On cubic bridgeless graphs whose edge-set cannot be covered by four perfect matchings

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Abstract

The problem of establishing the number of perfect matchings necessary to cover the edge-set of a cubic bridgeless graph is strictly related to a famous conjecture of Berge and Fulkerson. In this paper we prove that deciding whether this number is at most 4 for a given cubic bridgeless graph is NP-complete. We also construct an infinite family $\mathcal{F}$ of snarks (cyclically 4-edge-connected cubic graphs of girth at least five and chromatic index four) whose edge-set cannot be covered by 4 perfect matchings. Only two such graphs were known.

It turns out that the family $\mathcal{F}$ also has interesting properties with respect to the shortest cycle cover problem. The shortest cycle cover of any cubic bridgeless graph with m edges has length at least $\frac{4}{3}m$, and we show that this inequality is strict for graphs of $\mathcal{F}$. We also construct the first known snark with no cycle cover of length less than $\frac{4}{3}m + 2$.

Keywords: Berge-Fulkerson conjecture, perfect matchings, shortest cycle cover, cubic graphs, snarks.
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1 Introduction

Throughout this paper, a graph G always means a simple connected finite graph (without loops and parallel edges). A *perfect matching* of G is a 1-regular spanning subgraph of G. In this context, a *cover* of G is a set of perfect matchings of G such that each edge of G belongs to at least one of the perfect matchings. Following the terminology introduced in [3], the *excessive index* of G, denoted by $\chi'_e(G)$, is the least integer k such that the edge-set of G can be covered by k perfect matchings.

A famous conjecture of Berge and Fulkerson [7] states that the edge-set of every cubic bridgeless graph can be covered by 6 perfect matchings, such that each edge is covered precisely twice. The second author recently proved that this conjecture is equivalent to another conjecture of Berge stating that every cubic bridgeless graph has excessive index at most five [14].

Note that a cubic bridgeless graph has excessive index 3 if and only if it is 3-edge-colorable, and deciding the latter is NP-complete. Hägglund [8, Problem 3] asked if it is possible to give a characterization of all cubic graphs with excessive index 5. In Section 2, we prove that the structure of cubic bridgeless graphs with excessive index at least five is far from trivial. More precisely, we show that deciding whether a cubic bridgeless graph has excessive index at most four (resp. equal to four) is NP-complete.

The gadgets used in the proof of NP-completeness have many 2-edge-cuts, so our first result does not say much about 3-edge-connected cubic graphs. A *snark* is a non 3-edge-colorable cubic graph with girth (length of a shortest cycle) at least five that is cyclically 4-edge connected. A question raised by Fouquet and Vanherpe [6] is whether the Petersen graph is the unique snark with excessive index at least five. This question was answered by the negative by Hägglund using a computer program [8]. He proved that the smallest snark distinct from the Petersen graph having excessive index at least five is a graph $\mathring{H}$ on 34 vertices (see Figure 7). In Section 3 we show that the graph found by Hägglund is a special member of an infinite family $\mathcal{F}$ of snarks with excessive index precisely five.

In Section 4, we study shortest cycle covers of the graphs from our family $\mathcal{F}$. A *cycle cover* of a graph G is a covering of the edge-set of G by cycles (connected subgraphs with all degrees even), such that each edge is in at least one cycle. The *length* of a cycle cover is the sum of the number of edges in each cycle. The *Shortest Cycle Cover Conjecture* of Alon and Tarsi [1]

states that every bridgeless graph G has a cycle cover of length at most $\frac{7}{5}|E(G)|$. This conjecture implies a famous conjecture due to Seymour [15] and Szekeres [17], the *Cycle Double Cover Conjecture*, which asserts that every bridgeless graph has a cycle cover such that every edge is covered precisely twice (see [12]). It turns out that the Cycle Double Cover Conjecture also has interesting connections with the excessive index of snarks. Indeed, it was proved independently by Steffen [16] and Hou, Lai, and Zhang [10] that it is enough to prove the Cycle Double Cover conjecture for snarks with excessive index at least five.

The best known upper bound on the length of a cycle cover of a bridgeless graph G, $\frac{5}{3}|E(G)|$, was obtained by Alon and Tarsi [1] and Bermond, Jackson, and Jaeger [2]. For cubic bridgeless graphs there is a trivial lower bound of $\frac{4}{3}|E(G)|$, which is tight for 3-edge-colorable cubic graphs. Jackson [11] proved an upper bound of $\frac{64}{39}|E(G)|$, and Fan [5] improved it to $\frac{44}{27}|E(G)|$. The best known upper bound in the cubic case, $\frac{34}{21}|E(G)|$, was obtained by Kaiser, Král', Lidický, Nejedlý, and Šámal [13] in 2010.

Brinkmann, Goedgebeur, Hägglund, and Markström [4] proved using a computer program that the only snarks G on m edges and at most 36 vertices having no cycle cover of length $\frac{4}{3}m$ are the Petersen graph and the 34-vertex graph $\mathring{H}$ mentioned above. Moreover, these two graphs have a cycle cover of length $\frac{4}{3}m + 1$. They also conjectured that every snark G has a cycle cover of size at most $(\frac{4}{3} + o(1))|E(G)|$.

In Section 4, we show that all the graphs G in our infinite family $\mathcal{F}$ have shortest cycle cover of length more than $\frac{4}{3}|E(G)|$. We also find the first known snark with no cycle cover of length less than $\frac{4}{3}|E(G)| + 2$ (it has 106 vertices).

