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ASYMPTOTIC EXPANSION FOR THE CHARACTERISTIC FUNCTION OF A MULTISCALE STOCHASTIC VOLATILITY MODEL

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Abstract: We give the first order asymptotic correction for the characteristic function of the log-return of an asset price process whose volatility is driven by two diffusion processes on two different time scales. In particular we consider a fast mean reverting process with reverting scale $\frac{1}{\epsilon}$ and a slow mean reverting process with scale δ, and we perform the expansion for the associated characteristic function, at maturity time $T > 0$, in powers of $\sqrt{\epsilon}$ and $\sqrt{\delta}$. The latter result, according, e.g., to [2, 3, 8, 11], can be exploited to compute the fair price for an option written on the asset of interest.

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1. Introduction

We consider a multi-scale market model driven by two stochastic volatility fac-

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tors acting on different time scales and driven by two one dimensional diffusion process. This type of stochastic model has been first studied by Kabanov and Pergamenshchikov, see [9, Ch.4], starting from the Vasil'eva theorem, and then extensively applied to financial markets by Fouque et.al. in [4, 6]. In particular we focus our attention on the class of multi-scale stochastic volatility models defined by the following system:

$$\begin{cases} dS_t = rS_t dt + \sigma(X_t^2, X_t^3) S_t d\bar{W}_t^1, \\ dX_t^2 = \frac{1}{\epsilon} b(X_t^2) dt + \frac{1}{\sqrt{\epsilon}} \nu(X_t^2) d\bar{W}_t^2, \\ dX_t^3 = \delta c(X_t^3) dt + \sqrt{\delta} \mu(X_t^3) d\bar{W}_t^3, \end{cases} \tag{1}$$

where $t \in [0, T]$, $T > 0$ being the expiration time of our investment, S_t describes the time behaviour of the underlying asset, X_t^2 is a fast mean reversion process and X_t^3 a slow mean reversion process, see, e.g. [4, 6], for details. All the stochastic processes involved in (1), are real valued. Moreover ϵ, resp. δ, describes the fast, resp. slow, time scale of fluctuations for the diffusion X_t^2, resp. for the diffusionX_t^3, and $\bar{W}_t^i$, $i = 1, 2, 3$, are correlated standard Brownian motions under the real world probability measure $\mathbb{P}$. Further we assume that the *instantaneous interest rate* r is a positive constant, while the functions $\sigma : \mathbb{R}^2 \to \mathbb{R}$, b, ν, c and $\mu : \mathbb{R} \to \mathbb{R}$, are assumed to be measurable and sufficiently smooth, such that the *Feynman-Kac* theorem apply. We would like to underline that a financial analysis based on two volatility time scale is justified by considering real market data, see, e.g., [4, Sect. 3.6], for details.

2. Probabilistic and Financial Setting

We consider the system in (1) in the following probabilistic setting $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration generated by the process $\mathbf{S}_t = \left(S_t, X_t^2, X_t^3\right)$, while $\mathbb{P}$ represents the so called *real world* probability measure associated to the process $\mathbf{S}_t$.

For financial reasons, see, e.g., [4, Sect.4.1], and [12, Sect.5.4], we assume to deal with an arbitrage free market, so that at least one risk-free probability measure $\mathbb{Q}$ does exist. Applying the *Girsanov theorem* and exploiting the *Itô-Doeblin* lemma with respect to the log-normal stochastic process associated to

S_t, namely defining $X_t^1 := \log S_t$, we get

$$\begin{cases} dX_t^1 = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma\left(X_t^2, X_t^3\right) dW_t^1 \\ dX_t^2 = \left[\frac{1}{\epsilon} b\left(X_t^2\right) - \frac{1}{\sqrt{\epsilon}}\nu\left(X_t^2\right)\Lambda_1\left(X_t^2, X_t^3\right)\right] dt + \frac{1}{\sqrt{\epsilon}}\nu\left(X_t^2\right) dW_t^2 \\ dX_t^3 = \left[\delta c\left(X_t^3\right) - \sqrt{\delta}\mu\left(X_t^3\right)\Lambda_2\left(X_t^2, X_t^3\right)\right] dt + \sqrt{\delta}\mu\left(X_t^3\right) dW_t^3 \end{cases}, \tag{2}$$

