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Exponential quadrature rules for problems with time-dependent fractional source

Marco Caliari [a], Fabio Cassini [a,b],*

[a] Department of Computer Science, University of Verona, Strada Le Grazie, 15, 37134, Verona, Italy
[b] Istituto Nazionale di Alta Matematica, Piazzale Aldo Moro, 5, 00185, Roma, Italy

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ABSTRACT

In this manuscript, we propose newly-derived exponential quadrature rules for stiff linear differential equations with time-dependent fractional sources in the form $h(t^r)$, with $0 < r < 1$ and h a sufficiently smooth function. To construct the methods, the source term is interpolated at ν collocation points by a suitable non-polynomial function, yielding to time marching schemes that we call Exponential Quadrature Rules for Fractional sources (EQRFν). We write the integrators in terms of special instances of the Mittag–Leffler functions that we call fractional φ functions. We perform the error analysis of the schemes in the abstract framework of strongly continuous semigroups. Compared to classical exponential quadrature rules, which in our case of interest converge with order $1+r$ at most, we prove that the new methods may reach order $1+\nu r$ for proper choices of the collocation points. Several numerical experiments demonstrate the theoretical findings and highlight the effectiveness of the approach.

1. Introduction

The numerical time integration of stiff differential equations requires careful treatment due to stability issues that usually arise when employing standard explicit time marching methods. In the last years, exponential integrators proved to be a valuable way to effectively tackle the issue in many contexts. We mention here, among the others, [1–8] and the seminal manuscript [9]. In this paper, we focus our attention to the development of exponential integrators for linear ordinary differential equations (ODEs) in the form

$$\begin{cases} y'(t) = Ay(t) + g(t) = Ay(t) + h(t^r), \quad 0 < r < 1, \\ y(0) = y_0. \end{cases} \tag{1}$$

Here $t \in (0, T]$, y is the unknown function, $A : \mathcal{D}(A) \subset X \to X$ is a linear operator on the Banach space $(X, \|\cdot\|)$, and $g(t) = h(t^r)$ represents the time-dependent fractional source term for the given parameter r. Results on the existence and uniqueness of solution for this problem can be found, for instance, in [10, Section 4.2]. Stiff equations in the form (1) typically arise when considering linear partial differential equations (PDEs) as abstract ODEs on suitable function spaces [10], or when dealing with systems of ODEs coming from the semidiscretization in space of linear PDEs. A simple example is the one-dimensional heat equation, coupled with periodic or homogeneous Dirichlet boundary conditions and with heat source that grows in time at a *sublinear* rate r. In this case, the operator A is the second-order differential operator ∂_{xx}, together with the boundary conditions, or a suitable discretization thereof (e.g., by standard finite differences).

* Corresponding author at: Department of Computer Science, University of Verona, Strada Le Grazie, 15, 37134, Verona, Italy.
E-mail addresses: marco.caliari@univr.it (M. Caliari), fabio.cassini@univr.it, cassini@altamatematica.it (F. Cassini).

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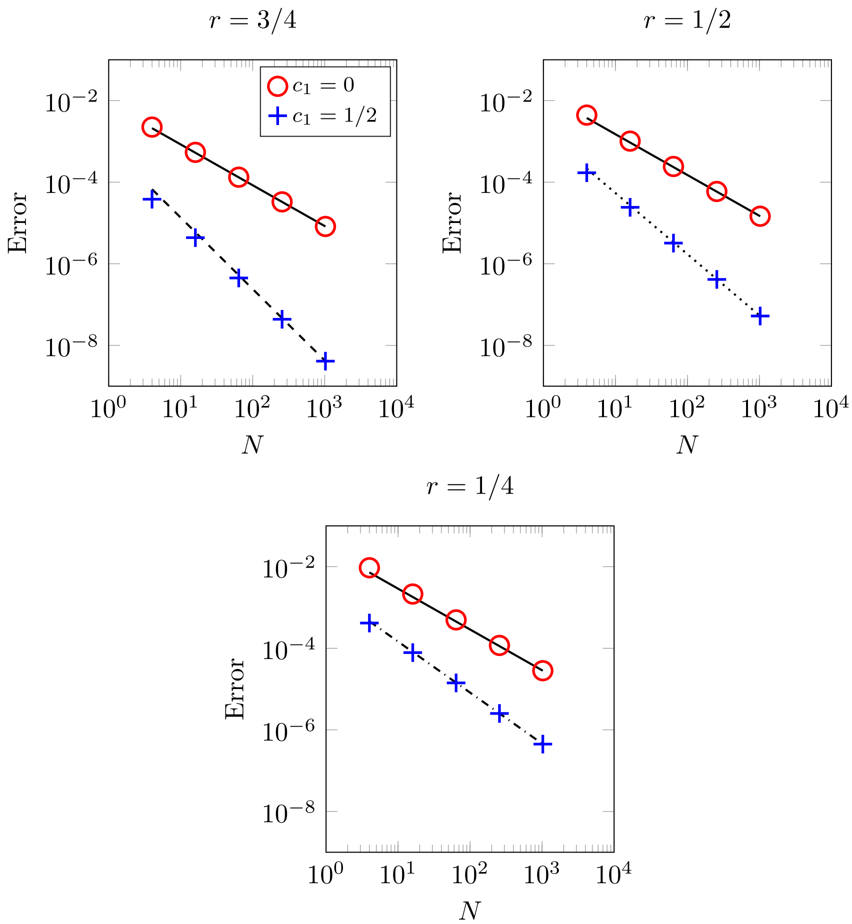

Fig. 1. Observed convergence rate of the error $|y(T) - y_N|$, with different values of r, for problem (3). The time marching is performed with scheme (2). The slope of the solid line is -1, that of the dashed line is -1.75, that of the dotted line is -1.5, and that of the dashdotted line is -1.25.