Notation   Let X, Y be subsets of the vertex-set of G. We denote by ∂X the set of edges with precisely one end-vertex in X and by $\partial(X, Y)$ the set of edges with one end-vertex in X and the other in Y. If X, Y form a partition of the vertex-set of G, the set $\partial(X, Y)$ is called an *edge-cut*, or a *k-edge-cut* if k is the cardinality of $\partial(X, Y)$. It is well-known that for any k-edge-cut $\partial(X, Y)$ and any perfect matching M of a cubic graph G, the numbers $|X|$, $|Y|$, k, and $|M \cap \partial(X, Y)|$ have the same parity. This will be used implicitly several times in the remaining of the paper.

2 NP-completeness

We denote by T the *Tietze graph*, that is the graph obtained from the Petersen graph by replacing one vertex by a triangle (see Figure 1).
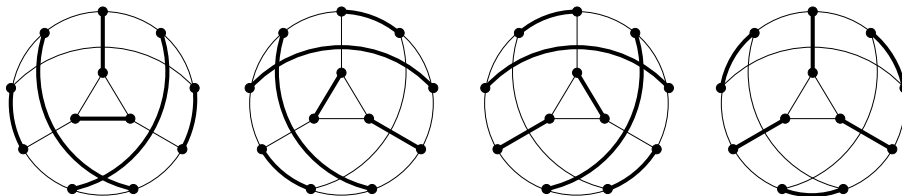

Figure 1: Four perfect matchings covering the edge-set of the Tietze graph.

It is not difficult to check that the Petersen graph has excessive index 5 (see also [6]), while the Tietze graph has excessive index 4 (see Figure 1 for a cover consisting of four perfect matchings). The next lemma describes how four perfect matchings covering the edge-set of T intersect the unique triangle of T.

Lemma 1. *For any cover $\mathcal{M}$ of T with $|\mathcal{M}| = 4$, each edge of the unique triangle of T is in precisely one perfect matching of $\mathcal{M}$.*

Proof. By parity, each perfect matching of $\mathcal{M}$ contains one or three edges incident to the unique triangle of T. If a perfect matching of $\mathcal{M}$ contains the three edges incident to T, then each of the three other perfect matchings of $\mathcal{M}$ must contain a distinct edge of T, hence the lemma holds. On the other hand, suppose for the sake of contradiction that each perfect matching of $\mathcal{M}$ contains precisely one edge incident to the triangle. Then by contracting the triangle we obtain four perfect matchings covering the Petersen graph, a contradiction. ☐

Before we prove the main result of this section, we need to describe an operation on cubic graphs. Given two cubic graphs G and H and two edges xy in G and uv in H, the *gluing*[1] of (G, x, y) and (H, u, v) is the graph obtained from G and H by removing edges xy and uv, and connecting x and u by an edge, and y and v by an edge. Note that if G and H are cubic and bridgeless, the resulting graph is also cubic and bridgeless.

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[1]This operation is sometimes called the *2-cut-connection*.

When the order of each pair (x, y) and (u, v) is not important, we simply say that *we glue the edge uv of H on the edge xy of G*. We obtain either the gluing of (G, x, y) and (H, u, v) or the gluing of (G, y, x) and (H, u, v), but which one does not matter.

We now study the complexity of bounding the excessive index of a cubic bridgeless graph. A cubic bridgeless graph has excessive index three if and only if it is 3-edge-colorable, and determining the latter is a well-known NP-complete problem (see [9]). We now prove that determining whether the excessive index is at most 4 (or equal to 4) is also hard.

Theorem 2. *Determining whether a cubic bridgeless graph G satisfies $\chi'_e(G) \leq 4$ (resp. $\chi'_e(G) = 4$) is an NP-complete problem.*

Proof. The proof proceeds by reduction to the 3-EDGE-COLORABILITY of cubic bridgeless graphs, which is NP-complete [9]. Our problem is certainly in NP, since a cover of G consisting of four perfect matchings gives a certificate that can be checked in polynomial time.

Now, let G be a cubic bridgeless graph, and let n be its number of vertices. Let G' be the graph obtained from G by replacing each vertex by a triangle. Note that G' is a cubic bridgeless graph with $3n$ vertices and n (vertex-disjoint) triangles. The $3n$ edges of G' contained in the n triangles are called the *new edges*, while the other are called the *original edges*. Let uv be an edge of the unique triangle of the Tietze graph T, and let H be the graph obtained from G' and $3n$ copies of T indexed by the $3n$ new edges of G', by gluing each new edge e of G on the edge uv of the copy T_e of T. Note that H is a cubic bridgeless graph with $39n$ vertices. We first remark that if H can be covered by k perfect matchings, then T can also be covered by k perfect matchings since for any copy T_e, any perfect matching of H contains 0 or 2 edges connecting T_e to G (and thus corresponds to a perfect matching of T). It follows that H is not 3-edge-colorable.

We now prove that G is 3-edge-colorable if and only if $\chi'_e(H) = 4$ (which is equivalent to $\chi'_e(H) \leq 4$ by the previous remark).