where the Λ_1 and Λ_2 are the combined market prices of volatility risk which determine the risk-neutral pricing measure $\mathbb{Q}$, see, e.g., [4, Ch.2] for details. Furthermore the $\mathbb{Q}$-Brownian motions $\left(W^1, W^2, W^3\right)$ are assumed to be correlated as follows

$$d\left\langle W^1 W^2\right\rangle_t = \rho_{12} dt, \quad d\left\langle W^1 W^3\right\rangle_t = \rho_{13} dt, \quad d\left\langle W^2 W^3\right\rangle_t = \rho_{23} dt\,,$$

with $|\rho_{12}| < 1$, $|\rho_{13}| < 1$, $|\rho_{23}| < 1$, $1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 > 0$, so that the covariance matrix $\{\rho_{i,j}\}_{i,j=1,\dots,3}$ is positive definite. In what follows, with a slight abuse of notation, we define $t := T - t$ the backward time in order to have the well-posedeness of the parabolic problems we will deal with, see, e.g. problem (5) below. In financial terms the variable t will denote the time to maturity, thus $t = 0$ will denote the maturity time. Moreover we assume the process X^2 to be mean-reverting and to admit a unique invariant distribution in the following sense.

Definition 2.1 (Invariant distribution)**.** Let us consider the process $X^2 = \{X_t^2 : t \in [0, T]\}$. An initial distribution Φ for X_0^2 is an *invariant distribution* for X^2 if for any $t > 0$, X^2 has the same distribution, namely

$$\frac{d}{dt}\mathbb{E}\left[g\left(X_t^2\right)\right] = \frac{d}{dt}\int \mathbb{E}\left[g\left(X_t^2\right)\middle| X_0^2 = x_2\right]\Phi\left(dx_2\right) = 0\,,\ \forall g \in C_b(\mathbb{R})\,. \tag{3}$$

The previous assumptions are not restrictive since they are satisfied if the stochastic volatility factors are driven by Ornstein-Uhlenbeck (OU) processes, as e.g. in the Cox-Ingersoll-Ross (CIR) setting.

Let Φ be the invariant distribution of the process X^2, then we define the space

$$L^2_{X^2} := \left\{ f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} |f(y)|^2 \Phi(y) < \infty \right\}\,, \tag{4}$$

moreover, for any $t \in [0, T]$, the *spot volatility* $\sigma : \mathbb{R}^2 \to \mathbb{R}$ of the process X^1 is driven by the two volatility processes X^2 and X^3 and its restriction to the first argument is required to be positive and smooth function belonging to $L^2_{X^2}$.

Our aim is to approximate the characteristic function of the log-normal value of the underlying asset S at expiring time T, namely the characteristic function of the random variable X^1_T. Our interest is motivated by the fact that the study of φ allows to numerically compute the fair price of an option written on the underlying X^1, see, e.g., [2, 3, 8, 11]. In what follows, if not otherwise stated, all the expectations are computed with respect to the risk neutral measure $\mathbb{Q}$. If we fix $t \in [0,T)$, we define the process $\{\mathbf{W}_s\}_{s\in[t,T]} = \left\{\left(W^1_s, W^2_s, W^3_s\right)\right\}_{s\in[t,T]}$, where, for $i = 1,2,3$, the process W^i is conditioned to start from $x_i \in \mathbb{R}$, we denote by $\bar{x} := (x_1, x_2, x_3)$, at time $t \in [0,T)$, and we define

$$\varphi(u;(s,\bar{x})) := \mathbb{E}\left[e^{iuX^1_T}\middle| \mathbf{W}_s = \bar{x}\right] =: \mathbb{E}^{s,\bar{x}}\left[e^{iuX^1_T}\right] ,$$

then, by the *Feynman-Kac* theorem, φ solves the following deterministic problem

$$\begin{cases} \mathcal{L}^{\epsilon,\delta}\varphi(u;(t,\bar{x})) = 0 \\ \varphi(u;(0,\bar{x})) = e^{iux_1} \end{cases} , \tag{5}$$

with

$$\mathcal{L}^{\epsilon,\delta} := \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3 ,$$

where the operators $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are defined as follows:

$$\begin{cases} \mathcal{L}_0 := \frac{1}{2}\nu^2(x_2)\partial_{x_2x_2} + b(x_2)\partial_{x_2} \\ \mathcal{L}_1 := \nu(x_2)\left[\rho_{12}\sigma(x_2,x_3)\partial_{x_1x_2} - \Lambda_1(x_2,x_3)\partial_{x_2}\right] \\ \mathcal{L}_2 := -\partial_t + \frac{1}{2}\sigma^2(x_2,x_3)\partial_{x_1x_1} + \left(r - \frac{1}{2}\sigma^2\left(x_2,x_3\right)\right)\partial_{x_1} \end{cases} , \tag{6}$$

$$\begin{cases} \mathcal{M}_1 := \mu(x_3)\left[\rho_{13}\sigma(x_2,x_3)\partial_{x_1x_3} - \Lambda_2(x_2,x_3)\partial_{x_3}\right] \\ \mathcal{M}_2 := \frac{1}{2}\mu^2(x_3)\partial_{x_3x_3} + c(x_3)\partial_{x_3} \\ \mathcal{M}_3 := \nu(x_2)\rho_{23}\mu(x_3)\partial_{x_2x_3} \end{cases} , \tag{7}$$

hence $\varphi(u;(T,\bar{x}))$ is the characteristic function of the random variable X^1_T.

3. Characteristic Function and its Expansion

Let us consider the formal expansion of the characteristic function φ in powers of $\sqrt{\epsilon}$ and $\sqrt{\delta}$, i.e. we consider $\sum_{i,j\geq 0}\left(\sqrt{\epsilon}\right)^i\left(\sqrt{\delta}\right)^j\varphi_{i,j}$, for suitable functions $\varphi_{i,j}$. In particular we would like to approximate φ considering the first order terms of the latter formal expansion, namely $\varphi_0, \varphi^\epsilon_{1,0}, \varphi^\delta_{0,1}$, defined as follows $\varphi_0 := \varphi_{0,0}$, $\varphi^\epsilon_{1,0} := \sqrt{\epsilon}\varphi_{1,0}$, $\varphi^\delta_{0,1} := \sqrt{\delta}\varphi_{0,1}$,

$$\tilde{\varphi} := \varphi_0 + \varphi^\epsilon_{1,0} + \varphi^\delta_{0,1} \; . \tag{8}$$

In order to identify the functions $\varphi_0, \varphi^\epsilon_{1,0}, \varphi^\delta_{0,1}$, we will proceed expanding first the function φ in powers of $\sqrt{\delta}$ and then in powers of $\sqrt{\epsilon}$. Expanding with respect to $\sqrt{\delta}$ gives us

$$\varphi = \varphi^\epsilon_0 + \sqrt{\delta}\varphi^\epsilon_1 + \delta\varphi^\epsilon_2 + \dots \; , \tag{9}$$

then, substituting (9) into (5), we get

$$\begin{aligned} &\left(\frac{\mathcal{L}_0}{\epsilon} + \frac{\mathcal{L}_1}{\sqrt{\epsilon}} + \mathcal{L}_2\right)\varphi^\epsilon_0 \\ &+ \sqrt{\delta}\left[\left(\frac{\mathcal{L}_0}{\epsilon} + \frac{\mathcal{L}_1}{\sqrt{\epsilon}} + \mathcal{L}_2\right)\varphi^\epsilon_1 + \left(\mathcal{M}_1 + \frac{\mathcal{M}_3}{\sqrt{\epsilon}}\right)\varphi^\epsilon_0\right] + \dots = 0 \quad . \end{aligned}$$

Equating then all the terms to 0, we obtain that φ^ϵ_0 is the unique solution to the problem

$$\begin{cases} \left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\varphi^\epsilon_0 = 0 \\ \varphi^\epsilon_0(u;(0,\bar{x})) = e^{iux_1} \; , \end{cases} \tag{10}$$

and φ^ϵ_1 is the unique solution to the problem

$$\begin{cases} \left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\varphi^\epsilon_1 = -\left(\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3\right)\varphi^\epsilon_0 \\ \varphi^\epsilon_1(u;(0,\bar{x})) = 0 \end{cases} \quad . \tag{11}$$

