular we would like to underline that, thanks to the approach developed in [20], we do not need to require any differentiability properties for the functions $F$, $G$ and $w$. In fact, the notion of gradient appearing in equation (4.5) is to be intend in a weak sense, which is exactly the notion of the *generalized directional gradient* we have reminded before, see [20]. In particular, the latter means that if $w$ is regular enough, then $\nabla w$ coincides with the standard notion of gradient, namely, with respect to the present case, it coincides with the *Fréchet derivative*, resp. with the *Gâteaux derivative*, if we assume $w$ to be *Fréchet* differentiable, resp. to be *Gâteaux* differentiable.

We thus have the following result.

**Proposition 4.2.** *Let us consider the optimal control problem* (4.1)-(4.2)*, then the associated HJB equation is represented by eq.* (4.5)*. Moreover, if assumptions 2.2-2.9-3.1-3.4-4.1 hold, then we have that the HJB equation* (4.5) *admits a unique mild solution in the sense of definition 4.1.1.*

*Proof.* The proof immediately follows from [20, Th. 5.1]. □

As a direct consequence of Proposition 4.2, we have the following

**Theorem 4.3.** *Let assumptions 2.2-2.9-3.1-3.4-4.1 hold, $w$ be a mild solution to the HJB equation* (4.5) *and $\rho$ is an element of the generalized directional gradient $\nabla^G w$. Then, for all ACS, we have have $J(t_0, \mathbf{u}_0, z) \geq w(t_0, \mathbf{u}_0)$, and the equality holds if and only if the following feedback law is verified by $z$ and $\mathbf{u}^z$*

$$z(t) = \Gamma\left(t, \mathbf{u}^z(t), G(t, \rho(t, \mathbf{u}^z(t))\right)\,, \quad \mathbb{P} - a.s. \ \text{for a.a. } t \in [t_0, T]\,. \tag{4.6}$$

*Moreover, if there exists a measurable function $\gamma : [0,T] \times \mathcal{X}^2 \times \mathcal{X}^2 \to \mathcal{Z}$ with*

$$\gamma(t, \mathbf{u}, \mathbf{v}) \in \Gamma(t, \mathbf{u}, \mathbf{v})\,, \quad t \in [0,T]\,,\ \mathbf{u}\,,\ \mathbf{v} \in \mathcal{X}^2\,,$$

*then there exists at least one ACS for which*

$$\bar{z}(t))\gamma(t, \mathbf{u}^z(t), \rho(t, \mathbf{u}^z(t)))\,, \quad \mathbb{P} - a.s. \ \text{for a.a. } t \in [t_0, T]\,.$$

*Eventually, we have that $\mathbf{u}^{\bar{z}}$ is a mild solution of equation* (4.1)*.*

*Proof.* See [20, Th. 7.2]. □

*Example* 4.1 (The heat equation with controlled stochastic boundary conditions on a graph). In what follows we give an example concerning the heat equation defined on a graph $\mathbb{G}$, as it has been defined in Sec. 2. On every nodes of $\mathbb{G}$ we assume local controlled dynamic boundary conditions. Hence, according with the setting introduced in 1, we have $m$ nodes and $n_0 = n$ nodes equipped with dynamic boundary conditions. We also assume to do not have any noise on the heat equation, whereas we assume the boundary condition to be perturbed by an additive Wiener process. Then, we are considering a system of the following form

$$\begin{cases} \dot{u}_j(t,x) = \left(c_j u_i'\right)'(t,x)\,, \quad t \geq 0\,, x \in (0,1)\,, j = 1, \dots, m\,, \\ u_j(t, \mathrm{v}_\alpha) = u_l(t, \mathrm{v}_\alpha) =: d_\alpha^u(t)\,, \quad t \geq 0\,, l, j \in \Gamma(\mathrm{v}_\alpha)\,, j = 1, \dots, m\,, \\ \dot{d}_\alpha^u(t) = -\sum_{j=1}^m \phi_{\alpha,j} c_j(\mathrm{v}_\alpha) u_j'(t, \mathrm{v}_\alpha) + b_\alpha d_\alpha^u(t) + \tilde{g}_\alpha(t)\left(z(t) + \dot{W}_\alpha^2(t)\right)\,, \quad t \geq 0\,, \alpha = 1, \dots, n\,, \\ u_j(0,x) = u_j^0(x)\,, \quad x \in (0,1)\,, j = 1, \dots, m\,, \\ d_i^u(0) = d_i^0\,, \quad i = 1, \dots, n\,. \end{cases} \tag{4.7}$$