Classical exponential integrators for the linear problem (1) are the so-called *exponential quadrature rules* [11], the simplest one being

$$y_{n+1} = \mathrm{e}^{\tau A} y_n + \tau\varphi_1(\tau A) g(t_n + c_1\tau), \quad \tau\varphi_1(\tau A) = \int_0^\tau \mathrm{e}^{(\tau-s)A} ds, \tag{2}$$

with c_1 a collocation point in $[0, 1]$. Here we introduced the time discretization $t_n = n\tau$, $n = 0, 1, \ldots, N$, with constant time step size τ. Under suitable assumptions (in particular g' being absolutely integrable, see [11, Theorem 1]), the time marching method (2) is first-order convergent. The scheme is actually second-order convergent if $c_1 = \frac{1}{2}$ and additional assumptions are verified, in particular g'' absolutely integrable (see [11, Theorem 2]). In the situation under investigation in this manuscript, however, the latter is in general not verified (since $0 < r < 1$), and therefore we expect an order reduction when employing scheme (2) with $c_1 = \frac{1}{2}$. This can be easily demonstrated numerically by considering the scalar ODE

$$\begin{cases} y'(t) = -y(t) + t^r, & t \in (0, T], \\ y(0) = 1. \end{cases} \tag{3}$$

By performing the time marching with scheme (2) up to $T = 0.1$ for different number of time steps $N = 2^j$ (with $j = 2, 4, 6, 8, 10$), we clearly see in Fig. 1 that the order of convergence for the case $c_1 = \frac{1}{2}$ drops to 1.75, 1.5, and 1.25 when $r = \frac{3}{4}$, $r = \frac{1}{2}$ and $r = \frac{1}{4}$, respectively. On the other hand, as expected, the integrator is first-order convergent for each r when $c_1 = 0$. It is worth mentioning that the order reduction phenomenon of exponential quadrature rules has already been studied in the literature in different situations. We refer, e.g., to [12] in the context of linear equations with time-dependent boundary conditions.

The simple numerical example just presented motivates the development and the investigation of exponential quadrature rules tailored for problems in the form (1). The remaining part of the manuscript is organized as follows. In Section 2, the main one, we introduce the abstract framework that we employ for the analysis and we present our proposed Exponential Quadrature Rules for Fractional sources with ν collocation points (EQRFν). The newly-derived methods are written in terms of linear combinations of special instances of the Mittag–Leffler functions, which we call fractional φ functions. We investigate in full details the integrators which originate by interpolating the source term at one and at two non-confluent points with a non-polynomial basis. The generalization of the procedure to v collocation points is addressed in Section 2.3. In particular, if the points satisfy a peculiar relation, we show in Theorem 4 that the proposed integrators EQRFv have convergence order $1+vr$ (compared to classical quadrature rules which have order $1 + r$). We then proceed in Section 3 by presenting several numerical experiments that demonstrate the theoretical findings. We finally draw the conclusions and discuss possible further developments in Section 4.

2. The proposed exponential quadrature rules

Classical exponential quadrature rules originate from the expression of the exact solution of problem (1) by means of the so-called *variation-of-constants* formula

$$y(t_{n+1}) = \mathrm{e}^{\tau A} y(t_n) + \int_0^\tau \mathrm{e}^{(\tau-s)A} g(t_n + s)ds = \mathrm{e}^{\tau A} y(t_n) + \int_0^\tau \mathrm{e}^{(\tau-s)A} h((t_n + s)^r)ds, \tag{4}$$

and by approximating the source term appearing in the integral with an interpolation *polynomial* of a certain degree. The choice of the interpolation points (or equivalently the nodes of a quadrature rule) determines in fact the numerical method. For instance, by employing a single point (i.e., using the approximation $g(t_n + c_1\tau) \approx g(t_n + s)$) we get scheme (2). Under suitable assumptions, the employment of more terms in the polynomial basis leads to different methods of higher order (see [11] for a thorough presentation). Our proposal is to embed the power r directly into the interpolation basis functions, that is considering $\{(t_n + s)^{jr}\}_{j=0}^{v-1}$.

As already mentioned in the introduction, for the analysis we will work in an abstract framework. A reader not familiar with the following formalism is invited to consult, for instance, [10,13].

Assumption 1. Let X be a Banach space with norm $\|\cdot\|$. Let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator. We assume that A generates a strongly continuous semigroup $\{\mathrm{e}^{tA}\}_{t\geq 0}$.

Note that, in our cases of interest, A is a differential operator (on a suitable space) considered with boundary conditions (e.g., $A = \mathrm{i}\partial_{xx}$ or $A = \partial_{xx}$ with periodic or homogeneous Dirichlet boundary conditions). Under Assumption 1, the following bound on the semigroup

$$\|\mathrm{e}^{tA}\| \leq C \tag{5}$$

holds in the time interval $t \in [0, T]$. Here and throughout the paper C is a generic constant that may have different values at different occurrences, but it is independent of the stiffness of the problem. Bound (5) directly implies that the φ_ℓ functions

$$\varphi_0(z) = \mathrm{e}^z, \quad \varphi_\ell(z) = \frac{1}{(\ell-1)!}\int_0^1 \mathrm{e}^{(1-\theta)z}\theta^{\ell-1}d\theta = \sum_{k=0}^{\infty}\frac{z^k}{(\ell+k)!}, \quad \ell \in \mathbb{N}, \tag{6}$$

are all bounded operators (that is, they satisfy $\|\varphi_\ell(tA)\| \leq C$). We state now some lemmas needed for the following analysis.

Lemma 1. *Assume that $h : [0, +\infty) \to X$ is k times differentiable, with bounded derivatives. Then, for $\sigma, \omega \geq 0$, we have*

$$h(\sigma^r) = \sum_{j=0}^{k-1}\frac{h^{(j)}(\omega^r)}{j!}(\sigma^r - \omega^r)^j + \frac{h^{(k)}(\gamma_\sigma^r)}{k!}(\sigma^r - \omega^r)^k,$$

where γ_σ is between σ and ω.

Proof. We take $x, y \geq 0$ and consider the Taylor expansion of $h(y)$ at x

$$h(y) = \sum_{j=0}^{k-1}\frac{h^{(j)}(x)}{j!}(y-x)^j + \frac{h^{(k)}(\lambda_y)}{k!}(y-x)^k,$$

where λ_y is between y and x. Now, we choose $y = \sigma^r$ and $x = \omega^r$. Since the power-r function is an increasing monotone function, we have guarantee of existence of γ_σ, between σ and ω, such that $\gamma_\sigma^r = \lambda_y$. This concludes the proof. □

Lemma 2. *Let $t_n = n\tau$, for $n = 1, \dots, N$, with $\tau = T/N$. Then the following bounds hold*

$$\sum_{j=1}^{n} t_j^{-p} < \begin{cases} \tau^{-p} + C\tau^{-1}, & \text{if } p > 0,\ p \neq 1, \\ \tau^{-p}(C + |\log \tau|), & \text{if } p = 1. \end{cases}$$

