SOME NEW REGULARITY PROPERTIES FOR THE MINIMAL TIME FUNCTION*

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Abstract. A minimal time problem with linear dynamics and convex target is considered. It is shown, essentially, that the epigraph of the minimal time function $T(\cdot)$ is φ-convex (i.e., it satisfies a kind of exterior sphere condition with locally uniform radius), provided $T(\cdot)$ is continuous. Several regularity properties are derived from results in [13], including twice a.e. differentiability of $T(\cdot)$ and local estimates on the total variation of DT.

Key words. Nonsmooth analysis, proximally smooth and φ-convex sets, small time controllability, functions with φ-convex epigraph.

AMS subject classifications. 49N05, 49J52

1. Introduction.

The regularity of the minimal time function $T(\cdot)$ is a widely studied topic (see, e.g., [6, 25, 7, 8, 9, 26, 4] and references therein), under different viewpoints. In particular, it is proved in [8] that with linear dynamics and convex target, $T(\cdot)$ is semiconvex provided the Petrov condition holds. The latter is equivalent to the Lipschitz continuity of $T(\cdot)$ near the target, and thus is a type of *strong* local controllability condition. Since $T(\cdot)$ is not necessarily convex (see p. 100 in [15]) even for a point-target, this is a natural regularity class for a linear minimum time problem.

Classical examples, however, exhibit minimal time functions that are not locally Lipschitz even though the system is small time locally controllable (see, e.g., [4, Example 2.7, p. 242]). Therefore, it is natural to seek conditions that identify regularity properties of $T(\cdot)$ in situations where $T(\cdot)$ is not locally Lipschitz. This motivated the results in [13], where a class of lower semicontinuous functions was studied whose epigraph satisfy an external sphere condition with locally uniform radius; this property, for general sets, is often referred to as *positive reach* [17], *φ-convexity* [16], or *proximal smoothness* [12]. Such functions are semiconvex if and only if they are locally Lipschitz, therefore are a good candidate to extend the result in [8] under more general controllability conditions. In [13], functions with φ-convex epigraph were shown to have several fine properties. In particular, a function in this class is of locally bounded variation; moreover, a.e. x admits a neighborhood where the function is indeed semiconvex, and as a consequence it is twice differentiable almost everywhere.

It will be shown below that the epigraph of $T(\cdot)$ is φ-convex, under suitable controllability assumptions. More precisely, we prove that, for a linear control problem with a convex target S, the epigraph of $T(\cdot)$ is φ-convex (Theorem 3.7), provided T is continuous. Our assumptions are satisfied in several situations, including, e.g., the case where the system fulfils the Kalman rank condition and the target is the origin. An example where Small Time Controllability does not hold, yet covered by Theorem

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3.7, is presented in §2.4.

Our analysis depends on a representation formula for the normal cone to sublevel sets of T, which is proved using simple tools of convex analysis together with Pontryagin's Maximum Principle. The techniques used here are essentially linear, due to the repeated use of explicit formulas. The main difficulty to handle is the possibility of having points where both the subdifferential and the superdifferential of T are empty, due to the lack of Lipschitz continuity. Finally, the regularity results in [13] are applied to $T(\cdot)$, and the corresponding properties of T are listed in Corollary 3.8.

We recall that for nonlinear dynamics, the semiconvexity of $T(\cdot)$ is generally not present (see, e.g., [7, Example 4.3]). However, in analogy with [7] and [9], one may expect regularity results of a similar nature under more restrictive assumptions on the target and dynamics. We mention that proving such a nonlinear result by methods analogous to ours must overcome two main difficulties: first, the existing nonlinear results rely either on the Lipschitz continuity of $T(\cdot)$ (see [8]) or are rather general, but provide substantially weaker estimates (see [9]); secondly, weaker controllability conditions lead to singularities of $T(\cdot)$ that are of both semiconvex and semiconcave type (see [5]) together with cusp points. Hence it is not clear how to obtain a nonlinear version of our Theorem 3.1, and this will be a topic of future research.

2. Preliminaries. This section briefly introduces concepts from nonsmooth analysis, geometric measure theory, and control theory.

2.1. Nonsmooth analysis. A standard reference for the nonsmooth concepts introduced here is [11]. Let $K \subseteq \mathbb{R}^n$ be closed. We denote, for $x \in \mathbb{R}^n$,

$$\begin{aligned} d_K(x) &= \min\{\|y-x\| : y \in K\} && \text{(the distance of } x \text{ from } K) \\ \pi_K(x) &= \{y \in K : \|y-x\| = d_K(x)\} && \text{(the projections of } x \text{ onto } K) \\ B(K,\rho) &= \{y \in \mathbb{R}^n : d_K(y) \le \rho\}. \end{aligned}$$

A vector v is a *proximal normal* to K at $x \in K$ (notated by $v \in N_K^P(x)$) if there exists $\sigma = \sigma(v,x) \ge 0$ such that

$$\langle v, y-x\rangle \le \sigma\|y-x\|^2 \quad \text{for all } y \in K. \tag{2.1}$$

For $v \neq 0$, then $v \in N_K^P(x)$ if and only if this there exists $\lambda > 0$ such that $\pi_K(x+\lambda v) = \{x\}$. If K is convex, then $N_K^P(x)$ equals the normal cone $N_K(x)$ to K at x as defined in convex analysis, namely the set of vectors $v \in \mathbb{R}^n$ for which

$$\langle v, y-x\rangle \le 0 \quad \text{for all } y \in K.$$

Suppose $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, and $\mathrm{epi}(f) := \{(x,\xi) : \xi \ge f(x)\}$ and $\mathrm{dom}(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$ are its epigraph and (effective) domain, respectively. Let $x \in \mathrm{dom}(f)$. A vector $\zeta \in \mathbb{R}^n$ is a *proximal subgradient* of f at x (notated by $\zeta \in \partial_P f(x)$) if $(\xi,-1) \in N^P_{\mathrm{epi}(f)}(x,f(x))$; equivalently (see [11, Theorem 1.2.5]), $\xi \in \partial_P f(x)$ if and only if there exist σ, $\eta > 0$ such that

$$f(y) \ge f(x) + \langle \zeta, y-x\rangle - \sigma\|y-x\|^2 \quad \text{for all } y \in B(x,\eta). \tag{2.2}$$

The following class of sets (see [12, §4]) will play a major role in our analysis.