Suppose first that G is 3-edge-colorable, and consider three perfect matchings M_1, M_2, M_3 covering the edges of G. Each perfect matching M_i, $i \in \{1, 2, 3\}$, can be naturally extended to a perfect matching M'_i of G' such that $\{M'_1, M'_2, M'_3\}$ covers the edges of G'. These three perfect matchings can be combined with the three leftmost perfect matchings of Figure 1, for each copy T_e glued with G', to obtain three perfect matchings of H covering all

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the edges of H except three in each copy of T. We add a fourth perfect matching of H, obtained by combining the perfect matching of G' consisting of the original edges of G, with the rightmost perfect matching of Figure 1 in each copy of T. Since all the edges of H are covered by these four perfect matchings of H, we have $\chi'_e(H) \leq 4$, and since H is not 3-edge-colorable, $\chi'_e(H) = 4$.

Suppose now that $\chi'_e(H) = 4$, and let M_1, M_2, M_3, M_4 be perfect matchings covering H. By Lemma 1, these perfect matchings correspond to four perfect matchings M'_1, M'_2, M'_3, M'_4 of G' covering the edges of G', and such that each new edge of G' is covered once. Since every vertex of G' is incident to two new edges and one original edge, it follows that each original edge is covered twice. For all $i \in \{1, 2, 3\}$, color the original edges with color i if they are covered by M'_4 and M'_i, or by M'_j and M'_k with $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$. Assume that some original edge xy is covered both by M'_4 and M'_1. Then the two new edges incident to x are covered by M'_2 and M'_3, respectively, and the last edge of the new triangle containing x is covered by one of M'_4 and M'_1. It follows that neither of the two edges incident to this triangle and distinct from xy is covered by M'_4 and M'_1, or by M'_2 and M'_3. By symmetry, it follows that each color class corresponds to a perfect matching of G, and so G is 3-edge-colorable. $\qquad\square$

3 An infinite family of snarks with excessive index 5

Recall that a *snark* is a cubic graph that is cyclically 4-edge-connected, has girth at least five, and is not 3-edge-colorable. In this section we show how to construct a snark G with excessive index at least 5 from three snarks G_0, G_1, G_2 each having excessive index at least 5. Taking $G_0 = G_1 = G_2$ to be the Petersen graph, we obtain the graph $\mathring{H}$ found by Hägglund using a computer program [8], and for which no combinatorial proof showing that its excessive index is 5 has been known up to now. Our proof holds for any graph obtained using this construction, thus we exhibit an infinite family of snarks with excessive index 5. This answers the question of Fouquet and Vanherpe [6] about the existence of snarks distinct from the Petersen graph having excessive index 5 in a very strong sense.

The windmill construction[2]   For $i \in \{0, 1, 2\}$, consider a snark G_i with an edge $x_i y_i$. Let x_i^0 and x_i^1 (resp. y_i^0 and y_i^1) be the neighbors of x_i (resp. y_i) in G_i. For $i \in \{0, 1, 2\}$, let H_i be the graph obtained from G_i by removing vertices x_i and y_i. We construct a new graph G from the disjoint union of H_0, H_1, H_2 and a new vertex u as follows. For $i \in \{0, 1, 2\}$, we introduce a set $A_i = \{a_i, b_i, c_i\}$ of vertices such that a_i is adjacent to x_{i+1}^0 and y_{i-1}^0, b_i is adjacent to x_{i+1}^1 and y_{i-1}^1, and c_i is adjacent to a_i, b_i and u (here and in the following all indices i are taken modulo 3).

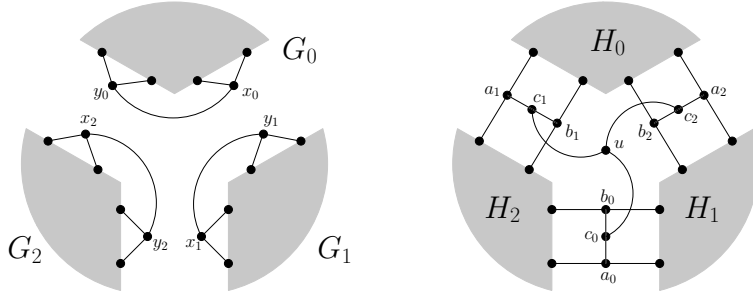The windmill construction is depicted in Figure 2.

Figure 2: The windmill construction of G (right) from G_0, G_1, and G_2.

Lemma 3. *If G_0, G_1, G_2 are snarks, any graph G obtained from G_0, G_1, G_2 by the windmill construction is cubic, cyclically 4-edge-connected, and has girth at least 5.*

Proof. It is trivial to check that G is cubic, bridgeless, and has girth at least five (we do not introduce cycles of length three or four, and the distance between any two vertices of any H_i is at least as large in G as in G_i). Note that for any $i \in \{0, 1, 2\}$, the graph $G \setminus (V(H_i) \cup V(H_{i+1}) \cup A_{i-1})$ is a subdivision of G_{i-1}. It follows that for any cyclic k-edge-cut in G, we can find a non-trivial edge-cut of cardinality at most k in some G_i. If $k \leq 3$ this non-trivial edge-cut is cyclic. Since all G_i's are cyclically 4-edge-connected, G is also cyclically 4-edge-connected. $\square$

Before we prove that G has excessive index at least five, we need a couple of definitions. By a slight abuse of notation, we denote the set of edges of

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[2]This construction should not be confused with *windmill graphs*, graphs obtained from disjoint cliques by adding a universal vertex.