Proof. We have

$$\sum_{j=1}^{n} t_j^{-p} = \tau^{-p}\sum_{j=1}^{n}\frac{1}{j^p} = \tau^{-p}H_n^{(p)} \leq \tau^{-p}H_N^{(p)},$$

where $H_n^{(p)}$ denotes the generalized harmonic number

$$H_n^{(p)} = \sum_{j=1}^{n} \frac{1}{j^p}.$$

Then, by employing the following bounds (see for instance [14, formulas (25)–(26)])

$$H_n^{(p)} < \begin{cases} 1 + Cn^{1-p}, & \text{if } p > 0,\ p \neq 1, \\ 1 + \log n, & \text{if } p = 1 \end{cases}$$

the result follows immediately. □

2.1. Exponential quadrature rule with one collocation point (EQRF1)

Starting from the variation-of-constants formula (4) and considering $h((t_n + c_1\tau)^r) \approx h((t_n + s)^r)$, for $s \in [0, \tau]$, we get the scheme

$$y_{n+1} = \mathrm{e}^{\tau A} y_n + \tau\varphi_1(\tau A)\alpha_{0,n}, \quad \alpha_{0,n} = h((t_n + c_1\tau)^r). \tag{7}$$

Obviously this method (labeled EQRF1) coincides with the simplest classical quadrature rule (2). We now prove the following.

Theorem 1. *Let Assumption 1 be valid, and consider the time marching scheme (7) for problem (1). Then, the following bounds on the error $\epsilon_n = y(t_n) - y_n$ hold uniformly on $t_n \in [0, T]$:*

1. *if h is differentiable with bounded derivative, then $\|\epsilon_n\| \leq C\tau$;*
2. *if h is twice differentiable with bounded derivatives, $h' \in \mathcal{D}(A)$, and $c_1 = \frac{1}{2}$, then $\|\epsilon_n\| \leq C\tau^{1+r}$.*

The constant C may depend on the final time T, but not on n.

Proof. The first statement directly follows from [11, Theorem 1]. For the second one, setting $c_1 = \frac{1}{2}$, we start by equivalently writing scheme (7) as

$$y_{n+1} = \mathrm{e}^{\tau A} y_n + \int_0^\tau \mathrm{e}^{(\tau - s)A} h\left(\left(t_n + \frac{\tau}{2}\right)^r\right) ds.$$

By subtracting this expression from the variation-of-constants formula (4) we get

$$\epsilon_{n+1} = \mathrm{e}^{\tau A}\epsilon_n + \delta_{n+1},$$

where we defined

$$\epsilon_n = y(t_n) - y_n \quad \text{and} \quad \delta_{n+1} = \int_0^\tau \mathrm{e}^{(\tau - s)A}\left(h((t_n + s)^r) - h\left(\left(t_n + \frac{\tau}{2}\right)^r\right)\right) ds.$$

Since $\epsilon_0 = 0$, the recursion leads to

$$\epsilon_n = \sum_{j=0}^{n-1} \mathrm{e}^{j\tau A}\delta_{n-j},$$

and by employing the bound on the semigroup (5) we get

$$\|\epsilon_n\| \leq C\left(\|\delta_1\| + \sum_{j=2}^{n} \|\delta_j\|\right).$$

For the term δ_1, by using Lemma 1, we get

$$\delta_1 = \int_0^\tau \mathrm{e}^{(\tau - s)A}\left(h(s^r) - h\left(\left(\frac{\tau}{2}\right)^r\right)\right) ds = \int_0^\tau \mathrm{e}^{(\tau - s)A} h'(\gamma_s^r)\left(s^r - \left(\frac{\tau}{2}\right)^r\right) ds,$$

where γ_s is between s and $\frac{\tau}{2}$, and therefore $\|\delta_1\| \leq C\tau^{1+r}$. For the remaining terms δ_j, we have

$$h((t_{j-1} + s)^r) - h\left(\left(t_{j-1} + \frac{\tau}{2}\right)^r\right) = h'\left(\left(t_{j-1} + \frac{\tau}{2}\right)^r\right)\left((t_{j-1} + s)^r - \left(t_{j-1} + \frac{\tau}{2}\right)^r\right) + \frac{h''(\gamma_s^r)}{2}\left((t_{j-1} + s)^r - \left(t_{j-1} + \frac{\tau}{2}\right)^r\right)^2.$$

Since $t_{j-1} \neq 0$, by Taylor expansion we have

$$(t_{j-1} + s)^r - \left(t_{j-1} + \frac{\tau}{2}\right)^r = r\left(t_{j-1} + \frac{\tau}{2}\right)^{r-1}\left(s - \frac{\tau}{2}\right) + \frac{r(r-1)}{2}\xi_{s,1}^{r-2}\left(s - \frac{\tau}{2}\right)^2,$$

and

$$\left((t_{j-1} + s)^r - \left(t_{j-1} + \frac{\tau}{2}\right)^r\right)^2 = r^2\xi_{s,2}^{2r-2}\left(s - \frac{\tau}{2}\right)^2,$$

with $\xi_{s,1}$ and $\xi_{s,2}$ between $t_{j-1}+s$ and $t_{j-1}+\frac{\tau}{2}$. Therefore, exploiting the boundedness of the derivatives of h, the bound on the semigroup, and the fact that $(t_{j-1}+s)^{r-1}\leq t_{j-1}^{r-1}$, we get

$$\|\delta_j\| \leq Ct_{j-1}^{r-1}\left\| \int_0^\tau \mathrm{e}^{(\tau-s)A}\left(s-\frac{\tau}{2}\right)ds\,h'\left(\left(t_{j-1}+\frac{\tau}{2}\right)^r\right)\right\| + Ct_{j-1}^{r-2}\int_0^\tau \left(s-\frac{\tau}{2}\right)^2 ds + Ct_{j-1}^{2r-2}\int_0^\tau\left(s-\frac{\tau}{2}\right)^2 ds.$$

By exploiting that

$$\int_0^\tau \mathrm{e}^{(\tau-s)A}\left(s-\frac{\tau}{2}\right)ds\,h'\left(\left(t_{j-1}+\frac{\tau}{2}\right)^r\right) = \tau^3\left(\varphi_3(\tau A)-\frac{1}{2}\varphi_2(\tau A)\right)Ah'\left(\left(t_{j-1}+\frac{\tau}{2}\right)^r\right),$$

we obtain

$$\|\delta_j\| \leq Ct_{j-1}^{r-1}\tau^3 + Ct_{j-1}^{r-2}\tau^3 + Ct_{j-1}^{2r-2}\tau^3,$$

since by assumption $h'\in\mathcal{D}(A)$. Here, we also exploited the boundedness of the φ functions. Finally, we conclude that

$$\|\epsilon_n\| \leq C\tau^{1+r} + C\tau^3\left(\sum_{j=2}^n t_{j-1}^{r-1} + \sum_{j=2}^n t_{j-1}^{r-2} + \sum_{j=2}^n t_{j-1}^{2r-2}\right) \leq C\tau^{1+r},$$

where we used the bounds in Lemma 2. □

2.2. Exponential quadrature rule with two collocation points (EQRF2)

The classical exponential quadrature rules with two (or more) collocation points still exhibit order reduction to $1+r$. This can be easily proved by using similar reasoning as above (see, for instance, Section 3.3 for a numerical confirmation). Here we employ the proposed approach to derive exponential integrators of order $1+2r$ using two collocation points. The generalization to ν points will be addressed in the next section.