Definition 2.1. *Suppose $K \subseteq \mathbb{R}^n$ is closed and $r > 0$. Then K is* r-proximally smooth *if the distance function d_K is continuously differentiable on $B(K,r) \setminus K$.*

Geometrically, in virtue of [12, Theorem 4.1], this means that every nonzero proximal normal to K is realized by an r-ball, i.e.,

$$\langle v, y-x\rangle \le \frac{1}{2r}\|y-x\|^2 \tag{2.3}$$

for all $x, y \in K$ and $v \in N_K^P(x)$, $\|v\| = 1$. Moreover, if K is proximally smooth, then the Clarke normal cone to K at x coincides with $N_K^P(x)$ for all $x \in K$, and in particular $N_K^P(x)$ is nontrivial (see [12]) at all points x on the boundary of K.

Proximal smoothness is rather restrictive for non-compact sets such as epigraphs. The following generalization allows for the constant in (2.3) to depend on x.

Definition 2.2. *Suppose $K \subseteq \mathbb{R}^n$ is closed and $\varphi : K \to [0, +\infty)$ is continuous. We say that K is φ-convex if*

$$\langle v, y-x\rangle \le \varphi(x)\|y-x\|^2. \tag{2.4}$$

for all $x, y \in K$ and $v \in N_K^P(x)$ with $\|v\| = 1$.
Comparing (2.3) and (2.4) reveals that K is r-proximally smooth if and only if it is φ-convex with $\varphi(x) = \frac{1}{2r}$ for all $x \in K$. Such sets are also referred to as prox-regular in [23], and several characterizations are known (see [17, 12, 23]). However, they will not be used here. We recall that, in particular, convex sets, or sets with a $\mathcal{C}^{1,1}$-boundary, are φ-convex.

If K is the epigraph of a continuous function $T(\cdot)$, then the φ-convexity condition (2.4) takes the form

$$\langle(\zeta,\xi),(y,\beta)-(x,\alpha)\rangle \le \varphi(x,\alpha)(\|\zeta\|+|\xi|)\left(\|y-x\|^2+|\beta-\alpha|^2\right) \tag{2.5}$$

for all $x, y \in \mathrm{dom}(T)$, $\alpha \ge T(x)$, $\beta \ge T(y)$, $(\zeta,\xi) \in N^P_{\mathrm{epi}(T)}(x,\alpha)$, with $\varphi : \mathrm{epi}(T) \to [0, +\infty)$ continuous.

2.2. Geometric measure theory. The study of some fine regularity properties of φ-convex sets and functions with φ-convex epigraph is taken up in [13], and will be quoted here below. Stating these require concepts from geometric measure theory, whose references are [1, 21].

For $0 \le k \le n$, the k-dimensional Hausdorff measure in $\mathbb{R}^n$ is denoted by $\mathcal{H}^k$. The Hausdorff dimension of a set E is $\mathcal{H}-\dim(E) := \inf\{k \ge 0 : \mathcal{H}^k(E) = 0\}$. A set $E \subseteq \mathbb{R}^n$ is *countably k-rectifiable* if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$\mathcal{H}^k\left(E \setminus \bigcup_{i=0}^{+\infty} f_i(\mathbb{R}^k)\right) = 0.$$

Let $\Omega \subset \mathbb{R}^n$ be open, and $u \in L^1(\Omega)$; we say that u is a *function of bounded variation in Ω* ($u \in BV(\Omega)$) if the distributional derivative of u is representable by a finite Radon measure in Ω, i.e., if

$$\int_\Omega u\frac{\partial\varphi}{\partial x_i}\,dx = -\int_\Omega \varphi\,dD_i u \text{ for all } \varphi \in C_c^\infty(\Omega),\ i = 1,\dots,n$$

for some finite Radon measure $Du = (D_1u, \dots, D_nu)$.

2.3. Control theory: generalities. We consider throughout the paper a linear control system of the form

$$\begin{cases} \dot{y}(t) &= Ay(t) + u(t) \quad \text{a.e. }, \\ u(t) &\in \mathcal{U} \quad \text{a.e.} \\ y(0) &= x, \end{cases} \tag{2.6}$$

where $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. The control set $\mathcal{U} \subset \mathbb{R}^n$ is compact and convex, and the control function $u(\cdot)$ is measurable. For all $t > 0$, we denote by $\mathcal{U}^t_{\mathrm{ad}}$ the set of admissible controls, i.e., the measurable functions $u : [0,t] \to \mathbb{R}^n$, such that $u(t) \in \mathcal{U}$ a.e. on $[0,t]$. For any $u(\cdot) \in \mathcal{U}^t_{\mathrm{ad}}$, the unique Carathéodory solution of (2.6) is denoted by $y^{x,u}(\cdot)$.

Suppose we are now given a closed nonempty set $S \subset \Omega$, which is called the *target set*. For fixed $x \notin S$, the *minimal time* $T(x)$ to reach S from x is defined by

$$T(x) := \inf\{T \geq 0 \,:\, \exists\, u(\cdot) \text{ such that } y^{x,u}(T) \in S\}.$$

When the set of controls $u(\cdot)$ steering x to S is empty, then $T(x) = +\infty$. Since the velocity sets $F(y) := \{Ay + u : u \in \mathcal{U}\}$ are convex, then standard arguments (see [10, Theorem 9.2.i, p. 311]) show the infimum is actually a minimum (provided it is finite); that is, there exists an optimal control steering x to S in the minimal time.

The reachable set from a point $x \in \Omega$ in time T is the set:

$$R^T(x) = \{y(T) : y(\cdot) \text{ satisfies (2.6)}\}$$

If $\bar{x} \in R^T(x)$, then $\bar{x}$ is *realized* by the control function $\bar{u}(\cdot)$ if $\bar{x} = y^{x,\bar{u}}(T)$. Note that $\bar{x} \in R^T(x)$ is realized by $\bar{u}(\cdot)$ if and only if the (equivalent) formulas

$$\bar{x} = e^{AT}x + \int_0^T e^{A(T-t)}\bar{u}(t)\,dt \quad \text{and} \quad x = e^{-AT}\bar{x} - \int_0^T e^{-At}\bar{u}(t)\,dt \tag{2.7}$$

hold. It is well known that $R^T(x)$ is convex and compact. It is convenient to also notate as $R^T_-(\bar{x})$ the reversed-time reachable set from a point $\bar{x}$, which is the reachable set associated to the dynamics $\dot{y} = -Ay - u$. Namely,

$$R^T_-(\bar{x}) = \{y(T) : \dot{y}(t) = -Ay(t) - u(t),\, u(\cdot) \in \mathcal{U}^T_{\mathrm{ad}} \text{ a.e.},\; y(0) = \bar{x}\}.$$

It is clear that $\bar{x} \in R^T(x)$ if and only if $x \in R^T_-(\bar{x})$. For $r > 0$, let

$$\begin{aligned} S(r) &= \{x \in \mathbb{R}^n : T(x) \leq r\}, \\ \mathcal{R} &= \{x \in \mathbb{R}^n : T(x) < +\infty\}, \end{aligned}$$

and observe

$$S(r) = \bigcup_{\bar{x}\in S,\, 0\leq T\leq r} R^T_-(\bar{x}).$$

Recall that a closed set $S \subseteq \Omega$ is *strongly invariant* for the system (2.6) if for all $x \in S$ and $T > 0$, one has $R^T(x) \subseteq S$. Analogously, S is *weakly invariant* (or *viable*) if for all $x \in S$ and all small $T > 0$, there exists a trajectory of (2.6) which remains in S for all $t \in [0,T]$.