G with exactly one end-vertex in $V(H_i)$ by ∂H_i (instead of $\partial(V(H_i))$). In what follows it will be useful to consider how each perfect matching M of G intersects ∂H_i. Since $|\partial H_i| = 4$, by parity we have that $|M \cap \partial H_i|$ is even. Observe that $|M \cap \partial H_i| \neq 4$, since otherwise u would be adjacent to both c_{i-1} and c_{i+1} in M. If $|M \cap \partial H_i| = 0$, we say that M is of *type* 0 on H_i. If $|M \cap \partial H_i| = 2$ we consider two cases: we say that M is of *type* 1 on H_i if $|M \cap \partial(H_i, A_{i-1})| = |M \cap \partial(H_i, A_{i+1})| = 1$, while M is of *type* 2 on H_i otherwise (in this case one of the two sets of edges $M \cap \partial(H_i, A_{i-1})$, $M \cap \partial(H_i, A_{i+1})$ has cardinality 2 and the other is empty). A perfect matching M of G of type 2 on H_0, type 1 on H_1 and type 0 on H_2 is depicted in Figure 3 (left).
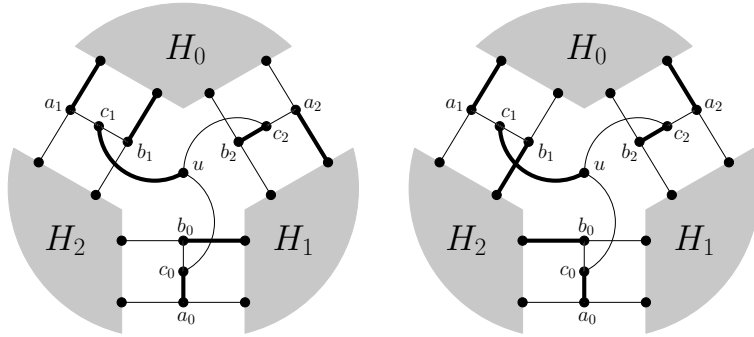

Figure 3: The two kinds of perfect matching of G (up to symmetry).

Observe first that each perfect matching of type 0 on H_i corresponds to a perfect matching of G_i containing $x_i y_i$, while each perfect matching of type 1 on H_i corresponds to a perfect matching of G_i avoiding $x_i y_i$. This observation has the following immediate consequence:

Lemma 4. *Assume that for some i and k, G can be covered by k perfect matchings, each of type 0 or 1 (and not all of type 1) on H_i. Then G_i can be covered by k perfect matchings.*

Note that a perfect matching M of G contains two edges of $\partial(H_{i-1} \cup H_{i+1}, A_i)$ if uc_i is an edge of M, and M contains one edge of $\partial(H_{i-1} \cup H_{i+1}, A_i)$ otherwise. It follows that up to symmetry (including a possible permutation of the H_i's), there are only two kinds of perfect matchings of G: perfect matching of types 0, 1, and 2 (see Figure 3, left), and perfect matching of types 0, 1, and 1 (see Figure 3, right). In particular, we have the following property:

Observation 5. *Every perfect matching of G is of type 2 on at most one H_i, type 0 on exactly one H_i, and type 1 on at least one H_i.*

We are now ready to prove our main theorem.

Theorem 6. *Let G_0, G_1, G_2 be snarks such that $\chi'_e(G_i) \geq 5$ for $i \in \{0, 1, 2\}$. Then any graph G obtained from G_0, G_1, G_2 by the windmill construction is a snark with $\chi'_e(G) \geq 5$.*

Proof. The graph G is cubic, cyclically 4-edge-connected, and has girth at least 5, as proved in Lemma 3. We now prove that G has excessive index at least five (which implies that G is not 3-edge-colorable, and thus a snark). For the sake of contradiction, let $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ be a cover of G, where M_1 might be identical to M_2 (in which case G is 3-edge-colorable).

By Observation 5, each perfect matching M_i is of type 0 on one H_j, so we can assume without loss of generality that M_1 and M_2 are of type 0 on H_0. By Lemma 4, at least one of M_3, M_4 is of type 2 on H_0, since otherwise G_0 would have excessive index at most four. If M_3, M_4 do not have the same type on H_0, then some edge of ∂H_0 is not covered, so it follows that both M_3, M_4 are of type 2 on H_0. More precisely, we can assume by symmetry that $|\partial(H_0, A_1) \cap M_3| = 2$ and $|\partial(H_0, A_2) \cap M_4| = 2$, *i.e.* uc_1 belongs to M_3 and uc_2 belongs to M_4. This implies that M_3 is of type 1 on H_1 and of type 0 on H_2, while M_4 is of type 0 on H_1 and of type 1 on H_2.

Since the edge uc_0 is not covered by $M_3 \cup M_4$, we can assume without loss of generality that M_1 contains uc_0. This implies that M_1 is of type 1 on both H_1 and H_2. Combining this with the conclusion of the previous paragraph, it follows by Lemma 4 that M_2 is of type 2 on H_1 and H_2, which contradicts Observation 5. $\square$

Let $\mathcal{F}$ be the family of all graphs inductively defined as being either the Petersen graph, or a graph obtained from three graphs G_0, G_1, G_2 of $\mathcal{F}$ by the windmill construction. Theorem 6 shows that $\mathcal{F}$ is an infinite family of snarks with excessive index at least 5.

We now prove that every graph from $\mathcal{F}$ has excessive index precisely five. We will indeed prove the stronger statement that every graph of $\mathcal{F}$ satisfies the Berge-Fulkerson conjecture, i.e. has six perfect matchings covering each edge precisely twice. Such a cover of a cubic bridgeless graph G, called a *Berge-Fulkerson cover*, can be equivalently defined as a coloring C of the edges of G with subsets of size two of $\{1, 2, 3, 4, 5, 6\}$ such that for each

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triple of edges e_1, e_2, e_3 sharing a vertex, the sets $C(e_i)$ are pairwise disjoint (equivalently, $C(e_1) \cup C(e_2) \cup C(e_3) = \{1, 2, 3, 4, 5, 6\}$).