We therefore consider the basis $\{1,(t_n+s)^r\}$ and perform the approximation

$$\alpha_{0,n} + (t_n+s)^r\alpha_{1,n} \approx h((t_n+s)^r). \quad s\in[0,\tau],$$

Here $\alpha_{0,n}$ and $\alpha_{1,n}$ are coefficients determined by imposing interpolation conditions on two non-confluent collocation points c_1 and c_2 in $[0,1]$. Simple calculations lead to

$$\alpha_{0,n} = h((t_n+c_1\tau)^r) - (t_n+c_1\tau)^r\alpha_{1,n}, \quad \alpha_{1,n} = \frac{h((t_n+c_2\tau)^r)-h((t_n+c_1\tau)^r)}{(t_n+c_2\tau)^r-(t_n+c_1\tau)^r}. \tag{8a}$$

Substituting the approximation into the variation-of-constants formula (4), we obtain

$$y_{n+1} = \mathrm{e}^{\tau A}y_n + \tau\varphi_1(\tau A)\alpha_{0,n} + \int_0^\tau \mathrm{e}^{(\tau-s)A}(t_n+s)^r ds\,\alpha_{1,n}. \tag{8b}$$

The final form of the numerical scheme is obtained by employing a generalization of the φ functions appearing in classical exponential quadrature rules. More in detail, we avail of the following integral formulation of the so-called Mittag–Leffler functions

$$E_{1,\beta}(z) = \frac{1}{\Gamma(\beta-1)}\int_0^1 \mathrm{e}^{(1-\theta)z}\theta^{\beta-2}d\theta, \quad \beta>1, \tag{9}$$

where $\Gamma(\cdot)$ denotes Euler's gamma function (see [15, Formula 1.100]). These functions are particular cases of two-parameter Mittag–Leffler functions $E_{\alpha,\beta}(z)$, typically encountered in the context of fractional differential equations [15–17]. Note also that the functions in (9) are related to the so-called Miller–Ross functions. Comparing definitions (6) and (9), we clearly see that the Mittag–Leffler functions $E_{1,\beta}$ generalize the φ functions, and therefore we choose to employ the notation

$$\varphi_\lambda(z) = E_{1,1+\lambda}(z) = \frac{1}{\Gamma(\lambda)}\int_0^1 \mathrm{e}^{(1-\theta)z}\theta^{\lambda-1}d\theta, \quad \lambda\in\mathbb{R}^+. \tag{10}$$

We call these functions *fractional φ functions*. Note that, as the classical φ functions, they are bounded operators, i.e., they satisfy $\|\varphi_\lambda(tA)\|\leq C$ for $t\in[0,T]$. We can now state the following.

Proposition 1. *Let $\lambda\in\mathbb{R}^+$. Then, we have*

$$\int_0^\tau \mathrm{e}^{(\tau-s)A}(t_n+s)^{\lambda-1}ds = \Gamma(\lambda)((t_n+\tau)^\lambda\varphi_\lambda((t_n+\tau)A) - t_n^\lambda \mathrm{e}^{\tau A}\varphi_\lambda(t_nA)).$$

Proof. By splitting the integration interval into two parts we get

$$\int_0^\tau \mathrm{e}^{(\tau-s)A}(t_n+s)^{\lambda-1}ds = \int_{-t_n}^\tau \mathrm{e}^{(\tau-s)A}(t_n+s)^{\lambda-1}ds - \mathrm{e}^{\tau A}\int_{-t_n}^0 \mathrm{e}^{-sA}(t_n+s)^{\lambda-1}ds.$$

For the first integral we employ the change of variable $(\tau - s) = (1-\theta)(t_n+\tau)$, while for the second one we perform the substitution $-s = (1-\theta)t_n$. Then, the right hand side becomes

$$\int_0^1 \mathrm{e}^{(1-\theta)(t_n+\tau)A}((t_n+\tau)\theta)^{\lambda-1}(t_n+\tau)d\theta - \mathrm{e}^{\tau A}\int_0^1 \mathrm{e}^{(1-\theta)t_nA}(t_n\theta)^{\lambda-1}t_n d\theta = (t_n+\tau)^\lambda \int_0^1 \mathrm{e}^{(1-\theta)(t_n+\tau)A}\theta^{\lambda-1}d\theta - t_n^\lambda \mathrm{e}^{\tau A}\int_0^1 \mathrm{e}^{(1-\theta)t_nA}\theta^{\lambda-1}d\theta.$$

The integral representation of the φ_λ function (10) concludes the proof. □

By inserting the result of Proposition 1 with $\lambda = 1+r$ into (8b), we finally get the proposed numerical scheme EQRF2, i.e.,

$$y_{n+1} = \mathrm{e}^{\tau A}y_n + \tau\varphi_1(\tau A)\alpha_{0,n} + \Gamma(1+r)((t_n+\tau)^{1+r}\varphi_{1+r}((t_n+\tau)A) - t_n^{1+r}\mathrm{e}^{\tau A}\varphi_{1+r}(t_nA))\alpha_{1,n}, \tag{11}$$

where $\alpha_{0,n}$ and $\alpha_{1,n}$ are defined in formula (8a). Remark that, by construction, this integrator is exact on problems for which $h(t^r) = t^r v$, being v time-independent (see the numerical experiments in Section 3.3). We now study the convergence of integrator (11).