A major tool in our analysis is the minimized Hamiltonian $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, given by

$$h(x,\zeta) = \langle Ax, \zeta\rangle + \min_{u\in\mathcal{U}} \langle u, \zeta\rangle. \tag{2.8}$$

It is known that a set S is weakly invariant for the dynamics (2.6) if $h(x,\zeta) \leq 0$ for all $x \in S$ and $\zeta \in N_S^P(x)$ (see [25], [11, Theorem 2.10]).

The *adjoint equation* associated with (2.6) is

$$\begin{cases} \dot{p}(t) = -A^\top p(t) \\ p(T) = \bar{p}, \end{cases} \tag{2.9}$$

and an adjoint arc is

$$p(t) = e^{A^\top (T-t)}\bar{p}, \tag{2.10}$$

which is the solution of (2.9). The Pontryagin's Maximum Principle is stated next.

Proposition 2.3 (Maximum Principle). *Suppose $\bar{x} \in R^T(x)$ is realized by $\bar{u}(\cdot)$. Then $\bar{x} \in$ bdry $R^T(x)$ (= the boundary of $R^T(x)$) if and only if there exists $\bar{p} \neq 0$ so that the solution $\bar{p}(\cdot)$ of (2.9) satisfies*

$$\langle \bar{p}(t), \bar{u}(t)\rangle = \max_{u\in U} \langle \bar{p}(t), u\rangle \tag{2.11}$$

for almost all $t \in [0,T]$. Moreover, in this case, $p(t) \in N_{R^t(x)}\big(y^{x,\bar{u}}(t)\big)$ for each $t \in [0,T]$.

A standard reference for the proof is [18], §13.

2.4. Continuity of the minimal time function. Continuity properties of the minimal time function is a widely studied topic, mainly in connection with controllability. We refer to Chapter IV, §1 in [4] and references therein for an introduction to the subject.

Definition 2.4. *The control system (2.6) is* small time controllable *(STC) near the target S if $S \subseteq int\ S(r)$ for all small $r > 0$.*

We collect some known results relating STC to continuity of $T(\cdot)$, with main emphasis on a target more general than a singleton, in the following theorem.

Theorem 2.5. *Assume (for simplicity) that S is compact.*

(1) *Suppose $S = \{0\}$ and $0 \in \operatorname{rel\,int} S(r)$ for all $r > 0$. Then $T(\cdot)$ is continuous on $\mathcal{R}$.*

(2) (generalized Petrov condition) *Suppose there exist $\delta > 0$ and a continuous nondecreasing function $\mu : [0,\delta] \to [0,+\infty)$ with the properties*

(a) *$\mu(0) = 0$, $\mu(\rho) > 0$ for $\rho > 0$, and $\int_0^\delta \frac{d\rho}{\mu(\rho)} < +\infty$,*

(b) *for all $x \in B(S,\delta) \setminus S$ there exists $\bar{s} \in \pi_S(x)$ such that*

$$h(x, x-\bar{s}) \leq -\mu(\|x-\bar{s}\|)\|x-\bar{s}\|. \tag{2.12}$$

Then the system (2.6) is STC near S and the minimal time function is continuous in a neighborhood of S.

(3) (second order Petrov condition) *Suppose that S is the closure of an open set with $\mathcal{C}^2$-boundary, and assume that there exist $\delta > 0$ and $\eta > 0$ such that for all $x \in B(S,\delta) \setminus S$*

(a) *$h(x, Dd_S(x)) \leq 0$,*

(b) $\langle Dd_S(x), A^2x\rangle + 2\left\langle \langle D^2 d_S(x), Ax\rangle, Ax\right\rangle \le -\eta$.
Then the system (2.6) is STC near S and the minimal time function is Hölder continuous with exponent $1/2$ in a neighborhood of S.

(4) Suppose $S = \{0\}$ and $\mathcal{U} = \{Bu : u \in \mathbb{R}^m,\ u \in [-1,1]^m\}$, $B \in Mat_{n\times m}(\mathbb{R})$. The following are equivalent for a fixed integer k, $k = 0, 1, \dots, n-1$.
  (a) $T(\cdot)$ is Hölder continuous in $\mathbb{R}^n$ with exponent $1/(k+1)$;
  (b) (Kalman rank condition)

$$\mathrm{rank}[B, AB, ..., A^kB] = n.$$

Proof. The proof of (1) is in [15, Theorem II.4.3]. Various versions of (2), obtained with different methods, can be found, e.g., in [25], [7], [8, Chapter 8, §8.2], [24], [19], [22], [20]. Condition (3) is a particular case of a controllability result contained in [20]. The proof of (4) can be found in [3, Chapter 2, §6]. □

We will consider a slightly more general situation, where the continuity of the minimal time function is not directly linked to a STC condition. We illustrate this with a simple example.