Lemma 7. *If G_0, G_1, G_2 are cubic bridgeless graphs with a Berge-Fulkerson cover, then any graph G obtained from G_0, G_1, G_2 by the windmill construction has a Berge-Fulkerson cover.*

Proof. Consider a Berge-Fulkerson cover C of G_0, and let e_1, e_2, e_3, e_4 be the four edges sharing a vertex with the edge $x_0 y_0$, such that e_1, e_2 are incident to x_0 and e_3, e_4 are incident to y_0. Let $\{\alpha, \beta\} = C(e_1)$ and $\{\gamma, \delta\} = C(e_2)$. Then at least one of γ, δ, say γ, is such that $C(e_3)$ and $C(e_4)$ are distinct from $\{\alpha, \gamma\}$. It follows that, after renaming the colors of C, we can assume without loss of generality that $i \in C(e_i)$ for all $1 \le i \le 4$, while $x_0 y_0$ is colored $\{5, 6\}$.
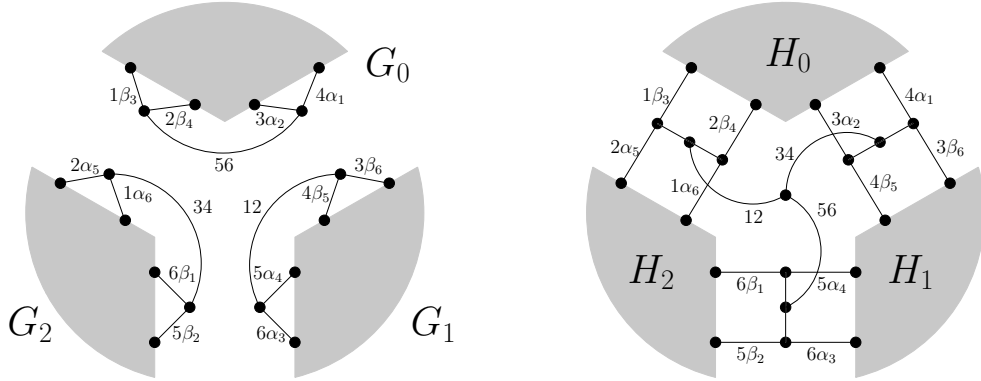

Figure 4: The extension of Berge-Fulkerson covers of G_0, G_1, G_2 to G. Colors are denoted by ij instead of $\{i, j\}$.

By symmetry, it follows that G_0, G_1, G_2 have a Berge-Fulkerson cover as depicted in Figure 4 (left), where for every $(i, j) \in \{(1, 2), (3, 4), (5, 6)\}$, $\{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = \{i, j\}$. These covers can be extended into a Berge-Fulkerson cover of G, as described in Figure 4 (right). In order not to overload the figure, we have omitted the colors of the edges $a_i c_i$ and $b_i c_i$, $0 \le i \le 2$. It can be checked that taking $C(a_0 c_0) = \{\beta_1, \alpha_4\}$, $C(b_0 c_0) = \{\beta_2, \alpha_3\}$, $C(a_1 c_1) = \{\beta_4, \alpha_6\}$, $C(b_1 c_1) = \{\beta_3, \alpha_5\}$, $C(a_2 c_2) = \{\alpha_2, \beta_5\}$, and $C(b_2 c_2) = \{\alpha_1, \beta_6\}$ defines a Berge-Fulkerson cover of G. $\qquad \square$

Theorem 6 and Lemma 7 have the following immediate corollary:

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Corollary 8. *The family $\mathcal{F}$ is an infinite family of snarks having excessive index precisely five.*

4 Shortest cycle cover

The length of a shortest cycle cover of a bridgeless graph G is denoted by $\mathrm{scc}(G)$. Note that for any cubic graph G, $\mathrm{scc}(G) \geq \frac{4}{3}|E(G)|$. The purpose of this section is to show that for any graph G in the family $\mathcal{F}$ constructed in the previous section, $\mathrm{scc}(G) > \frac{4}{3}|E(G)|$. Given some integer k, a *k-parity subgraph* of a cubic graph G is a spanning subgraph of G in which all vertices have degree one, except k vertices that have degree 3. We first observe the following:

Observation 9. *For any cubic graph G and any cycle cover $\mathcal{C}$ of G, $\mathcal{C}$ has length $\frac{4}{3}|E(G)|$ if and only if the set of edges covered twice by $\mathcal{C}$ is a perfect matching of G; and $\mathcal{C}$ has length $\frac{4}{3}|E(G)| + 1$ if and only if the set of edges covered twice by $\mathcal{C}$ is a 1-parity subgraph of G.*

Proof. For each vertex v of G, let $d_{\mathcal{C}}(v)$ be the sum of the degree of v in each of the cycles of $\mathcal{C}$. Since $\mathcal{C}$ is a cycle cover and G is cubic, for each vertex v, $d_{\mathcal{C}}(v) \geq 3$ and $d_{\mathcal{C}}(v)$ is an even integer. In particular $d_{\mathcal{C}}(v) \geq 4$. It follows that $\mathcal{C}$ has length $\frac{4}{3}|E(G)|$ if and only if for each vertex v, $d_{\mathcal{C}}(v) = 4$. This is equivalent to the fact that two edges incident to v are covered once by $\mathcal{C}$, and the third edge is covered twice by $\mathcal{C}$. Hence, $\mathcal{C}$ has length $\frac{4}{3}|E(G)|$ if and only if the set of edges covered twice by $\mathcal{C}$ is a perfect matching of G.