Miming what we have done during previous section, we rewrite (4.7) as an abstract Cauchy problem on the Hilbert space $\mathcal{X}^2$, obtaining

$$\begin{cases} d\mathbf{u}(t)^z = \mathcal{A}\mathbf{u}^z(t)dt + G(t, \mathbf{u}^z(t))\left(Rz(t) + dW(t)\right)\,, \quad t \in [t_0, T]\,, \\ \mathbf{u}^z(t_0) = \mathbf{u}_0 \in \mathcal{X}^2\,, \end{cases} \tag{4.8}$$

where $R : \mathbb{R}^n \to \mathcal{X}^2$ is the immersion of the boundary space $\mathbb{R}^n$ into the product space $\mathcal{X}^2 := X^2 \times \mathbb{R}^n$. In the present setting the control $z$ takes values in $\mathbb{R}^n$, and $\mathcal{Z}$

is a subset of $\mathbb{R}^n$. Then, if we consider a cost functional of the form (4.2), we have, by Prop. 4.2 and Theorem 4.4, the existence of at least one ACS for the HJB equation (4.5) which is associated to the stochastic control problem (4.8)-(4.2). Moreover, we can derive the following

*Theorem* 4.4. *Let assumptions 2.2-2.9-3.1-3.4-4.1 hold, and let $w$ be a mild solution to the HJB equation* (4.5)*, and $\rho$ be an element of the generalized directional gradient $\nabla^G w$. Then, for all ACS, we have have $J(t_0, \mathbf{u}_0, z) \geq w(t_0, \mathbf{u}_0)$, and the equality holds if and only of the following feedback law is verified by $z$ and $\mathbf{u}^z$*

$$z(t) = \Gamma\left(t, \mathbf{u}^z(t), G(t, \rho(t, \mathbf{u}^z(t))\right) , \quad \mathbb{P} - a.s. \text{ for a.a. } t \in [t_0, T] . \tag{4.9}$$

*Besides, if there exists a measurable function $\gamma : [0, T] \times \mathcal{X}^2 \times \mathcal{X}^2 \to \mathcal{Z}$, with*

$$\gamma(t, \mathbf{u}, \mathbf{v}) \in \Gamma(t, \mathbf{u}, \mathbf{v}) , \quad t \in [0, T] , \mathbf{u} , \mathbf{v} \in \mathcal{X}^2 ,$$

*then there exists at least one ACS such that*

$$\bar{z}(t))\gamma(t, \mathbf{u}^z(t), \rho(t, \mathbf{u}^z(t))) , \quad \mathbb{P} - a.s. \text{ for a.a. } t \in [t_0, T] .$$

*Eventually, we have that $\mathbf{u}^{\bar{z}}$ is a mild solution to the eq.* (4.1)*.*

## 5. Conclusions

In the present paper, we have generalized previously obtained results concerning different evolution problems on networks, by taking into account a diffusion problem on a graph which has been endowed with non-local boundary static and dynamic conditions, and also considering a stochastic perturbation. We would like to underline that assumptions we made throughout the paper, could be relaxed taking into account the particular geometry of the graph, as it can be constructed according with the peculiarities of the concrete problem in which one is interested.

A second possible generalization of the results presented here, consists in considering time-non-local boundary conditions. The latter, leads to a problem that, as it is standard when dealing with delay equations, can be studied by introducing a suitable path space, with its associated corresponding operator. The price to pay regards the regularity of the leading operator, which is no longer analytic. This implies that the Gaussian estimate, obtained in the present work, does not hold, hence the *Hilbert-Schmidt* property of the semigroup has to be proved with different techniques.