We expand heuristically φ^ϵ_0 in powers of $\sqrt{\epsilon}$

$$\varphi^\epsilon_0 = \varphi_0 + \sqrt{\epsilon}\varphi_{1,0} + \epsilon\varphi_{2,0} + \epsilon^{3/2}\varphi_{3,0} + \dots \; , \tag{12}$$

then we use such expansion in eq. (10), obtaining

$$\begin{aligned} &\frac{1}{\epsilon}\mathcal{L}_0\varphi_0 + \frac{1}{\sqrt{\epsilon}}\left(\mathcal{L}_0\varphi_{1,0} + \mathcal{L}_1\varphi_0\right) + 1\left(\mathcal{L}_0\varphi_{2,0} + \mathcal{L}_1\varphi_{1,0} + \mathcal{L}_2\varphi_0\right) \\ &+\sqrt{\epsilon}\left(\mathcal{L}_0\varphi_{3,0} + \mathcal{L}_1\varphi_{2,0} + \mathcal{L}_2\varphi_{1,0}\right) + \dots = 0 \; . \end{aligned}$$

Equating the term $\epsilon^{-\mathbf{1}}$ and $\sqrt{\epsilon^{-\mathbf{1}}}$ to 0 we recover the independence of φ_0 and $\varphi_{1,0}$ from the variable x_2. The next order term gives us the following

$$\mathcal{L}_2\varphi_0 + \mathcal{L}_1\varphi_{1,0} + \mathcal{L}_0\varphi_{2,0} = \mathcal{L}_2\varphi_0 + \mathcal{L}_0\varphi_{2,0} = 0 \; , \tag{13}$$

that is a Poisson equation for the function $\varphi_{2,0}$ in the x_2 variable.

Proposition 3.1. *The Poisson equation* (13) *has a solution if and only if*

$$\langle \mathcal{L}_2\varphi_0\rangle = \langle \mathcal{L}_2\rangle \varphi_0 = 0 \quad \text{with} \quad \langle f\rangle := \int f(x_2)\Phi(dx_2)\,,$$

Φ being the invariant distribution of the process X^2 as in Definition 2.1.

Proof. Let now $\mathcal{L}$ be the infinitesimal generator of the process X^2 and $\mathbf{P}_t$ the Markov transition semigroup for the process X^2, see e.g. [4] for details.

From the homogeneity in time, i.e. $\frac{d}{dt}\mathbf{P}_t f(x_2) = \mathcal{L}\mathbf{P}_t f(x_2)$ and by eq. (3), we get

$$\begin{aligned} 0 &= \frac{d}{dt}\int \mathbb{E}^{x_2}\left[f\left(X^2\right)\right]\Phi\left(dx_2\right) \\ &= \int \mathcal{L}\mathbf{P}_t f(x_2)\Phi(dx_2) = \int \mathbf{P}_t f(x_2)\mathcal{L}^*\Phi(dx_2)\,, \end{aligned} \tag{14}$$

where we have denoted by $\mathcal{L}^*$ the adjoint operator of $\mathcal{L}$ with respect to the scalar product defined on the space $L^2_{X^2}$ defined in (4). Since eq. (14) has to be satisfied for any f, we have that if an invariant distribution Φ exists, then the centering condition $\mathcal{L}^*\Phi = 0$ holds.

Averaging now eq. (13), with respect to the invariant distribution Φ, integrating by parts and eventually using the centering condition we get

$$\begin{aligned} \langle \mathcal{L}_2\varphi_0\rangle &= -\langle \mathcal{L}_0\varphi_{2,0}\rangle = -\int \left(\mathcal{L}_0\varphi_{2,0}\right)\Phi(x_2)dx_2 \\ &= \int \varphi_{2,0}\left(\mathcal{L}_0^*\Phi(x_2)\right)dx_2 = 0\,, \end{aligned}$$

since Φ satisfies both

$$\lim_{x_2\to\pm\infty}\Phi(x_2) = 0, \qquad \lim_{x_2\to\pm\infty}\Phi'(x_2) = 0\,. \qquad \square$$

