

OPTIMAL PORTFOLIO ALLOCATION WITH CVAR: A ROBUST APPROACH

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ABSTRACT. The paper discuss the sensitivity to the presence of outliers of the portfolio optimization procedure based on the expected shortfall as a measure of risk. A robust approach based on the forward search is then suggested which seems to give quite good results.

1 INTRODUCTION

The main objective of portfolio selection is the construction of portfolios that maximize expected returns at a certain level of risk. In the classical Markowitz mean-variance efficient frontier problem, estimates of the expected returns and the covariance matrix of the assets are used to calculate the optimal allocation weights (Markowitz, 1959). The drawbacks of the classical mean-variance approach have been widely discussed (Michaud, 1989). It is well known that asset returns are not normal and, therefore, the mean and the variance alone do not fully describe the characteristics of the joint asset distribution. As a consequence, especially in cases of strong nonnormality, the classical mean-variance approach will not be a satisfactory portfolio allocation model. Indeed, it is sometimes considered useless because it can lead to financially irrelevant optimal portfolios (Alexander and Baptista, 2002). Among the reasons of this drawback a relevant role is played by the influence of extreme returns (Huang *et al.*, 2008). A sensitivity analysis of the mean-variance model, together with a robust alternative, has been carried out by Grossi and Laurini (2011).

Another criticism which is commonly made to the Markowitz is the use of the historical standard deviation as a measure of risk. Several alternative measures of risk have been proposed (e.g., value of risk and expected shortfall, also called conditional value at risk) and optimizers based on these measures have been implemented (Rockafeller and Uryasev, 2002). Some of these measures are more appropriate than classical volatility (the return standard deviation) for long-tailed return models and are more realistic than the classical normal model. In addition, the computation of optimal portfolios based on these measures does not even require the return covariance matrix.

In this paper we discuss the problem of statistical robustness of optimization methods based on Spectral Risk Measures and show that the latter are not robust, meaning that a few extreme assets prices or returns can lead to “sub-optimal” portfolios. We then introduce a robust estimator based on the forward search (Atkinson *et al.*, 2004) of input parameters in the maximization procedure and show that it is far more stable than the classical version based on maximum likelihood estimator (MLE).

2 PORTFOLIO SELECTION WITH DIFFERENT MEASURES OF RISK

To introduce the general problem, let us suppose to have N risky assets, whose observed prices for T periods are $p_{it}, t = 1, \dots, T, i = 1, \dots, N$ and let $x = (x_1, \dots, x_N)'$ be the vector of portfolio weights. The assets returns are given by a matrix $Y = (y_1, \dots, y_N)$, where $y_i = (y_{i1}, \dots, y_{iT})'$ and $y_{it} = \ln(p_{it}/p_{i,t-1}) \approx (p_{it}/p_{i,t-1}) - 1$ with expected returns given by a $N \times 1$ vector μ and $N \times N$ expected covariance matrix Σ . The expected return and variance of the portfolio can be written as $\mu_p = x'\mu$ and $\sigma_p^2 = x'\Sigma x$, respectively.

For a given level of risk tolerance γ , the classical mean-variance optimization problem can be formulated as

$$\min_x (x'\Sigma x - \gamma x'\mu), \quad (1)$$

subject to the constraints $x \geq 0$ (meaning that all the weights are strictly non negative) and $x'\mathbf{1}_N = 1$, where $\mathbf{1}_N$ is a $N \times 1$ vector of ones. The constraint of no short-selling ($x \geq 0$) is very frequently imposed as many funds and institutional investors are not allowed to sell stocks short. We will make use of the no short-selling constraint throughout the paper. When this constraint is removed, it is easily proved that, using the Lagrange method, for any $\gamma \geq 0$, the maximization problem has an analytical solution. If we gradually increase γ from zero and for each instance solve the optimization problem, we end up calculating each portfolio along the efficient frontier. Loosely, the efficient frontier is the line connecting the upper boundary of the set of feasible portfolios that have the maximum return for a given level of risk.