Theorem 2. *Let Assumption 1 be valid and assume that h is twice differentiable with bounded derivatives. Consider the non-confluent points c_1 and c_2 in $[0,1]$. Then, the following bound*

$$\|y(t_n) - y_n\| \le \begin{cases} C\tau^{\min\{1+2r,2\}}, & \text{if } r \ne \frac{1}{2}, \\ C\tau^2(1+|\log\tau|), & \text{if } r = \frac{1}{2}, \end{cases}$$

for the time marching scheme (11) holds uniformly on $t_n \in [0,T]$. The constant C may depend on the final time T, but not on n.

Proof. By comparing the exact solution expressed by means of the variation-of-constants formula (4) and the numerical one (see also formula (8)), we get the recursion

$$\epsilon_{n+1} = \mathrm{e}^{\tau A}\epsilon_n + \delta_{n+1},$$

where

$$\epsilon_n = y(t_n) - y_n \quad \text{and} \quad \delta_{n+1} = \int_0^\tau \mathrm{e}^{(\tau-s)A}\rho_n(s)ds,$$

with

$$\rho_n(s) = h((t_n+s)^r) - (\alpha_{0,n} + (t_n+s)^r\alpha_{1,n}). \tag{12}$$

Since $\epsilon_0 = 0$, similarly to the proof of Theorem 1, we get

$$\|\epsilon_n\| \le C\left(\|\delta_1\| + \sum_{j=2}^n \|\delta_j\|\right).$$

Expanding $h((t_{j-1}+s)^r)$ around $t_{j-1}+c_1\tau$ according to Lemma 1, and doing simple algebraic manipulations, we get

$$\rho_{j-1}(s) = ((t_{j-1}+s)^r - (t_{j-1}+c_1\tau)^r)\left(h'(\gamma_s^r) - \frac{h((t_{j-1}+c_2\tau)^r) - h((t_{j-1}+c_1\tau)^r)}{(t_{j-1}+c_2\tau)^r - (t_{j-1}+c_1\tau)^r}\right),$$

where γ_s is between $t_{j-1}+c_1\tau$ and $t_{j-1}+s$. Expanding now $h((t_{j-1}+c_2\tau)^r)$ around $t_{j-1}+c_1\tau$ we obtain

$$\rho_{j-1}(s) = ((t_{j-1}+s)^r - (t_{j-1}+c_1\tau)^r)(h'(\gamma_s^r) - h'(\eta^r)),$$

where η is between $t_{j-1}+c_1\tau$ and $t_{j-1}+c_2\tau$. Then, by using the bound on the semigroup (5) and the fact that h'' is bounded (i.e., h' is Lipschitz), we get

$$\begin{aligned}\|\delta_j\| &\le C\int_0^\tau \|\rho_{j-1}(s)\|ds \le C\int_0^\tau |(t_{j-1}+s)^r - (t_{j-1}+c_1\tau)^r||\gamma_s^r - \eta^r|ds \\ &\le C\tau \max_{0\le s\le\tau}|(t_{j-1}+s)^r - (t_{j-1}+c_1\tau)^r| \max_{0\le s\le\tau}|\gamma_s^r - \eta^r| \le C\tau((t_{j-1}+\tau)^r - t_{j-1}^r)^2.\end{aligned}$$

Therefore $\|\delta_1\| \le C\tau^{1+2r}$. For the remaining δ_j, by Taylor expansion of $(t_{j-1}+\tau)^r$ around t_{j-1} we get

$$\|\delta_j\| \le C\tau^3 t_{j-1}^{2r-2}.$$

Finally, using Lemma 2, we conclude that

$$\|\epsilon_n\| \le C\tau^{1+2r} + C\tau^3\sum_{j=2}^n t_{j-1}^{2r-2} \le \begin{cases} C\tau^{1+2r} + C\tau^2, & \text{if } r \ne \frac{1}{2}, \\ C\tau^2 + C\tau^2(1+|\log\tau|), & \text{if } r = \frac{1}{2}, \end{cases}$$

which yields the result. □

The rate of convergence can actually be improved if we impose additional conditions on the collocation points and smoothness assumptions.

Theorem 3. *Let Assumption 1 be valid and assume that h is three times differentiable with bounded derivatives. In addition, assume that h'' lies in $\mathcal{D}(A)$ and that the non-confluent points c_1 and c_2 in $[0,1]$ satisfy the relation*

$$\frac{1}{3}-\frac{1}{2}(c_1+c_2)+c_1c_2=0. \tag{13}$$

Then, the following bound

$$\|y(t_n)-y_n\|\le C\tau^{1+2r}$$

for the time marching scheme (11) holds uniformly on $t_n\in[0,T]$. The constant C may depend on the final time T, but not on n.

Proof. By comparing the exact solution with the numerical one, from the recursion on $\epsilon_n=y(t_n)-y_n$ we get

$$\epsilon_n=\sum_{j=0}^{n-1}\mathrm{e}^{j\tau A}\delta_{n-j},\quad \delta_{n+1}=\int_0^\tau \mathrm{e}^{(\tau-s)A}\rho_n(s)ds,$$

where $\rho_n(s)$ is defined in formula (12). As in the proof of Theorem 2, $\|\delta_1\|\le C\tau^{1+2r}$. Consider now $j>1$. Employing Lemma 1 and expanding the function h around $t_{j-1}+c_1\tau$ up to the second derivative, we get

$$\rho_{j-1}(s)=-\frac{h''(\eta^r)}{2}((t_{j-1}+c_2\tau)^r-(t_{j-1}+c_1\tau)^r)((t_{j-1}+s)^r-(t_{j-1}+c_1\tau)^r)+\frac{h''(\gamma_s^r)}{2}((t_{j-1}+s)^r-(t_{j-1}+c_1\tau)^r)^2,$$

where η is between $t_{j-1}+c_1\tau$ and $t_{j-1}+c_2\tau$, while γ_s is between $t_{j-1}+c_1\tau$ and $t_{j-1}+s$. Now, by writing $(t_{j-1}+c_2\tau)^r-(t_{j-1}+c_1\tau)^r$ as $\left((t_{j-1}+s)^r-(t_{j-1}+c_1\tau)^r\right)+\left((t_{j-1}+c_2\tau)^r-(t_{j-1}+s)^r\right)$, we obtain the following expression for δ_j

$$\begin{aligned}\delta_j&=\int_0^\tau \mathrm{e}^{(\tau-s)A}\rho_{j-1}(s)ds\\ &=\underbrace{\int_0^\tau \mathrm{e}^{(\tau-s)A}((t_{j-1}+s)^r-(t_{j-1}+c_1\tau)^r)^2\left(\frac{h''(\gamma_s^r)}{2}-\frac{h''(\eta^r)}{2}\right)ds}_{\psi_j}\\ &\quad+\underbrace{\int_0^\tau \mathrm{e}^{(\tau-s)A}((t_{j-1}+s)^r-(t_{j-1}+c_1\tau)^r)((t_{j-1}+s)^r-(t_{j-1}+c_2\tau)^r)ds\frac{h''(\eta^r)}{2}}_{\omega_j}.\end{aligned}$$