EXAMPLE 1: Let $\alpha > 1$ and $S = \{(x, y) \in \mathbb{R}^2 : y \ge |x|^\alpha\}$. Let $\mathcal{U} = [-1, 1]$ and consider the linear control system

$$\begin{cases} \dot{x} &=& u \in \mathcal{U} \\ \dot{y} &=& 0 \end{cases} \tag{2.13}$$

None of the conditions listed in Theorem 2.5 is satisfied in a neighborhood of S, and actually $\mathcal{R} = \mathbb{R} \times [0, +\infty)$ is not a neighborhood of S. Let $x > 0$, $0 \le y < x^\alpha$. Then $T(x, y) = x - y^{1/\alpha}$, which is continuous on $\mathcal{R}\setminus S$, but not locally Lipschitz. We observe that for all (x, y), there exists a control $u(x, y)$ (actually $u(x, y) = -\mathrm{sgn}(x)$) such that $A(x, y) + u(x, y) = (u(x, y), 0)$ points towards S. However, the angle between the vector pointing to S and the external normal to S is not uniformly bounded away from 0, and in fact this angle tends to 0 as $(x, y) \to (0, 0)$. We estimate its rate of convergence to 0 along the x-axis. Let $\xi(x) = x + \alpha x^{2\alpha-1}$. Observe that the segment joining $(\xi(x), 0)$ and (x, x^α) is orthogonal to the graph of $y = x^\alpha$ at (x, x^α). Moreover, $\xi(x) \sim x$ for $x \to 0$ and

$$d_S((\xi(x), 0)) = x^\alpha\sqrt{1 + \alpha^2 x^{2(\alpha-1)}} \sim x^\alpha \text{ for } x \to 0.$$

Finally,

$$\begin{aligned} \min_{u\in[-1,1]} \left\langle (u, 0), \frac{(\xi(x), 0) - (x, x^\alpha)}{d_S(\xi(x), 0)} \right\rangle &= \frac{x - \xi(x)}{d_S(\xi(x), 0)} \\ &= -\frac{\alpha x^{\alpha-1}}{\sqrt{1 + \alpha^2 x^{2(\alpha-1)}}} \\ &\le -\mathrm{const}\,(d_S(\xi(x), 0))^{\frac{\alpha-1}{\alpha}}. \quad \square \end{aligned}$$

In this example, the angle satisfies an estimate of the type (2.12). However, this estimate does not hold in an entire neighborhood of S, and the continuity of T in $\mathcal{R}$ is not covered by any of the statements in Theorem 2.5. The forthcoming paper [20] contains a result covering the Hölder continuity of T in $\mathcal{R}$ also in the above example.

3. The epigraph of the minimal time function, and differentiability properties. We repeat the setting we are concerned with. We consider the linear system

$$\begin{cases} \dot{y}(t) &= Ay(t)+u(t) \quad \text{a.e.} \\ y(0) &= x \\ u(t) &\in \mathcal{U} \quad \text{a.e.} \end{cases} \tag{3.1}$$

with $\mathcal{U} \subseteq \mathbb{R}^n$ compact and convex. Let $S \neq \emptyset$ be the target set.

Let $\delta > 0$ be given, and set $\mathcal{R}_\delta = S(\delta)\backslash S$. We make the following further assumptions:
(H1) S is closed and convex, and $h(x,\zeta) \leq 0$ for all $x \in S$ and $\zeta \in N_S(x)$;
(H2) $T(\cdot)$ is continuous in $S(\delta)$.
Observe that (H1) and (H2) do not imply the Small Time Controllability, because $S(\delta)$ is not required to be a neighborhood of S. Such a situation is illustrated by Example 1 in §2.4.

The following result is an easy consequence of (H1).

Proposition 3.1. *Under the above assumption (H1), the sets $S(r)$ are compact and convex, and if $r_1 \leq r_2$ we have $S(r_1) \subseteq S(r_2)$. Therefore $\mathcal{R}$ is convex.*
We need a few technical lemmas. A version of some of them (Lemmas 3.2 and 3.3) already appeared in [14, §2]. We repeat the proofs here, in order to make this paper more self contained. The first two ones concern a representation of the normal cone to the level sets of T and of the proximal subdifferential of T.

Lemma 3.2. *Let (H1) hold, and let $r \geq 0$, $x \in \mathbb{R}^n$ with $T(x) = r$, and $\bar{x} \in S \cap R^r(x)$. Then*

$$N_{S(r)}(x) = \Big\{ -e^{A^\top r}\bar{p} : \bar{p} \in [-N_S(\bar{x})] \cap N_{R^r(x)}(\bar{x}) \Big\}, \tag{3.2}$$

and therefore the righthand side is independent of $\bar{x} \in S \cap R^r(x)$.

Proof. (See also [14, Theorems 4 and 8]).
"$\subseteq$." Let $\zeta \in N_{S(r)}(x)$. Then, by convexity,

$$\langle \zeta, y - x\rangle \leq 0 \quad \forall y \in S(r). \tag{3.3}$$

Let $\bar{u}(\cdot) \in \mathcal{U}^r_{ad}$ be an admissible control that realizes $\bar{x}$, and thus (2.7) holds with $T = r$. The rest of the proof is broken into two claims.
Claim 1: $e^{-A^\top r}\zeta \in N_S(\bar{x})$.
Proof of Claim 1. Let $\bar{y} \in S$, and define

$$y = e^{-Ar}\bar{y} - \int_0^r e^{-At}\bar{u}(t)\,dt, \tag{3.4}$$

which therefore belongs to $S(r)$. We have

$$\begin{aligned}\langle e^{-A^\top r}\zeta, \bar{y}-\bar{x}\rangle &= \langle \zeta, e^{-Ar}\bar{y} - e^{-Ar}\bar{x}\rangle \\ &= \langle \zeta, y - x\rangle \quad \text{(by (2.7) and (3.4))} \\ &\leq 0 \quad \text{(by (3.3) and since } y \in S(r)).\end{aligned}$$

It follows that $e^{-A^\top r}\zeta \in N_S(\bar{x})$.

Claim 2. $-e^{-A^\top r}\zeta \in N_{R^r(x)}(\bar{x})$.
Proof. First note that $x \in R^r_-(\bar{x}) \subseteq S(r)$, and therefore $\zeta \in N_{R^r_-(\bar{x})}(x)$. By Proposition 2.3 applied to the reversed time data $-A$ and $-\mathcal{U}$, we have that, for all $t \in [0, r]$,

$$\left\langle -e^{-A^\top t}\zeta, \bar{u}(t) \right\rangle = \max_{u \in U} \left\langle -e^{-A^\top t}\zeta, u \right\rangle. \tag{3.5}$$

Now suppose $\bar{y} \in R^r(x)$, so that

$$\bar{y} = e^{Ar}x + \int_0^r e^{A(r-t)}u(t)\,dt$$

for some $u(\cdot) \in \mathcal{U}^r_{ad}$. We have

$$\begin{aligned}\left\langle -e^{-A^\top r}\zeta, \bar{y} - \bar{x} \right\rangle &= \left\langle -e^{-A^\top r}\zeta, \int_0^r e^{A(r-t)}\big(u(t) - \bar{u}(t)\big)\,dt \right\rangle \\ &= \int_0^r \left\langle -e^{-A^\top t}\zeta, u(t) - \bar{u}(t) \right\rangle\,dt \\ &\leq 0,\end{aligned}$$

where the last inequality follows from (3.5). The validity of Claim 2 is now established.