Therefore we have to solve the following problem

$$\begin{cases} \langle \mathcal{L}_2\rangle \varphi_0 = 0 \\ \varphi_0(u;(0,x_1)) = e^{iux_1} \end{cases}, \tag{15}$$

where

$$\begin{aligned} \langle \mathcal{L}_2\rangle &= -\partial_t + \frac{1}{2}\left\langle \sigma^2\right\rangle \partial_{x_1x_1} + \left(r - \frac{1}{2}\left\langle \sigma^2\left(x_2,x_3\right)\right\rangle\right)\partial_{x_1} \\ &= -\partial_t + \frac{1}{2}\bar{\sigma}^2\partial_{x_1x_1} + \left(r - \bar{\sigma}^2\right)\partial_{x_1}\,, \end{aligned} \tag{16}$$

where $\bar{\sigma}^2$ stands for the so called *average effective volatility*, and it is defined as follows

$$\bar{\sigma}^2 := \bar{\sigma}^2(x_3) = \left\langle \sigma^2(x_2, x_3) \right\rangle = \int \sigma^2(x_2, x_3)\Phi(dx_2)\,.$$

By the *Feynman-Kac* theorem we have that φ_0 is the characteristic function of $X_t^1, t \in [0,T]$, given $\bar{\sigma}^2$, moreover it can be computed analytically by, e.g., the separation of variables method, obtaining

$$\varphi_0(u;(t,x_1)) = \exp\left\{ i\left[x_1 + \left(r - \bar{\sigma}^2\right)t\right]u - \frac{\bar{\sigma}^2 u^2}{2}t \right\}\,, \tag{17}$$

which evaluated at maturity time $t = T$ gives us

$$\varphi_0(u;(T,x_1)) = \exp\left\{ i\left[x_1 + \left(r - \bar{\sigma}^2\right)T\right]u - \frac{\bar{\sigma}^2 u^2}{2}T \right\}\,,$$

which agrees with the fact that $\varphi_0(u;(T,x_1))$ is the characteristic function for

$$X_T^1 \sim \mathcal{N}\left(x_1 + \left(r - \bar{\sigma}^2\right)T, \bar{\sigma}^2 T\right)\,,$$

with constant volatility $\bar{\sigma}^2$.

Turning back to the formal expansion of ϕ_0^ϵ in eq. (12), we have that the term of order $\sqrt{\epsilon}$ is related to the solution of the following Poisson equation in the x_2 variable

$$\mathcal{L}_0\varphi_{3,0} + \mathcal{L}_1\varphi_{2,0} + \mathcal{L}_2\varphi_{1,0} = 0\,.$$

Proceeding as before, we require the centering condition to hold, namely

$$\langle \mathcal{L}_2\varphi_{1,0} + \mathcal{L}_1\varphi_{2,0} \rangle = \langle \mathcal{L}_2 \rangle \varphi_{1,0} + \langle \mathcal{L}_1\varphi_{2,0} \rangle = 0\,,$$

and by eq. (13) we have

$$\varphi_{2,0} = -\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle\right)\varphi_0\,.$$

Since our aim is to retrieve the term $\varphi_{1,0}^\epsilon$ we multiply everything by $\sqrt{\epsilon}$, so that the function $\varphi_{1,0}^\epsilon$ turns to be the classical solution of

$$\begin{cases} \langle \mathcal{L}_2 \rangle \varphi_{1,0}^\epsilon = \mathcal{A}^\epsilon \varphi_0 \\ \varphi_{1,0}^\epsilon(u;(0,x_1,x_3)) = 0 \end{cases}\,, \tag{18}$$

where $\mathcal{A}^\epsilon := \sqrt{\epsilon}\left\langle \mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)\right\rangle$.

In order to compute the operator $\mathcal{A}^\epsilon$ we let $\phi(x_2,x_3)$ be a solution of $\mathcal{L}_0\phi(x_2,x_3) = \sigma^2 - \bar{\sigma}^2$ with respect to the x_2 variable, obtaining

$$\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle\right) = \frac{1}{2}\left[\phi(x_2,x_3)\partial_{x_1x_1} - \phi(x_2,x_3)\partial_{x_1}\right] ,$$