In the present paper, we replace, as risk indicator, volatility with the Expected Shortfall (or Conditional VaR, CVaR) of the portfolio. Remind that the Expected Shortfall of V of order $\alpha \in (0, 1)$ is

$$ES_\alpha(V) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_u(V) du, \quad (2)$$

where $\text{VaR}_u(V) = -\inf\{v : P(V \leq v) \geq u\}$ is the Value-at-Risk of V of order u . Uryasev *et al.* (2000) proved a very useful result for the computation of ES:

- the function

$$G_\alpha(z, V) = z + \frac{1}{\alpha} \mathbb{E}[(-z - V)^+] \quad (3)$$

is jointly convex in (z, V) ($z \in \mathbb{R}$ and V is in a space of random variables) for any α

- In particular

$$(z, x) \mapsto z + \frac{1}{\alpha} \mathbb{E}[(-z - x'Y)^+]$$

is jointly convex in (z, x)

- the expected shortfall is computed through

$$ES_\alpha(V) = \min_{z \in Y} G_\alpha(z, V)$$

and $\text{VaR}_\alpha(V) = z^*$ is the solution of the minimization problem.

The problem 1 becomes

$$\begin{cases} \min_x \text{ES}_\alpha(x'Y) - \gamma x' \mu = \min_{x,z} \{z + \alpha^{-1} \mathbb{E}[(-z - x'Y)^+] - \gamma x' Y\} \\ \sum_{i=1}^N x_i = 1 \\ x_i \geq 0, \forall i \end{cases} \quad (4)$$

In the numerical optimization program, the trick of introducing dummy variables is used. Note that we have built the estimated frontier, as the problem (4) is not analytically solvable in general, so that no true frontier can be exactly computed (of course a true frontier exists, in principle).

3 SENSITIVITY ANALYSIS OF PORTFOLIO OPTIMIZATION

In this section, it would be useful to analyze how the CVaR is affected by the presence of outliers. It would be a preliminary step to the application of the spectral risk measures in asset allocation problems. At this step of the analysis the non-robust version of the CVAR is used to study the influence of units on portfolio weights. Also efficient frontiers should be analyzed on non-contaminated and contaminated data selecting some values of the risk aversion parameter.

A deeply studied problem of portfolio allocation (Broadie, 1993) is given by Y Gaussian with off-diagonal elements $\rho_{ij} = 0.3$ and components given by

$$\mu_Y = (0.006, 0.01, 0.014, 0.018, 0.022)' \quad (5)$$

$$\sigma_Y = (0.085, 0.08, 0.095, 0.09, 0.1)' \quad (6)$$

We will consider such parameters.

In Figure 1 the “true” efficient frontier (bold line) obtained through a Gaussian optimization is compared with efficient frontiers (thinner lines) estimated on data simulated from a Gaussian distribution with mean vector μ_Y in (5) and covariance matrix obtained from the variance vector σ_Y in (6) and constant correlation $\rho_{ij} = 0.3$. As it can be seen the range of the estimated frontiers in both axes are approximately as large as the domains of the true frontier.

Figure 2 has been drawn similarly to Figure 1, but the estimated frontiers has been obtained from contaminated data. Contamination of simulated data has been carried out introducing outliers at random positions according to the following scheme. Let U be a discrete random set of indices belonging to $\{1, \dots, T\}$ which gives the positions of the p outliers and R a multivariate Gaussian distribution with vector of means equal to zero and covariance matrix $\theta \times I_p$. The parameter θ gives the magnitude of the contamination and I_p is a p sized identity matrix. Finally the contaminated data-set is $\tilde{Y}_t = Y_t$, for $t \notin U$ and $\tilde{Y}_t = R_t$ for $t \in U$, where $Y_t \sim N(\mu, \Sigma)$, and the vector μ and matrix Σ , reported above, are taken from Broadie (1993). Notice that the scale of the axes in the two Figures are the same. In the case of contaminated data the estimated frontiers are more scattered around the true frontier and the domain of estimated frontiers has been inflated by the presence of outliers in the data.

This simulation experiment proves that a robust estimation procedure of optimal portfolio weights is needed.