Then, for the first integral term ψ_j we have

$$\|\psi_j\|\le C\int_0^\tau((t_{j-1}+s)^r-(t_{j-1}+c_1\tau)^r)^2\|h''(\gamma_s^r)-h''(\eta^r)\|ds\le C\tau^4 t_{j-1}^{3r-3}.$$

Here, similarly to the proof of the previous theorem, we used the fact that h'' is Lipschitz and the Taylor expansion of $(t_{j-1}+\tau)^r$ around t_{j-1}. For the second integral term ω_j, using again Taylor expansions, we get

$$\begin{aligned}\|\omega_j\|=&\left\|\int_0^\tau \mathrm{e}^{(\tau-s)A}\left(r(t_{j-1}+c_1\tau)^{r-1}(s-c_1\tau)+\frac{r(r-1)}{2}\gamma_s^{r-2}(s-c_1\tau)^2\right)\right.\\ &\left.\times\left(r(t_{j-1}+c_2\tau)^{r-1}(s-c_2\tau)+\frac{r(r-1)}{2}\eta_s^{r-2}(s-c_2\tau)^2\right)ds\frac{h''(\eta^r)}{2}\right\|,\end{aligned}$$

where γ_s is between $t_{j-1}+c_1\tau$ and $t_{j-1}+s$ and η_s is between $t_{j-1}+c_2\tau$ and $t_{j-1}+s$. Then, by using the triangle inequality we obtain

$$\begin{aligned}\|\omega_j\|\le C&\underbrace{\left\|((t_{j-1}+c_1\tau)(t_{j-1}+c_2\tau))^{r-1}\int_0^\tau \mathrm{e}^{(\tau-s)A}(s-c_1\tau)(s-c_2\tau)ds\,h''(\eta^r)\right\|}_{\omega_{j,1}}\\ &+C\underbrace{\left\|(t_{j-1}+c_2\tau)^{r-1}\int_0^\tau \mathrm{e}^{(\tau-s)A}\gamma_s^{r-2}(s-c_1\tau)^2(s-c_2\tau)ds\,h''(\eta^r)\right\|}_{\omega_{j,2}}\\ &+C\underbrace{\left\|(t_{j-1}+c_1\tau)^{r-1}\int_0^\tau \mathrm{e}^{(\tau-s)A}\eta_s^{r-2}(s-c_1\tau)(s-c_2\tau)^2ds\,h''(\eta^r)\right\|}_{\omega_{j,3}}\\ &+C\underbrace{\left\|\int_0^\tau \mathrm{e}^{(\tau-s)A}\gamma_s^{r-2}\eta_s^{r-2}(s-c_1\tau)^2(s-c_2\tau)^2ds\,h''(\eta^r)\right\|}_{\omega_{j,4}}.\end{aligned} \tag{14}$$

Concerning the first term $\omega_{j,1}$, by exploiting that

$$\int_0^\tau \mathrm{e}^{(\tau-s)A}(s-c_1\tau)(s-c_2\tau)ds\,h''(\eta^r)=\left(\tau^3\left(\frac{1}{3}-\frac{1}{2}(c_1+c_2)+c_1c_2\right)+\tau^4(2\varphi_4(\tau A)-(c_1+c_2)\varphi_3(\tau A)+c_1c_2\varphi_2(\tau A))A\right)h''(\eta^r)$$

we get

$$\|\omega_{j,1}\| \le C t_{j-1}^{2r-2}\tau^4,$$

since h'' lies in $\mathcal{D}(A)$ and c_1 and c_2 satisfy relation (13). Concerning the remaining terms, using similar estimates as in the proof of Theorem 1, we have

$$\|\omega_{j,2}\| \le C t_{j-1}^{2r-3}\tau^4, \quad \|\omega_{j,3}\| \le C t_{j-1}^{2r-3}\tau^4, \quad \|\omega_{j,4}\| \le C t_{j-1}^{2r-4}\tau^5.$$

Summarizing, for $j > 1$ we get

$$\|\delta_j\| \le C\tau^4 t_{j-1}^{3r-3} + C\tau^4 t_{j-1}^{2r-2} + C\tau^4 t_{j-1}^{2r-3} + C\tau^5 t_{j-1}^{2r-4},$$

and therefore

$$\|\epsilon_n\| \le C\left(\|\delta_1\| + \sum_{j=2}^{n}\|\delta_j\|\right) \le C\tau^{1+2r} + C\tau^4\sum_{j=2}^{n} t_{j-1}^{3r-3} + C\tau^4\sum_{j=2}^{n} t_{j-1}^{2r-2} + C\tau^4\sum_{j=2}^{n} t_{j-1}^{2r-3} + C\tau^5\sum_{j=2}^{n} t_{j-1}^{2r-4}.$$

Finally, using Lemma 2, we conclude

$$\|\epsilon_n\| \le C\tau^{1+2r}$$

which is the statement. □

We notice that relation (13) is satisfied, for instance, by the Gauss and Gauss–Radau quadrature nodes.

2.3. Exponential quadrature rule with ν collocation points (EQRFν)

We generalize now the procedure to ν collocation points by considering the interpolation

$$\sum_{j=0}^{\nu-1} \alpha_{j,n}(t_n+s)^{jr} \approx h((t_n+s)^r), \quad s \in [0,\tau],$$

where $\alpha_{0,n}, \dots, \alpha_{\nu-1,n}$ are the coefficients given by the interpolation conditions on ν non-confluent points $c_1, \dots, c_\nu$ in $[0,1]$. Substituting the approximation into the variation-of-constants formula (4) and using Proposition 1, we get the integrator

$$y_{n+1} = e^{\tau A} y_n + \tau\varphi_1(\tau A)\alpha_{0,n} + \sum_{j=1}^{\nu-1} \Gamma(1+jr)\left((t_n+\tau)^{1+jr}\varphi_{1+jr}((t_n+\tau)A) - t_n^{1+jr} e^{\tau A}\varphi_{1+jr}(t_n A)\right)\alpha_{j,n}, \tag{15}$$

that we label EQRFν. The theorems stated above generalize to the following.