It is clear that the "$\subseteq$" inclusion in (3.2) follows from Claims 1 and 2.

"$\supseteq$". Let $\bar{x} \in S \cap R^r(x)$, and let $\bar{p} \in [-N_S(\bar{x})] \cap N_{R^r(x)}(\bar{x})$. Let $y \in S(r)$ and $\bar{y} \in S \cap R^r(y)$. Respectively, let $u(\cdot)$, $\bar{u}(\cdot) \in \mathcal{U}^T_{ad}$ realize $\bar{y}$, $\bar{x}$, and thus

$$\begin{aligned} y &= e^{-Ar}\bar{y} - \int_0^r e^{-At}u(t)\,dt \\ x &= e^{-Ar}\bar{x} - \int_0^r e^{-At}\bar{u}(t)\,dt. \end{aligned} \tag{3.6}$$

We have

$$\begin{aligned}\left\langle -e^{A^\top r}\bar{p}, y - x \right\rangle &= \left\langle -\bar{p}, e^{Ar}\big(y - x\big) \right\rangle \\ &= \left\langle -\bar{p}, \bar{y} - \bar{x} \right\rangle + \int_0^r \left\langle -p(t), \bar{u}(t) - u(t) \right\rangle\,dt\end{aligned}$$

by (3.6). Since $-\bar{p} \in N_S(\bar{x})$, the first term on the righthand side of the previous expression is nonpositive. By Maximum Principle, the second term is also nonpositive. Hence the assertion $-e^{A^\top r}\bar{p} \in N_{S(r)}(x)$ follows, and the proof is concluded. □

LEMMA 3.3. *Let the assumption (H1) hold. Let $x \in S(r)$, $T(x) = r > 0$ and let $\bar{x} \in S \cap R^r(x)$. Then a vector ζ belongs to $\partial_P T(x)$ if and only if*

$$h(x, \zeta) = -1$$

and

$$-e^{-A^\top r}\zeta \in \big[-N_S(\bar{x})\big] \cap N_{R^r(x)}(\bar{x}).$$

Proof. By Theorem 5.1 in [26],

$$\partial_P T(x) = N_{S(r)}(x) \bigcap \{\zeta : h(x,\zeta) = -1\}. \tag{3.7}$$

Then the statement follows from Lemma 3.2. □

The next three lemmas concern the Hamiltonian, mainly in connection with normal vectors to the epigraph of T.

LEMMA 3.4. *Let $r > 0$, $x_0 \in S(r)$. If $\zeta \in N_{S(r)}(x_0)$, then $h(x_0, \zeta) \le 0$.*

Proof. By contradiction, let $\zeta \in N_{S(r)}(x_0)$ be such that $h(x_0,\zeta) > 0$. By definition of Hamiltonian, we have $\zeta \neq 0$. Let $x(\cdot)$ be an optimal trajectory starting from $x(0) = x_0$ and let $u(\cdot)$ be an optimal control realizing $x(\cdot)$. Let $z = x_0 + \zeta$. We are now going to contradict the Dynamic Programming Principle. Indeed, by convexity of $S(r)$, it is enough to show that there exists $\eta > 0$ such that $x(t) \in B(z, \|\zeta\|)$ for all $t \in (0,\eta)$. In fact this implies that $x(t) \notin S(r)$ for all $t \in (0,\eta)$, i.e. there exists $0 < \bar{t} < T(x_0)$ such that $T(x(\bar{t})) > T(x_0)$, which is against the optimality of $x(\cdot)$. We have:

$$\begin{aligned}
\frac{d}{dt}\|x(t) - z\|^2 &= \frac{d}{dt}\left\langle x(t) - x_0 - \zeta, x(t) - x_0 - \zeta\right\rangle \\
&= 2\langle \dot{x}(t), x(t) - x_0 - \zeta\rangle \\
&= 2\langle \dot{x}(t), x(t) - x_0\rangle - 2\langle Ax(t) + u(t), \zeta\rangle \\
&\le 2\left(K^2 t - h(x(t),\zeta)\right),
\end{aligned}$$

where K is a bound on $\|\dot{x}\|$. According to our hypothesis $h(x(0),\zeta) > 0$, so for small t we have by continuity $\frac{d}{dt}\|x(t) - z\|^2 < 0$, which implies $x(t) \in B(z, \|\zeta\|)$. □

LEMMA 3.5. *Let $(H1)$ hold. Let $r > 0$, and let $x_0 \in \mathbb{R}^n$ be such that $T(x_0) = r$. If $(\zeta, 0) \in N^P_{\mathrm{epi}(T)}(x_0, T(x_0))$, then $\zeta \in N_{S(r)}(x_0)$ and $h(x_0,\zeta) \le 0$.*

Proof. In view of Lemma 3.4, it is enough to show that $\zeta \in N_{S(r)}(x_0)$. To this aim, observe that there exists $\sigma > 0$ such that

$$\langle (\zeta,0), (y,\xi) - (x_0, T(x_0))\rangle \le \sigma(\|x_0 - y\|^2 + |T(x_0) - \xi|^2) \tag{3.8}$$

for all $(y,\xi) \in \mathrm{epi}(T)$. In particular for $y \in S(r)$ and $\xi = r$ the inequality (3.8) yields

$$\langle \zeta, y - x_0\rangle \le \sigma\|x_0 - y\|^2,$$

and this says that $\zeta \in N^P_{S(r)}(x_0)$. Since $S(r)$ is convex, this fact is equivalent to $\zeta \in N_{S(r)}(x_0)$. The proof is concluded. □

LEMMA 3.6. *Let $r > 0$, $x_0 \in S(r)$, $T(x_0) = r$. If $(\zeta, -1) \in N^P_{\mathrm{epi}(T)}(x_0, T(x_0))$ then $h(x_0,\zeta) = -1$.*

Proof. By hypothesis, $\zeta \in \partial_P T(x_0)$; then apply Lemma 3.3. □

The following is the main result of the paper.

THEOREM 3.7. *Consider the system (3.1) with the assumptions (H1), (H2) Then there exists a continuous function φ such that the epigraph of $T_{|\mathcal{R}_\delta}$ is φ-convex.*

Proof. The proof consists of two steps. In the first step we establish an inequality of the type (2.5) for a particular choice of points in $\mathrm{epi}(T)$, by assuming that S is compact. In the second one, we show that the inequality proved in the first step holds in general.