Similarly, $\mathcal{C}$ has length $\frac{4}{3}|E(G)| + 1$ if and only if there is a vertex u with $d_{\mathcal{C}}(u) = 6$ and for each vertex $v \neq u$, $d_{\mathcal{C}}(v) = 4$. Note that as above, if $d_{\mathcal{C}}(v) = 4$, then two edges incident to v are covered once by $\mathcal{C}$, and the third edge is covered twice by $\mathcal{C}$. In particular, no edge of G is covered more than twice by $\mathcal{C}$. Since $d_{\mathcal{C}}(u) = 6$, the only possibility is that each of the three edges incident to u is covered twice by $\mathcal{C}$. Equivalently, the set of edges covered twice by $\mathcal{C}$ is a 1-parity subgraph of G. $\qquad\square$

Let G_0, G_1, G_2 be snarks and let G be a snark obtained from G_0, G_1, G_2 by the windmill construction. Recall that the *type* of a perfect matching of G on H_i was defined in the previous section. We define the type of a 1-parity subgraph P of G on H_i similarly: If $|P \cap \partial H_i| = j$ with $j \in \{0, 4\}$, we say that P is of *type j* on H_i (as it was observed in the previous section, type 4 cannot

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occur when P is a perfect matching). If $|P \cap \partial H_i| = 2$ we consider two cases: we say that P is of *type* 1 on H_i if $|P \cap \partial(H_i, A_{i-1})| = |P \cap \partial(H_i, A_{i+1})| = 1$, while P is of *type* 2 on H_i otherwise (in this case one of the two sets of edges $P \cap \partial(H_i, A_{i-1})$, $P \cap \partial(H_i, A_{i+1})$ has cardinality 2 and the other is empty).

Let $\mathcal{C}$ be a cycle cover of G. For any edge e, we denote by $\mathcal{C}(e)$ the set of cycles of $\mathcal{C}$ containing the edge e. In the next two lemmas, we fix $i, j \in \{0, 1, 2\}$ with $i \neq j$, and call e_j and f_j the edges connecting H_i to a_j and b_j respectively.

Lemma 10. *Assume G has a cycle cover $\mathcal{C}$ of length $\frac{4}{3}|E(G)|$. Let M be the perfect matching of G consisting of the edges covered twice by $\mathcal{C}$. If there are distinct cycles $x, y \in \mathcal{C}$, such that either (i) M is of type 0 on H_i and $\{\mathcal{C}(e_j), \mathcal{C}(f_j)\} = \{\{x\}, \{y\}\}$, or (ii) M is of type 1 on H_i and $\{\mathcal{C}(e_j), \mathcal{C}(f_j)\} = \{\{x\}, \{x, y\}\}$, then G_i has a cycle cover $\mathcal{C}_i$ of length $\frac{4}{3}|E(G_i)|$.*

Proof. In both cases we obtain a cycle cover of G_i such that the set of edges covered twice is a perfect matching of G_i (see Figure 5, where cycles of G_i are represented by dashed or dotted lines for more clarity). By Observation 9 this cycle cover has length $\frac{4}{3}|E(G_i)|$. Note that in case (ii) (see Figure 5, right), x and z might be the same cycle. $\square$
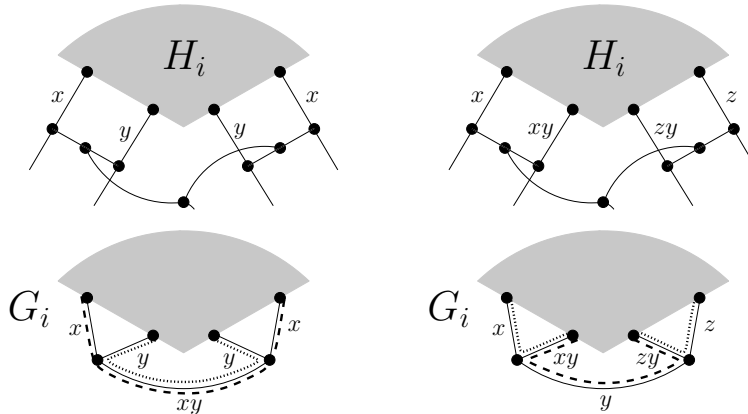

Figure 5: Constructing cycle covers of G_i from cycle covers of G.

Given a cycle cover $\mathcal{C}$ of G of length $\frac{4}{3}|E(G)| + 1$, the unique vertex of degree 3 in the 1-parity subgraph of G associated to $\mathcal{C}$ will be denoted by $t_\mathcal{C}$ (or simply t if the cycle cover is implicit). The same proof as above, applied this time to 1-parity subgraphs, gives the following slightly different lemma.