## References


1. S. Albeverio and L. Di Persio; Some stochastic dynamical models in neurobiology: recent developments, *European Communications in Mathematical and Theoretical Biology*, vol. 14, (2011).
2. S. Albeverio, L. Di Persio and E. Mastrogiacomo; Invariant measures for stochastic differential equations on networks, *Spectral analysis , Differential Equations and Mathematical Physics: A Festschrift in Honor of Fritz Gesztesy's 60th Birthday*, The American Mathematical Society , (2013).
3. W. Arendt; *Semigroups and evolution equations: functional calculus, regularity and kernel estimates.* Handbook of Differential Equations: Evolutionary Equations 1 (2002).
4. V. Barbu, F. Cordoni and L. Di Persio; Optimal control of stochastic FitzHugh-Nagumo equation, *International Journal of Control*, (2015) 1-11.
5. S. Bonaccorsi, C. Marinelli and G. Ziglio; Stochastic FitzHugh-Nagumo equations on networks with impulsive noise. *Electron. J. Probab* 13 (2008): 1362-1379.
6. S. Bonaccorsi and E. Mastrogiacomo. Analysis of the stochastic Fitzhugh-Nagumo system. *Infinite Dimensional Analysis, Quantum Probability and Related Topics* 11.03 (2008): 427-446.
7. S. Bonaccorsi and G. Ziglio. A semigroup approach to stochastic dynamical boundary value problems. *Systems, control, modeling and optimization.* Springer US, 2006. 55-65.
8. V. Burenkov; *Sobolev spaces on domains.* Vol. 137. Stuttgart: Teubner, 1998.
9. S. Cardanobile and D. Mugnolo; Analysis of a FitzHugh-Nagumo-Rall model of a neuronal network. *Mathematical Methods in the Applied Sciences* 30.18 (2007): 2281-2308.
10. S. Cardanobile and D. Mugnolo; Qualitative properties of coupled parabolic systems of evolution equations. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* 7.2 (2008): 287-312.
11. S. Cardanobile, D. Mugnolo and R. Nittka. Well-posedness and symmetries of strongly coupled network equations. *Journal of Physics A: Mathematical and Theoretical* 41.5 (2008): 055102.
12. F. Cordoni and L. Di Persio; A stochastic dynamic boundary problems with time delayed boundary conditions on a network. (submitted).
13. G. Da Prato and J. Zabczyk; *Stochastic equations in infinite dimensions.* Vol. 152. Cambridge university press, 2014.
14. G. Da Prato and J. Zabczyk; *Ergodicity for infinite dimensional systems.* Vol. 229. Cambridge University Press, 1996.
15. E. B. Davies; *Heat kernels and spectral theory.* Vol. 92. Cambridge University Press, 1990.
16. E. B. Davies; Heat kernel bounds, conservation of probability and the Feller property. *Journal d'Analyse Mathématique* 58.1 (1992): 99-119.
17. L. Di Persio and G. Ziglio; Gaussian estimates on networks with applications

to optimal control. *Netw. Heterog. Media* 6 (2011), no. 2, 279-296 (2011).
18. K. J. Engel and R. Nagel; *One-parameter semigroups for linear evolution equations.* Vol. 194. Springer Science & Business Media, 2000.
19. W. H. Fleming and H. M. Soner; *Controlled Markov processes and viscosity solutions.* Vol. 25. Springer Science & Business Media, 2006.
20. M. Fuhrman and G. Tessitore; Generalized directional gradients, backward stochastic differential equations and mild solutions of semilinear parabolic equations. *Applied Mathematics and Optimization* 51.3 (2005): 279-332.
21. G. Lumer; Connecting of local operators and evolution equations on networks. *Potential Theory* Copenhagen 1979. Springer Berlin Heidelberg, 1980. 219-234.
22. D. Mugnolo; A variational approach to strongly damped wave equations. *Functional analysis and evolution equations.* Birkhäuser Basel, 2008. 503-514.
23. D. Mugnolo; Gaussian estimates for a heat equation on a network. *Netw. Heterog. Media* 2 (2007), no. 1, 55-79. (2010).
24. D. Mugnolo; *Semigroup methods for evolution equations on networks.* Springer, 2014.
25. D. Mugnolo and S. Romanelli. Dynamic and generalized Wentzell node conditions for network equations. *Mathematical methods in the applied sciences* 30.6 (2007): 681-706.
26. E. Ouhabaz; *Analysis of Heat Equations on Domains.*(LMS-31). Princeton University Press, 2009.
27. R. Tumulka; The analogue of Bohm-Bell processes on a graph. *Physics Letters A* 348.3 (2006): 126-134.