Theorem 4. *Let Assumption 1 be valid, and consider the non-confluent points $c_1, \dots, c_\nu$ in $[0,1]$. Then, the following bounds on the error $\epsilon_n = y(t_n) - y_n$ for the time marching scheme (15) hold uniformly on $t_n \in [0,T]$:*

1. *if h is ν times differentiable with bounded derivatives, then*

$$\|\epsilon_n\| \le \begin{cases} C\tau^{\min\{1+\nu r,\nu\}}, & \text{if } r \ne \frac{\nu-1}{\nu}, \\ C\tau^{\nu}(1+|\log\tau|), & \text{if } r = \frac{\nu-1}{\nu}; \end{cases}$$

2. *if h is $\nu+1$ times differentiable with bounded derivatives, $h^{(\nu)} \in \mathcal{D}(A)$, and the points satisfy*

$$\frac{1}{\nu+1} - \frac{c_1+\cdots+c_\nu}{\nu} + \sum_{i<j}\frac{c_i c_j}{\nu-1} - \sum_{i<j<k}\frac{c_i c_j c_k}{\nu-2} + \cdots + (-1)^\nu c_1\cdots c_\nu = 0, \tag{16}$$

then $\|\epsilon_n\| \le C\tau^{1+\nu r}$.

The constant C may depend on the final time T, but not on n.

Proof. The proof is very similar to that of Theorems 2 and 3, and employs Lemma 1 and Taylor expansions. In particular, for the second statement, we exploit the identity

$$\int_0^1 (s-c_1)\cdots(s-c_\nu)ds = \frac{1}{\nu+1} - \frac{c_1+\cdots+c_\nu}{\nu} + \sum_{i<j}\frac{c_i c_j}{\nu-1} - \sum_{i<j<k}\frac{c_i c_j c_k}{\nu-2} + \cdots + (-1)^\nu c_1\cdots c_\nu$$

in the following formula

$$\int_0^\tau e^{(\tau-s)A}(s-c_1\tau)\cdots(s-c_\nu\tau)ds\, h^{(\nu)}(\eta^r) = \int_0^\tau (s-c_1\tau)\cdots(s-c_\nu\tau)ds\, h^{(\nu)}(\eta^r) + \int_0^\tau \varphi_1((\tau-s)A)(\tau-s)(s-c_1\tau)\cdots(s-c_\nu\tau)ds\, A h^{(\nu)}(\eta^r).$$

This is used in the estimate of the term analogous to $\omega_{j,1}$ in formula (14). We omit the remaining details. □

Notice that relation (16) is satisfied by any quadrature rule with ν nodes and degree of exactness at least ν, such as Gauss–Lobatto (for $\nu \geq 3$), Gauss–Radau (for $\nu \geq 2$), and Gauss.

Remark 1. In the second statement of Theorem 4, the assumption $h^{(\nu)} \in \mathcal{D}(A)$ can be removed if $\{e^{tA}\}_{t\geq 0}$ is an *analytic* semigroup, e.g., $A = \partial_{xx}$ with homogeneous Dirichlet boundary conditions. This follows from the fact that in the analytic case the bounds

$$\|(tA)^{\kappa} e^{tA}\| \leq C \quad \text{and} \quad \left\| \tau A \sum_{j=1}^{n-1} e^{j\tau A} \right\| \leq C$$

hold for $\kappa \geq 0$ and $0 \leq t \leq T$ (see [11, Lemma 1]). We numerically confirm this in Section 3.2.

3. Numerical experiments

In this section we present some numerical examples that demonstrate the theoretical findings and highlight the effectiveness of the proposed procedure. We perform the experiments by suitably semidiscretizing in space one-dimensional evolutionary PDEs, leading to a system of ODEs in the form (1) where A is the discretization matrix. All the simulations have been realized in MathWorks Matlab® R2022a on an Intel Core i7-10750H CPU (16 GB of RAM).

3.1. On the computation of the arising matrix functions

The EQRFν schemes require the computation of the fractional φ functions (10) evaluated at a matrix argument. Being special instances of the Mittag–Leffler functions, they could be approximated by employing techniques available in the literature (such as those explained in [18,19]). However, in our first three examples (see Sections 3.2, 3.3, and 3.4) we employ a Fourier pseudospectral semidiscretization in space, and therefore all the needed matrix functions are computed in a scalar fashion in Fourier space. In particular, we exploit the algorithm for general Mittag–Leffler functions of scalar arguments described in [20] to approximate the fractional φ functions. Notice that in these examples we are just interested in numerically demonstrating the bounds derived in the previous sections, and hence we do not discuss about the computational cost of the procedures.

On the other hand, in the last example (see Section 3.5) we focus on the computational aspect. In fact, for that experiment we perform the semidiscretization in space with standard second-order finite differences and we diagonalize once and for all the arising discretization matrix. Then, we can compute the matrix functions needed by EQRF2 (11) on scalar arguments. This approach, however, has a non-negligible computational burden (see Fig. 6), since the scalar fractional φ functions depend on the current time and must be recomputed at each time step. This contrasts what happens for the classical exponential quadrature rules, e.g., the scheme

$$y_{n+1} = e^{\tau A} y_n + \tau \left(\frac{c_2}{c_2 - c_1} \varphi_1(\tau A) - \frac{1}{c_2 - c_1} \varphi_2(\tau A) \right) g(t_n + c_1\tau) + \tau \left(-\frac{c_1}{c_2 - c_1} \varphi_1(\tau A) + \frac{1}{c_2 - c_1} \varphi_2(\tau A) \right) g(t_n + c_2\tau). \tag{17}$$

An alternative approach for EQRF2 is briefly described in the following. We notice that the needed linear combination of fractional φ functions can be computed *at once* by using its integral representation, see formulation (8) and Proposition 1. In this sense, at each time step EQRF2 requires to compute

$$\int_0^{\tau} e^{(\tau - s)\Lambda} (t_n + s)^r ds,$$

being Λ the diagonal matrix containing the eigenvalues. To this aim, if $t_n = 0$ we approximate the integral by using the Gauss–Jacobi quadrature formula with quadrature weight s^r, while for $t_n > 0$ we employ the Gauss–Legendre rule. In any case, we fix the number of quadrature nodes σ_i to 16. Note that, since the time step size τ is constant, this allows to compute the quantities $e^{(\tau - \sigma_i)\Lambda}$ once and for all before the actual time integration starts. In fact, in our numerical example, this procedure results in consistent computational savings without sacrificing accuracy (see the outcome in Section 3.5). The study of a general, detailed, and more sophisticated technique to compute (linear combinations of) fractional φ functions is a subject of ongoing work and is far from the scope of this manuscript.