STEP 1: Let S be compact. We claim that there exists $K = K(\delta) > 0$ with the following property: for all $x_1, x_2 \in \mathcal{R}_\delta$, for all $(\zeta, \xi) \in N^P_{\mathrm{epi}(T)}(x_1, T(x_1))$ with $\xi \in \{0, -1\}$ it holds

$$\langle (\zeta,\xi),(x_2,T(x_2))-(x_1,T(x_1))\rangle \le K(\|\zeta\|+|\xi|)(\|x_2-x_1\|^2+|T(x_2)-T(x_1)|^2). \quad (3.9)$$

Proof of Step 1.

Let $r_1 = T(x_1)$, $r_2 = T(x_2)$. Let u_i be an optimal control steering x_i to $\bar{x}_i \in S$ in time r_i for $i = 1, 2$. Take $(\zeta, \xi) \in N^P_{\mathrm{epi}(T)}(x_1, T(x_1))$.

We have the following possibilities:

1. $\xi = -1$: in this case $\zeta \in \partial_P T(x_1)$ and, by Lemma 3.3, we have $h(x_1, \zeta) = -1$ and there exists $p \in N_{R^{r_1}(x_1)}(\bar{x}_1) \cap [-N_S(\bar{x}_1)]$ such that $\zeta = -e^{A^\top r_1} p$.
2. $\xi = 0$: in this case, by Lemma 3.5 we have $\zeta \in N_{S(r_1)}(x_1)$ and $h(x_1, \zeta) \le 0$, and by Lemma 3.2 there exists $p \in N_{R^{r_1}(x_1)}(\bar{x}_1) \cap [-N_S(\bar{x}_1)]$ such that $\zeta = -e^{A^\top r_1} p$.

In both cases, we have the existence of $p \in N_{R^{r_1}(x_1)}(\bar{x}_1) \cap [-N_S(\bar{x}_1)]$ such that $\zeta = -e^{A^\top r_1} p$. By Pontryagin Maximum Principle,

$$\langle p(t), u_1(t)\rangle = \max_{u\in\mathcal{U}} \langle p(t), u\rangle$$

for a. e. t, where $p(t) = e^{A^\top (r_1 - t)} p$, and $\zeta = -p(0)(= -p)$.

Now suppose $r_2 \le r_1$ and define

$$\begin{aligned} y &:= e^{A(r_1-r_2)} x_1 + \int_0^{r_1-r_2} e^{A(r_1-r_2-t)} u_1(t)\,dt \\ &= e^{-Ar_2}\bar{x}_1 - \int_{r_1-r_2}^{r_1} e^{A(r_1-r_2-t)} u_1(t)\,dt. \end{aligned}$$

We have:

$$\begin{aligned} \langle \zeta, x_2 - x_1\rangle &= \langle p(r_1-r_2) - p(0), x_2 - x_1\rangle + \langle -p(r_1-r_2), x_2 - y\rangle + \\ &\quad + \langle -p(r_1-r_2), y - x_1\rangle \\ &=: (I) + (II) + (III) \end{aligned}$$

We estimate separately each term of the above sum:

$$\begin{aligned} |(I)| &= \left|\langle (e^{A^\top r_2} - e^{A^\top r_1})p, x_2 - x_1\rangle\right| = \left|\langle (e^{A^\top r_2}(Id - e^{A^\top (r_1-r_2)})p, x_2 - x_1\rangle\right| \\ &\le k_2'(r_1-r_2)\|p\|\,\|x_2-x_1\| \le k_2''\|p\|(\|x_2-x_1\|^2 + |r_2-r_1|^2), \end{aligned}$$

where $k_2', k_2'' \in \mathbb{R}$ are positive constants, and Id denotes the identity matrix. Furthermore, observe that $\|p\| \le k\|\zeta\|$, with k independent of ζ, r_1, r_2 because δ is finite. So it holds

$$|(I)| \le k_2\|\zeta\|(\|x_2-x_1\|^2 + |r_2-r_1|^2),$$

where k_2 is a positive constant independent of $x_2, x_1, r_2, r_1, \zeta$.

Let us now consider (II). First observe that

$$\begin{aligned} x_2 - y &= e^{-r_2 A}(\bar{x}_2 - \bar{x}_1) + \int_{r_1 - r_2}^{r_1} e^{A(r_1 - r_2 - t)} u_1(t)\, dt - \int_0^{r_2} e^{-At} u_2(t)\, dt \\ &= e^{-r_2 A}(\bar{x}_2 - \bar{x}_1) + \int_{r_1 - r_2}^{r_1} e^{A(r_1 - r_2 - t)} (u_1(t) - u_2(t - r_1 + r_2))\, dt. \end{aligned}$$

Then

$$(II) = \langle -e^{A^\top r_2} p, x_2 - y \rangle = \langle -p, \bar{x}_2 - \bar{x}_1 \rangle + \int_{r_1 - r_2}^{r_1} \langle p(t), u_2(t - r_1 + r_2) - u_1(t) \rangle\, dt.$$

By observing that $-p \in N_S(\bar{x}_1)$ and by Maximum Principle, we have that $(II) \leq 0$.

Let us now consider (III). First, observe that:

$$y - x_1 = \int_0^{r_1 - r_2} \dot{x}_1(t)\, dt = \int_0^{r_1 - r_2} (Ax_1(t) + u_1(t))\, dt,$$

where

$$x_1(t) := e^{At} x_1 + \int_0^t e^{A(r_1 - t)} u_1(t)\, dt$$

is the optimal trajectory associated with x_1 and $u_1(t)$.
Let us define

$$k_3'' = \max\{\|A\|\|x\| + \|u\| : x \in \mathcal{R}_\delta,\, u \in \mathcal{U}\}.$$

Then we have that

$$(III) = \int_0^{r_1 - r_2} \langle p(t) - p(r_1 - r_2), Ax_1(t) + u_1(t) \rangle\, dt + \int_0^{r_1 - r_2} \langle -p(t), Ax_1(t) + u_1(t) \rangle\, dt.$$

We have also the following estimate, valid for all $t \in [0, r_1 - r_2]$:

$$\begin{aligned} |\langle p(t) - p(r_1 - r_2), Ax_1(t) + u_1(t) \rangle| &\leq k_3'' \|p(t) - p(r_1 - r_2)\| \\ &\leq k_3' \|p\| (r_1 - r_2) \\ &\leq k_3 \|\zeta\| (r_1 - r_2). \end{aligned}$$