hence the operator $\mathcal{A}^\epsilon$ can be computed as follows

$$\begin{aligned}\mathcal{A}^\epsilon &= \sqrt{\epsilon}\left\langle\mathcal{L}_1\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle\right)\right\rangle \\ &= \sqrt{\epsilon}\left\langle \nu(x_2)\rho_{12}\sigma(x_2,x_3)\partial_{x_1x_2}\frac{1}{2}\phi(x_2,x_3)\partial_{x_1x_1}\right\rangle \\ &- \left\langle \nu(x_2)\rho_{12}\sigma(x_2,x_3)\partial_{x_1x_2}\frac{1}{2}\phi(x_2,x_3)\partial_{x_1}\right\rangle \\ &- \left\langle \nu(x_2)\Lambda_1(x_2,x_3)\partial_{x_2}\frac{1}{2}\phi(x_2,x_3)\partial_{x_1x_1}\right\rangle \\ &+ \left\langle \nu(x_2)\Lambda_1(x_2,x_3)\partial_{x_2}\frac{1}{2}\phi(x_2,x_3)\partial_{x_1}\right\rangle \\ &= \mathbf{A}_1^\epsilon(x_3)\left[\partial_{x_1x_1x_1} - \partial_{x_1x_1}\right] - \mathbf{A}_2^\epsilon(x_3)\left[\partial_{x_1x_1} - \partial_{x_1}\right] ,\end{aligned} \tag{19}$$

where

$$\begin{aligned}\mathbf{A}_1^\epsilon(x_3) &:= \frac{\rho_{12}\sqrt{\epsilon}}{2}\langle\nu(x_2)\sigma(x_2,x_3)\partial_{x_2}\phi(x_2,x_3)\rangle , \\ \mathbf{A}_2^\epsilon(x_3) &:= \frac{\sqrt{\epsilon}}{2}\langle\nu(x_2)\Lambda_1(x_2,x_3)\partial_{x_2}\phi(x_2,x_3)\rangle .\end{aligned}$$

It follows that the action of $\mathcal{A}^\epsilon$ on φ_0, reads as follows

$$\mathcal{A}^\epsilon\varphi_0 = \left[-\mathbf{A}_1^\epsilon(x_3)u^2(iu-1) + \mathbf{A}_2^\epsilon(x_3)u(u+i)\right]\varphi_0 = -C_u^\epsilon(x_3)\varphi_0 ,$$

where

$$-C_u^\epsilon(x_3) := \left[-\mathbf{A}_1^\epsilon(x_3)u^2(iu-1) + \mathbf{A}_2^\epsilon(x_3)u(u+i)\right] . \tag{20}$$

In what follows we will write C_u^ϵ for $C_u^\epsilon(x_3)$, keeping in mind that C_u^ϵ depends only on the x_3 variable and not on x_1 and x_2. In the following proposition we will give an analytic solution to the problem (18).

Proposition 3.2. *The correction term $\varphi_{1,0}^\epsilon$ solution to the problem* (18) *is explicitly given by*

$$\varphi_{1,0}^\epsilon = -t\mathcal{A}^\epsilon\varphi_0 = tC_u^\epsilon\varphi_0 , \tag{21}$$

where $\mathcal{A}^\epsilon$ is given in (19), *C_u^ϵ is given in* (20) *and φ_0 is given as in* (17).

Proof. In order to show that a solution for the eq. (21) is also a solution to the equation (18) it is enough to note that

$$\begin{aligned}\langle \mathcal{L}_2 \rangle \varphi^\epsilon_{1,0} &= \langle \mathcal{L}_2 \rangle (tC^\epsilon_u \varphi_0) = C^\epsilon_u \langle \mathcal{L}_2 \rangle t\varphi_0 = -C^\epsilon_u \varphi_0 + C^\epsilon_u t \langle \mathcal{L}_2 \rangle \varphi_0 \\ &= -C^\epsilon_u \varphi_0 = \mathcal{A}^\epsilon \varphi_0 \,,\end{aligned}$$

where we have used the fact that since C^ϵ_u does not depend neither on x_1 nor t, it can be taken out of the differential operator, and that $\langle \mathcal{L}_2 \rangle \varphi_0 = 0$. Moreover, since φ_0 is defined as a Fourier transform, we have $C^\epsilon_u \varphi_0$ remains bounded as $t \to 0$ so that $\lim_{t\to 0} tC^\epsilon_u \varphi_0 = 0$ and the initial condition holds. □

Since our aim is to retrieve the characteristic function of the random variable X^1_T, we take $t = T$ in eq. (21), so that, by eq. (17), we get the following

$$\varphi^\epsilon_{1,0} = -C^\epsilon_u(x_3) T \varphi_0 \,, \tag{22}$$

with C^ϵ_u given in (20).