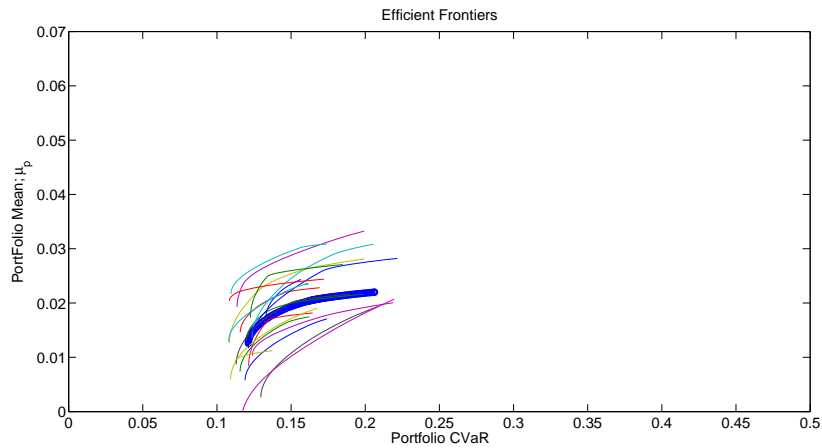


Figure 1. True efficient frontier for the covariance matrix of Broadie (1993) (bold line) compared with efficient frontiers estimated on simulated data (thin lines). Data are simulated from a Gaussian distribution with Broadie’s mean vector and covariance matrix.

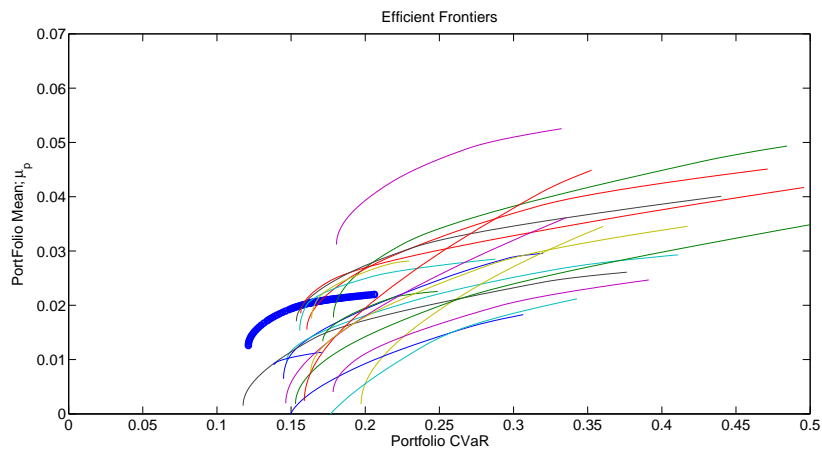


Figure 2. True efficient frontier for the covariance matrix of Broadie (1993) (bold line) compared with efficient frontiers estimated on simulated contaminated data (thin lines). Data are simulated from a Gaussian distribution with Broadie’s mean vector and covariance matrix and contaminated with additive outliers at random positions.

4 ROBUST PORTFOLIO OPTIMIZATION

The last step is to compute forward-search robust weights for each observation. The portfolio optimization procedure will then be applied to a transformed weighted matrix of returns. Finally, the robust efficient frontier will be compared with the non-robust frontier.

Our target is to compute weights $w_t \in [0, 1]$, for each observation in the multiple time series $y_t = (y_{1t}, \dots, y_{Nt})'$, $t = 1, \dots, T$, with the forward search method (Atkinson *et al.*, 2004). A similar procedure has been applied in a previous paper by Grossi and Laurini (2011) to get a robust version of the covariance matrix in the classical Markowitz problem. The weights will then be used to obtain a weighted version of the matrix Y of returns such that the most outlying observations get small weight. For multivariate data, standard methods for outlier detection are based on the squared Mahalanobis distance for the t -th observation: $d_t^2 = (y_t - \hat{\mu})' \hat{\Sigma}^{-1} (y_t - \hat{\mu})$, where both mean-vector μ and covariance matrix Σ are estimated. One of the main pitfalls of the classical Mahalanobis distance as an outlier detection tool, is the bias on the estimation of μ and Σ caused by the presence of multiple outliers. This “masking effect” of multiple outliers, is overcome by the forward search (see, Atkinson *et al.*, 2004). The goal of the forward search is the detection of units which are different from the main bulk of the observations, called Clean Data Set (CDS) and to assess the effect of these units on inferences made about the correct model.