Lemma 11. *Assume G has a cycle cover $\mathcal{C}$ of length $\frac{4}{3}|E(G)| + 1$. Let P be the 1-parity subgraph of G consisting of the edges covered twice by $\mathcal{C}$. If there are distinct cycles $x, y \in \mathcal{C}$, such that either (i) P is of type 0 on H_i and $\{\mathcal{C}(e_j), \mathcal{C}(f_j)\} = \{\{x\}, \{y\}\}$, or (ii) P is of type 1 on H_i and $\{\mathcal{C}(e_j), \mathcal{C}(f_j)\} = \{\{x\}, \{x, y\}\}$, then G_i has a cycle cover $\mathcal{C}_i$ of length at most $\frac{4}{3}|E(G_i)| + 1$. Moreover, if $t_{\mathcal{C}}$ does not lie in H_i or its neighborhood in G, then $\mathcal{C}_i$ has length at most $\frac{4}{3}|E(G_i)|$.*
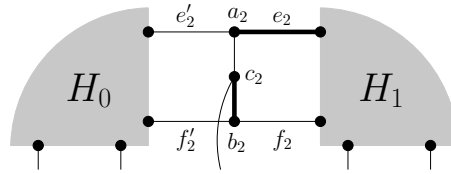

Figure 6: Notation in the proof of Theorem 12.

We now use Lemma 10 to prove the main result of this section.

Theorem 12. *Let G_0, G_1, G_2 be snarks such that $\mathrm{scc}(G_i) > \frac{4}{3}|E(G_i)|$ for $i \in \{0, 1, 2\}$. Then any graph G obtained from G_0, G_1, G_2 by the windmill construction is a snark with $\mathrm{scc}(G) > \frac{4}{3}|E(G)|$.*

Proof. Assume that G has a cycle cover $\mathcal{C}$ of length $\frac{4}{3}|E(G)|$, and let M be the perfect matching consisting of the edges covered twice by $\mathcal{C}$. By Observation 5, we can assume that M is of type 0 on H_0 and type 1 on H_1. Let e_2, f_2 be the edges connecting H_1 to a_2, b_2 and let e_2', f_2' be the edges connecting H_0 to a_2, b_2 (see Figure 6). By symmetry we can assume that e_2 and b_2c_2 are in M. Let $x, y \in \mathcal{C}$ be the two cycles covering e_2. Then one of them, say x, also covers b_2c_2, and therefore covers either f_2 or f_2'. If x covers f_2' we obtain a contradiction with Lemma 10, case (i), and if x covers f_2 we obtain a contradiction with Lemma 10, case (ii). $\square$

Recall that beside the Petersen graph, only one snark G with $\mathrm{scc}(G) > \frac{4}{3}|E(G)|$ was known [4] (and the proof of it was computer-assisted). Theorem 12 has the following immediate corollary:

Corollary 13. *The family $\mathcal{F}$ is an infinite family of snarks such that for any $G \in \mathcal{F}$, $\mathrm{scc}(G) > \frac{4}{3}|E(G)|$.*
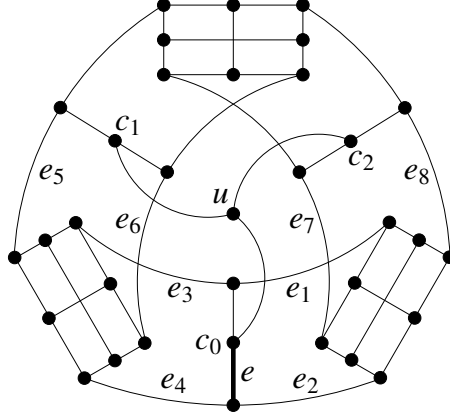
Figure 7: The graph $\mathring{H}$ in Lemma 14. The bold edge e will be used in the proof of Theorem 15.

We now prove that in a cycle cover $\mathcal{C}$ of the graph $\mathring{H}$ of Figure 7 of length $\frac{4}{3}|E(\mathring{H})| + 1$, some vertices cannot play the role of $t_{\mathcal{C}}$ (the only vertex of degree 3 in the 1-parity subgraph of $\mathring{H}$ associated to $\mathcal{C}$).

Lemma 14. *Let G_0, G_1, G_2 be three copies of the Petersen graph. Then the graph $\mathring{H}$ obtained from G_0, G_1, G_2 by the windmill construction is a snark such that:*

1. $\mathrm{scc}(\mathring{H}) = \frac{4}{3}|E(\mathring{H})| + 1$

2. For every cycle cover $\mathcal{C}$ of $\mathring{H}$ of minimum length, the vertex $t_{\mathcal{C}}$ cannot be one of the vertices c_i (see Figure 7).