We now heuristically expand the function φ^ϵ_1 in powers of $\sqrt{\epsilon}$

$$\varphi^\epsilon_1 = \varphi_{0,1} + \sqrt{\epsilon}\varphi_{1,1} + \epsilon\varphi_{2,1} + \epsilon^{3/2}\varphi_{3,1} + \dots \tag{23}$$

Substituting expansion (23) into eq. (11) we get

$$\begin{aligned}&\frac{1}{\epsilon}\mathcal{L}_0\varphi_{0,1} + \frac{1}{\sqrt{\epsilon}}\left(\mathcal{L}_0\varphi_{1,1} + \mathcal{L}_1\varphi_{0,1} + \mathcal{M}_3\varphi_0\right) \\ &+ \left(\mathcal{L}_0\varphi_{2,1} + \mathcal{L}_1\varphi_{1,1} + \mathcal{L}_2\varphi_{0,1} + \mathcal{M}_1\varphi_0 + \mathcal{M}_3\varphi_{1,0}\right) \\ &+ \sqrt{\epsilon}\left(\mathcal{L}_0\varphi_{3,1} + \mathcal{L}_1\varphi_{2,1} + \mathcal{L}_2\varphi_{1,1} + \mathcal{M}_1\varphi_{1,0} + \mathcal{M}_3\varphi_{2,0}\right) + \dots = 0\,.\end{aligned} \tag{24}$$

The first two terms in equation (24) give us the independence from the y variable whereas the third term gives us

$$\begin{aligned}&\mathcal{L}_0\varphi_{2,1} + \mathcal{L}_1\varphi_{1,1} + \mathcal{L}_2\varphi_{0,1} + \mathcal{M}_1\varphi_0 + \mathcal{M}_3\varphi_{1,0} \\ &= \mathcal{L}_0\varphi_{2,1} + \mathcal{L}_2\varphi_{0,1} + \mathcal{M}_1\varphi_0 = 0\,,\end{aligned}$$

which admits a solution iff

$$\langle \mathcal{L}_2\varphi_{0,1} + \mathcal{M}_1\varphi_0 \rangle = 0\,. \tag{25}$$

Multiplying eq. (25) by $\sqrt{\delta}$ and averaging $\mathcal{M}_1$ with respect to the invariant distribution Φ, we get

$$\begin{aligned}\sqrt{\delta}\langle \mathcal{M}_1 \rangle \varphi_0 = \sqrt{\delta}\mu(x_3)\langle \rho_{13}\mu(x_3)\langle \sigma(x_2,x_3)\rangle \partial_{x_1}\partial_{x_3} \\ - \Lambda_2(x_2,x_3)\rangle \partial_{x_3}\varphi_0 = 2\mathcal{A}^\delta \varphi_0 \,,\end{aligned}$$

where, since $\partial_{x_3}\varphi_0 = \partial_\sigma\varphi_0\sigma'(x_3)$, we have defined

$$\mathcal{A}^\delta := \mathbf{A}_1^\delta(x_3)\partial_\sigma\partial_{x_1} - \mathbf{A}_2^\delta(x_3)\partial_\sigma \ , \tag{26}$$

$\mathbf{A}_1^\delta, \mathbf{A}_2^\delta$ being defined as follows

$$\begin{aligned}\mathbf{A}_1^\delta(x_3) &:= \frac{\rho_{13}g(x_3)\sqrt{\delta}}{2}\left\langle f(x_2,x_3)\right\rangle \bar{\sigma}'(x_3) \ ,\\ \mathbf{A}_2^\delta(x_3) &:= \frac{g(x_3)\sqrt{\delta}}{2}\left\langle \Lambda_2(x_2,x_3)\right\rangle \bar{\sigma}'(x_3) \ .\end{aligned}$$

Moreover since the action of $\mathcal{A}^\delta$ on φ_0 is given by

$$\mathcal{A}^\delta\varphi_0 = t\left[-\left(2\bar{\sigma}ui + \bar{\sigma}u^2\right)\left(\mathbf{A}_1^\delta(x_3)iu - \mathbf{A}_2^\delta(x_3)\right)\right]\varphi_0 = tC_u^\delta\varphi_0 \ , \tag{27}$$

where

$$C_u^\delta := C_u^\delta(x_3,\bar{\sigma}) = \left[-\left(2\bar{\sigma}ui + \bar{\sigma}u^2\right)\left(\mathbf{A}_1^\delta(x_3)iu - \mathbf{A}_2^\delta(x_3)\right)\right] \ , \tag{28}$$

we have that the first order slow scale correction $\varphi_{0,1}^\delta$ is the unique classical solution to the problem

$$\begin{cases}\left\langle \mathcal{L}_2\right\rangle \varphi_{0,1}^\delta = -2\mathcal{A}^\delta\varphi_0\\ \varphi_{0,1}^\delta(u;(0,x_1,x_3)) = 0\end{cases} \ . \tag{29}$$