Given the best subset $S_*^{(m)}$ of size $m \geq m_0$ detected at step m , we can calculate a set of T squared Mahalanobis distances, defined as

$$d_{t(m)}^2 = (y_t - \hat{\mu}_m^*)' (\hat{\Sigma}_m^*)^{-1} (y_t - \hat{\mu}_m^*), \quad t = 1, \dots, T, \quad (7)$$

where $\hat{\mu}_m^*$ and $\hat{\Sigma}_m^*$ are the mean and covariance matrix estimated on the m -sized subset. The distance introduced in equation (7) is the forward search version of the Mahalanobis distance. In the second step of the forward search, we increase the size of the initial CDS selecting observations with small value of (7) and so are unlikely to be outliers. Thus, with the forward search algorithm the data are ordered according to their degree of closeness to the CDS, with observations furthest from it joining the CDS in the last steps of the procedure. When μ and Σ are estimated by MLE on the whole sample, the classical Mahalanobis distances follow a scaled beta distribution. But in equation (7) the Mahalanobis distances are estimated from a subset of m observations which do not include the observation being tested. In such a case, the reference null distribution would be (see, Riani *et al.*, 2009):

$$d_{t(m)}^2 \sim [T/(T-1)][N(m-1)/(m-N)]F_{N, T-N}, \quad (8)$$

where N is the number of columns of Y .

For the computation of the weights we compare the trajectories of $d_{(t)}^{*2}$ during the forward search with confidence bands from the F distribution. Formally, for the t -th unit at step m , we define the squared Euclidean distance as: $\pi_m^{(t)} = 0$ if $d_{t(m)}^2 \in [0, F_\delta]$, $\pi_m^{(t)} = (d_{t(m)}^2 - F_\delta)^2$ if $d_{t(m)}^2 > F_\delta$, where $d_{t(m)}^2$ has been defined in (7) and F_δ is the δ percentile of the $F_{N, T-N}$ distribution.

Then, for the t -th observation, we have

$$\pi_t = \frac{\sum_{m=m_0}^T \pi_m^{(t)}}{T - m_0 + 1}. \quad (9)$$

The final outlyingness index π_{it} for each return y_{it} is then obtained as follows:

$$\pi_{it} = \pi_t \frac{\theta_{it}}{\sum_{i=1}^N \theta_{it}} \quad (10)$$

where $\theta_{it} = |y_{it}|/MAD(\mathbf{y}_i)$ and $MAD(\mathbf{y}_i)$ is the median absolute deviation from the median of \mathbf{y}_i .

Finally, the main goal of under-weighting the most extreme observations is obtained through the computation of a weight, in the interval $[0, 1]$, with the following mapping of (10): $w_{it} = \exp(-\pi_{it})$. The weights are computed for each observation at the end of the forward search.

The next step of the procedure is based on building a weighted matrix of returns Y^* , with generic element $y_{it}^* = y_{it} w_{it}^{1/2}$. The weighted matrix Y^* will be finally used as input matrix in the estimation procedure of optimal portfolio weights.

The efficient frontiers robustly estimated on contaminated data are very similar to those obtained in Figure 1.

5 FINAL REMARKS

In this paper a new robust method for estimating optimal portfolio allocation has been introduced based on the forward search. The results are quite promising. One open issue is the reference distribution for determining the threshold used to get the weight for each observation. Moreover a suitable metric to measure the distance between frontiers must be introduced. Grossi and Laurini (2011) suggested a Root Mean Error measuring the distance between frontiers considering different values of the tolerance parameter, but separately computed for the return and the volatility of portfolios.

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