Proof. The first part of the lemma was proved in [4]. Assume now that the second part fails for some cycle cover $\mathcal{C}$ of length $\frac{4}{3}|E(\mathring{H})| + 1$, and let P be the 1-parity subgraph associated to $\mathcal{C}$. By symmetry, we suppose $t_{\mathcal{C}} = c_0$. It implies that all three edges incident to c_0 are in P. As a consequence, each edge e_i, $1 \leq i \leq 4$, is covered only once by $\mathcal{C}$ (see Figure 7). It follows also that one edge incident to c_1 and distinct from $c_1 u$ is covered twice, and by parity this implies that e_5 and e_6 are covered once. By a symmetric argument, e_7 and e_8 are covered once, therefore P is of type 0 on H_1 and H_2. The edges incident to c_0 are covered by three different cycles, so at least one of the pairs e_1, e_2 and e_3, e_4 is such that the two edges in the pair are not covered by the same cycle of $\mathcal{C}$. This contradicts Lemma 11, case (i), since the Petersen graph has no cycle cover of length 20. $\square$
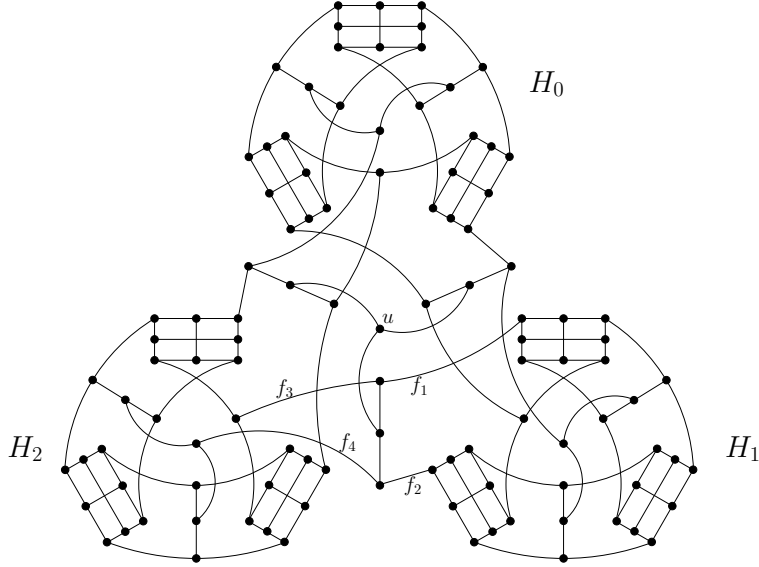
Figure 8: A snark G with $\mathrm{scc}(G) \geq \frac{4}{3}|E(G)| + 2$.

We now use Lemma 14 to exhibit the first known snark G satisfying $\mathrm{scc}(G) > \frac{4}{3}|E(G)| + 1$.

Theorem 15. *Any cycle cover of the graph G depicted in Figure 8 has length at least $\frac{4}{3}|E(G)| + 2$.*

Proof. The graph G is obtained from three copies G_0, G_1, G_2 of the graph $\mathring{H}$ of Figure 7 by the windmill construction, where each H_i is obtained from G_i by removing the endpoints of the edge e (see Figure 7). By Theorem 12, $\mathrm{scc}(G) > \frac{4}{3}|E(G)|$. Assume for the sake of contradiction that G has a cycle cover $\mathcal{C}$ of length $\frac{4}{3}|E(G)| + 1$, and let P be the 1-parity subgraph associated to $\mathcal{C}$ and $t = t_{\mathcal{C}}$ be the only vertex of degree three in P.

We claim that if P is of type 0 or 1 on H_i, then t lies in H_i. For the sake of contradiction, assume that P is of type 0 or 1 on H_i, while t lies outside of H_i (note that t might be adjacent to some vertex of H_i). Then P induces a perfect matching of G_i. The graph G_i itself was obtained from three copies G'_0, G'_1, G'_2 of the Petersen graph by the windmill construction. By Observation 5, P is of type 0 on some G'_i, and type 1 on some G'_j. Therefore we can assume, following the notation of Figure 7, that $e, e_1 \in P$ and $e_2, e_3, e_4 \notin P$. Since the Petersen graph has no cycle cover of length 20, we can assume by Lemma 10, case (i), that e_3 and e_4 belong to the same

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cycle of $\mathcal{C}$, and by Lemma 10, case (ii), that the cycle containing e_2 does not contain e_1. It can be checked that this is a contradiction.

Now, if t lies inside some H_i, remark that P satisfies the conclusion of Observation 5 (since the proof of this observation only considers the intersection of P and the edges outside the H_i's). In particular, P is of type 0 on some H_j, and type 1 on some H_k. This contradicts the claim of the previous paragraph. Similarly, if t is outside of the H_i's, but distinct from u (see Figure 8), it can be checked that P is of type 1 on some H_i, which again contradicts the previous paragraph. It follows that t coincides with u and P is of type 2 or 4 on each H_i. This implies that P is of type 2 on each H_i, so either the two edges f_1,f_2 or the two edges f_3,f_4 belong to P. Then P induces a 1-parity subgraph of G_1 (in the first case) or G_2 (in the second case), such that the unique vertex of degree three of this 1-parity subgraph coincides with a vertex c_j of G_i, contradicting Lemma 14. $\qquad\square$

5   Open problems

Hägglund proposed the following two problems (Problems 3 and 4 in [8]):

1. Is it possible to give a simple characterization of cubic graphs G with $\chi'_e(G) = 5$?

2. Are there any cyclically 5-edge-connected snarks G with excessive index at least five distinct from the Petersen graph?

While the former problem has a negative answer by Theorem 2 of the present paper (unless $P = NP$), the latter one is still open, since each element of the infinite family $\mathcal{F}$ contains cyclic 4-edge-cuts. The edge-connectivity also plays an important role in the proof of Theorem 2, in particular the gadgets we use have many 2-edge-cuts. Hence we leave open the problem of establishing whether it is possible to give a simple characterization of 3-edge-connected or cyclically 4-edge-connected cubic graphs with excessive index 5.

In Section 4, we have shown interesting properties of $\mathcal{F}$ with respect to cycle covers. In particular, Theorem 15 proves the existence of a snark $G \in \mathcal{F}$ with no cycle cover of length less than $\frac{4}{3}|E(G)|+2$. We believe that there exist snarks in $\mathcal{F}$ for which the constant 2 can be replaced by an arbitrarily large number. On the other hand, recall that Brinkmann, Goedgebeur, Hägglund, and Markström [4] conjectured that every snark G has a cycle cover of size at most $(\frac{4}{3} + o(1))|E(G)|$.

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