So the first integral in (III) can be majorized by

$$k_3 \|\zeta\| |r_1 - r_2|^2,$$

where k_3 is a positive constant independent of $x_1, x_2, r_1, r_2, \zeta$. By Maximum Principle, the second integral in (III) is

$$\int_0^{r_1 - r_2} [\langle -p(t), Ax_1(t) \rangle - \max_{u \in \mathcal{U}} \langle p(t), u \rangle]\, dt.$$

The following estimates hold, for a suitable constant k_4, independent of $x_1, x_2, r_1, r_2, \zeta$:

$$\begin{aligned}\int_0^{r_1-r_2} \langle -p(t), Ax_1(t)\rangle &= \int_0^{r_1-r_2} \Big[\langle p(0)-p(t), Ax_1(t)\rangle + \\ &\quad + \langle -p(0), A(x_1(t)-x_1)\rangle\Big]\, dt + \\ &\quad + \int_0^{r_1-r_2} \langle -p(0), Ax_1\rangle\, dt \\ &\le k_4\|\zeta\||r_1-r_2|^2 + \int_0^{r_1-r_2} \langle \zeta, Ax_1\rangle\, dt, \\ \int_0^{r_1-r_2} -\max_{u\in\mathcal{U}}\langle p(t), u\rangle\, dt &= \int_0^{r_1-r_2} \min_{u\in\mathcal{U}}\langle p(0)-p(t), u\rangle\, dt + \\ &\quad + \int_0^{r_1-r_2} \min_{u\in\mathcal{U}}\langle -p(0), u\rangle\, dt \\ &\le k_4\|\zeta\||r_1-r_2|^2 + \int_0^{r_1-r_2} \min_{u\in\mathcal{U}}\langle \zeta, u\rangle\, dt.\end{aligned}$$

Therefore,

$$(III) \le k_4'\|\zeta\|\,|r_1-r_2|^2 + (r_1-r_2)h(x_1,\zeta).$$

Now we have to distinguish two cases:

1. if $\xi = -1$, then $h(x_1,\zeta) = -1$ and so putting together the estimates on (I), (II) and (III) we obtain that

   $$\langle \zeta, x_2 - x_1\rangle \le r_2 - r_1 + k'\|\zeta\||r_1-r_2|^2 + k''\|\zeta\|(\|x_2-x_1\|^2 + |r_2-r_1|^2),$$

   which may be written, for a suitable constant k_5 independent of $x_1, x_2, r_1, r_2, \zeta$, as

   $$\langle (\zeta,-1), (x_2, T(x_2)) - (x_1, T(x_1))\rangle \le k_5(\|\zeta\|+1)(\|x_2-x_1\|^2 + |T(x_2)-T(x_1)|^2)$$

   for all $x_1, x_2 \in \mathcal{R}_\delta$ and for all $(\zeta,-1) \in N^P_{\mathrm{epi}(T)}(x_1)$.

2. if $\xi = 0$, then $h(x_1,\zeta) \le 0$ and so

   $$(III) \le k_4'\|\zeta\|\,|r_1-r_2|^2$$

   Putting the estimates together, we obtain

   $$\langle \zeta, x_2 - x_1\rangle \le k'\|\zeta\|\,|r_1-r_2|^2 + k''\|\zeta\|(\|x_2-x_1\|^2 + |r_2-r_1|^2),$$

   which may be written, for a suitable constant k_5 independent of $x_1, x_2, r_1, r_2, \zeta$, as

   $$\langle (\zeta,0), (x_2, T(x_2)) - (x_1, T(x_1))\rangle \le k_5\|\zeta\|(\|x_2-x_1\|^2 + |T(x_2)-T(x_1)|^2)$$

   for all $x_1, x_2 \in \mathcal{R}_\delta$ and for all $(\zeta,0) \in N^P_{\mathrm{epi}(T)}(x_1)$.

In both cases, we obtain (3.9).

The case $r_2 > r_1$ is similar. Let $u_i(\cdot) \in \mathcal{U}^{r_i}_{\mathrm{ad}}$ be controls steering x_i to $\overline{x}_i \in S$ in the optimal times r_i, $i = 1, 2$, together with adjoint arcs $p_i : [0, r_i] \to \mathbb{R}^n$,

$p_i(t) = e^{A^\top(r_i-t)}\overline{p}_i$. Now set $\tilde{p}(t) = e^{A^\top(r_2-t)}\overline{p}_1$ for $t \in [0, r_2]$, and observe that, for $t \in [r_2-r_1, r_2]$, $u_1(t-(r_2-r_1)) \in \operatorname{Argmax}_{u\in\mathcal{U}}\langle \tilde{p}(t), u\rangle$. Choose now, for $t \in [0, r_2-r_1]$, $\bar{u}(t) \in \mathcal{U}$ such that $\bar{u}(t) \in \operatorname{Argmax}_{u\in\mathcal{U}}\langle \tilde{p}(t), u\rangle$, and set

$$\tilde{u}(t) = \begin{cases} \bar{u}(t) & t \in [0, r_2 - r_1] \\ u_1(t-(r_2-r_1)) & t \in (r_2 - r_1, r_2]. \end{cases}$$

Define

$$y = e^{-A(r_2-r_1)}x_1 - \int_0^{r_2-r_1} e^{-At}\bar{u}(t)\, dt = e^{-Ar_2}\bar{x}_1 - \int_0^{r_2} e^{-At}\tilde{u}(t)\, dt.$$

Now the estimates proceed analogously to the previous case $r_2 \le r_1$, with $\tilde{p}, \tilde{u}$ in place of p_1, u_1. The proof of Step 1 is concluded.

STEP 2: Let $T : \mathbb{R}^n \to \mathbb{R}^+ \cup \{+\infty\}$ be lower semicontinuous and proper, with a φ-convex domain $D = \{x \in \mathbb{R}^n : T(x) < +\infty\}$ and such that:

1. T is continuous on D;
2. for all $R > 0$ there exists $\sigma = \sigma(R) > 0$ such that for all $x, y \in D \cap \bar{B}(0, R)$ and for all $(\zeta, \xi) \in N^P_{\mathrm{epi}(T)}(x, T(x))$ with $\xi \in \{0, 1\}$ it holds

$$\langle(\zeta, \xi), (y, T(y)) - (x, T(x))\rangle \le \sigma(\|\zeta\| + |\xi|)(\|y - x\|^2 + |T(y) - T(x)|^2).$$

Then there exists a continuous $\overline{\varphi}$ such that $\mathrm{epi}(T)$ is $\overline{\varphi}$-convex.

Proof of step 2.