Proposition 3.3. *The correction term $\varphi_{0,1}^\delta$, solution to the problem* (29) *is given by*

$$\varphi_{0,1}^\delta = t^2C_u^\delta\varphi_0 \ ,$$

with $\mathcal{A}^\delta$ given in (26) *and φ_0 given in* (17).

Proof. By direct computation we have

$$\begin{aligned}\left\langle \mathcal{L}_2\right\rangle \varphi_{0,1}^\delta &= \left\langle \mathcal{L}_2\right\rangle\left(t^2C_u^\delta(x_3,\bar{\sigma})\varphi_0\right) = C_u^\delta(x_3,\bar{\sigma})\left\langle \mathcal{L}_2\right\rangle\left(t^2\varphi_0\right)\\ &= C_u^\delta(x_3,\bar{\sigma})\left[-2t\varphi_0 + t^2\left\langle \mathcal{L}_2\right\rangle\varphi_0\right] = -2C_u^\delta(x_3,\bar{\sigma})t\varphi_0 = -2\mathcal{A}^\delta\varphi_0 \ ,\end{aligned}$$

where we have used that $\left\langle \mathcal{L}_2\right\rangle\varphi_0 = 0$, hence the initial condition $\varphi_{0,1}^\delta(u;(0,x_1,x_3)) = 0$ clearly holds. □

Thus, exploiting Proposition 3.3, we evaluate $\varphi^{\delta}_{0,1}$ at time $t = T$ to be

$$\varphi^{\delta}_{0,1} = T^2 C^{\delta}_u \varphi_0 \ . \tag{30}$$

Gathering previous results, namely eq. (17), Proposition 3.3 and Proposition 3.2, we have that the first order approximation for the characteristic function of the process X^1 at maturity time 0, see eq. (8), reads as follows

$$\tilde{\varphi} = \varphi_0 \left[1 + T C^{\epsilon}_u + T^2 C^{\delta}_u \right] \ . \tag{31}$$

4. Conclusion

We derive a formal approximation for the characteristic function of the log-normal price X^1 at maturity time, see eq. (31). In particular, we derive an explicit expression, see eq. (31), for the first order correction of the characteristic function of the random variable X^1_T, $\left\{X^1_t\right\}_{t\in[0,T]}$ being the log-normal process associated to the stochastic behaviour of a given underlying asset $\{S_t\}_{t\in[0,T]}$ price process, see eq. (1). This can be used numerically to retrieved the probability to end up in the money and therefore find the fair price of an option written on the underlying X^1, see, e.g., [2, 3, 8, 11].

Moreover we would like to underline that the result stated in eq. (31) can be used, see the result in [4, Th.4.10], to obtain a bound for the difference between the exact solution φ and the approximate solution $\tilde{\varphi}$. In fact we have that, for fixed (t, x_1, x_2, x_3), there exists a constant $C > 0$ such that we can estimate the error of the approximation given in (31), as follows

$$|\varphi - \tilde{\varphi}| \leq C(\epsilon + \delta) \ , \ \forall \epsilon \leq 1 \ , \ \delta \leq 1 \ .$$

We would like to stress that, although assumptions made in [4, Th.4.10] look particularly strong, in fact they are not. The latter is due to the fact that in many financial applications the type of stochastic processes which are used to model volatility movements satisfy our requirements. In particular we have that both the Ornstein-Uhlenbeck process and the Cox-Ingersoll-Ross process, can be used as volatility processes in our setting. The assumptions on the volatility function $\sigma(x_2, x_3)$ can be relaxed with particular choices of volatility factors such as σ being an exponential function or a square root. Furthermore if we freeze the slow scale process X^3 and we assume $\sigma = \sqrt{X^2}$ with X^2 to be a CIR process, we retrieve the well known Heston model which turns to satisfy our requirements.

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