We have to prove that given $(x, \alpha), (y, \beta) \in \mathrm{epi}(T)$ with $\|x\|, \|y\| \le R$ and $(\zeta, \xi) \in N^P_{\mathrm{epi}(T)}(x, \alpha)$ with $\xi \in \{0, -1\}$, there exists $\sigma' = \sigma'(R) > 0$ such that

$$\langle(\zeta, \xi), (y, \beta) - (x, \alpha)\rangle \le \sigma'(\|\zeta\| + |\xi|)(\|y - x\|^2 + |\alpha - \beta|^2).$$

Let $\alpha > T(x)$. Two cases may occur:

1. if $(x, \alpha) \in \operatorname{int}\mathrm{epi}(T)$, then $N^P_{\mathrm{epi}(T)}(x, \alpha) = \{(0, 0)\}$, and there is nothing to prove;
2. suppose $(x, \alpha) \in \operatorname{bdry}\mathrm{epi}(T)$. Let $(\zeta, \xi) \ne (0, 0)$ be such that $(\zeta, \xi) \in N^P_{\mathrm{epi}(T)}(x, \alpha)$. Without loss of generality, suppose that $\|(\zeta, \xi)\| = 1$. Assume that (ζ, ξ) is realized by an r-ball, with $2r\sigma \le 1$. We claim that $\xi = 0$. In fact, by contradiction, let $\xi \ne 0$; since (ζ, ξ) is normal to an epigraph, we necessarily have that $\xi < 0$. Then there exists $0 < \varepsilon < \alpha - T(x)$ such that

$$\|(x, \alpha - \varepsilon) - (x + r\zeta, \alpha + r\xi)\|^2 < r^2.$$

This means that $(x, \alpha - \varepsilon) \in B((x + r\zeta, \alpha + r\xi), r)$, which is a contradiction since $(x, \alpha - \varepsilon) \in \mathrm{epi}(T)$. So, if $(x, \alpha) \in \operatorname{bdry}\mathrm{epi}(T)$ and $0 \ne (\zeta, 0) \in N^P_{\mathrm{epi}(T)}(x, \alpha)$, by the continuity of T on D and the same argument of Lemma 3.5 we have that $\zeta \in N_D$. Since D is φ-convex,

$$\langle \zeta, y - x\rangle \le \varphi(x)\|\zeta\|\,\|y - x\|^2 \qquad \text{for all } x, y \in D,$$

and so

$$\langle(\zeta, 0), (y, \beta) - (x, \alpha)\rangle \le (\sigma \vee \varphi(x))\|\zeta\|(\|y - x\|^2 + |\alpha - \beta|^2).$$

It remains to consider the case $\alpha = T(x)$. Define

$$z = x + \frac{1}{2\sigma}\frac{\zeta}{\|(\zeta,\xi)\|}, \qquad \chi = T(x) + \frac{1}{2\sigma}\frac{\xi}{\|(\zeta,\xi)\|}.$$

Let $(y,\beta) \in \mathrm{epi}(T)$ with $\beta > T(y)$ and $y \neq x$. The segment connecting (z,χ) and (y,β) contains a point (y',β') which lies on the boundary of $\mathrm{epi}(T)$, so $\beta' = T(y')$. Thus we have:

$$d((x,T(x)),(z,\chi)) < d((y',T(y)),(z,\chi)) < d((y,\beta),(z,\chi)).$$

By direct computation the desired inequality follows. By the arbitrariness of R, the proof is concluded. □

Remark. 1) The problem in Example 1 satisfies the assumptions of Theorem 3.7, although Small Time Controllability does not hold.
2) If (H1) and (H2) are valid in the whole of $\mathcal{R}$, then there exists a continuous function φ such that the epigraph of T is φ-convex. Indeed, it is enough to apply Theorem 3.7 in $\mathcal{R}_\delta$ for all $\delta > 0$.

In [13], functions with φ-convex epigraph were studied. As a corollary of the above result, we list some regularity properties of the minimal time function, which are direct consequences of Theorem 3.7 and of [13].

Corollary 3.8. *Let the assumption of Theorem 3.7 hold. Then:*

1. *for a.e. $x \in \mathcal{R}_\delta$, there exists $\varepsilon = \varepsilon(x)$ such that T is semiconvex on $B(x,\varepsilon(x))$;*
2. *in particular, T is twice differentiable a.e. on $\mathcal{R}_\delta$, in the sense that for a.e. $x \in \mathcal{R}_\delta$ there exists a symmetric $n \times n$ matrix X_x such that*

$$DT(y) = DT(x) + X_x(y-x) + o(\|y-x\|)$$

for $y \to x$, $y \in \mathrm{dom}(DT)$ and, as $y \to x$, $y \in \mathrm{dom}(T)$,

$$\left| T(y) - T(x) - \langle DT(x), y-x\rangle - \frac{1}{2}\langle X_x(y-x), y-x\rangle \right| = o(\|y-x\|^2); \quad (3.10)$$

3. *for a.e. $x \in \mathcal{R}_\delta$, there exist $\epsilon = \epsilon(x) > 0$ and $c = c(x) \geq 0$ such that for all $\nu \in \mathbb{R}^n$, with $\|\nu\| = 1$, we have $\frac{\partial^2 T}{\partial \nu^2} \geq -c$ in the sense of distributions in $B(x,\varepsilon)$;*
4. *set, for $1 \leq k \leq n$,*

$$\Sigma_k = \{x \in \mathrm{int\,dom}(T) : \mathcal{H} - dim(\partial_P T(x)) \geq k\};$$

then Σ_k is countably $\mathcal{H}^{n-k}$-rectifiable;

5. *let $\mathrm{int\,dom}(T)$ be nonempty; then, for all open set $\Omega \subseteq \mathrm{int\,dom}(T)$, $T \in BV(\Omega)$; moreover, for a.e. $x \in \Omega$, there exists $\varepsilon = \varepsilon(x)$ such that $DT \in BV(B(x,\varepsilon))$.*

Proof. Extend T to $\mathbb{R}^n$ by setting $T(x) = +\infty$ if $x \notin \mathcal{R}_\delta$. By standard arguments, T is lower semicontinuous on $\mathbb{R}^n$. By Theorem 3.7, $\mathrm{epi}(T)$ is φ-convex. Then the statements (1)-(5) are direct consequences of corresponding properties proved in [13], to which all the following citations refer.
Statement (1) follows from Theorem 6.1; (2) and (3) are Corollaries 6.1 and 6.2, respectively; (4) is Proposition 5.1, while (5) is Propositions 7.1 and 7.2. □

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