Finally, we mention that the φ functions required by the classical exponential quadrature rule (17) could be computed by standard algorithms (see, e.g., [4,21–23]). However, as for the fractional φ functions, we in fact compute them on scalar arguments by exploiting the diagonalization procedure.

3.2. Results with one collocation point (EQRF1)

We consider here the integrator discussed in Section 2.1, i.e., the classical exponential quadrature rule with the single node $c_1 = \frac{1}{2}$ that we label EQRF1 G. As test problem, we take

$$\begin{cases} \partial_t y(t,x) = \zeta \partial_{xx} y(t,x) + t^{\frac{3}{4}} v(x), \\ y(0,x) = \sin(2\pi x) \end{cases} \tag{18}$$

in the spatial domain $\Omega = (0,1)$ with periodic boundary conditions. The final simulation time is $T = 3$. We semidiscretize the problem with a Fourier pseudospectral decomposition using 500 modes. We perform the first test with $\zeta = \mathrm{i}$ and $v(x) = 1/(2 + \cos(2\pi x))$. This

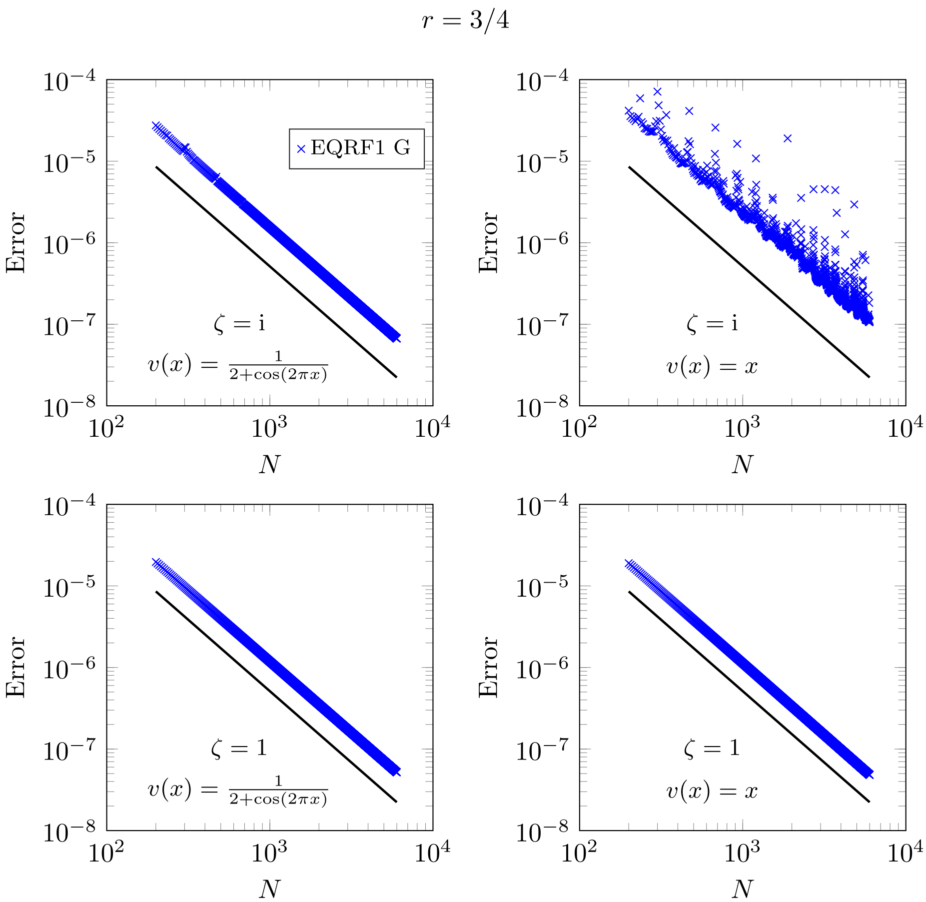

Fig. 2. Observedconvergence rate of the error $\|y(T) - y_N\|_\infty$, with different choices of ζ and $v(x)$, for problem (18). The time marching is performed with scheme (7) setting $c_1 = \frac{1}{2}$. The slope of the solid line is −1.75.

choice satisfies all the requirements of the second statement in Theorem 1. Therefore, since $r = \frac{3}{4}$, we expect order of convergence $1 + r = \frac{7}{4}$, which is indeed demonstrated in Fig. 2 (top left). On the other hand, by taking $\zeta = \mathrm{i}$ and $v(x) = x$, the latter does not satisfy the boundary conditions, and therefore the requirement on the domain of the operator in the second statement of Theorem 1 is violated. Indeed, the results presented in Fig. 2 (top right) show an oscillatory behavior of the error as the number of time steps increases. Note that this requirement is not needed if the semigroup generated by the operator is analytic (see Remark 1). In fact, we numerically demonstrate this by setting $\zeta = 1$ and $v(x) = 1/(2 + \cos(2\pi x))$ (Fig. 2, bottom left), and $\zeta = 1$ and $v(x) = x$ (Fig. 2, bottom right).

3.3. Exactness of EQRF2

We consider now the problem

$$\begin{cases} \partial_t y(t,x) = \mathrm{i}\partial_{xx} y(t,x) + \dfrac{t^r}{2+\cos(2\pi x)}, \\ y(0,x) = \sin(2\pi x), \end{cases} \tag{19}$$

with periodic boundary conditions in the spatial domain $\Omega = (0, 1)$ and different choices of r. The time interval is $[0, T]$, with $T = 1$. The spatial discretization is again performed with a Fourier pseudospectral method using 500 modes. As time integrator, we consider the proposed method EQRF2 (11) with $c_1 = 0$ and $c_2 = 1$ that we label EQRF2 T. In addition, we test the classical exponential quadrature rule (17) again with $c_1 = 0$ and $c_2 = 1$ (labeled CEQR2 T). Notice that the selected collocation points correspond to those of the trapezoidal quadrature rule. The results for different values of r are presented in Fig. 3. As mentioned at the beginning of Section 2.2, the CEQR2 T method suffers from order reduction (to $1 + r$, in particular), while the error of the proposed integrator EQRF2 T is around 10^{-14} (which is in fact the error of the routine employed for computing the fractional φ functions). This is expected, since scheme (11) is exact for problem (19).

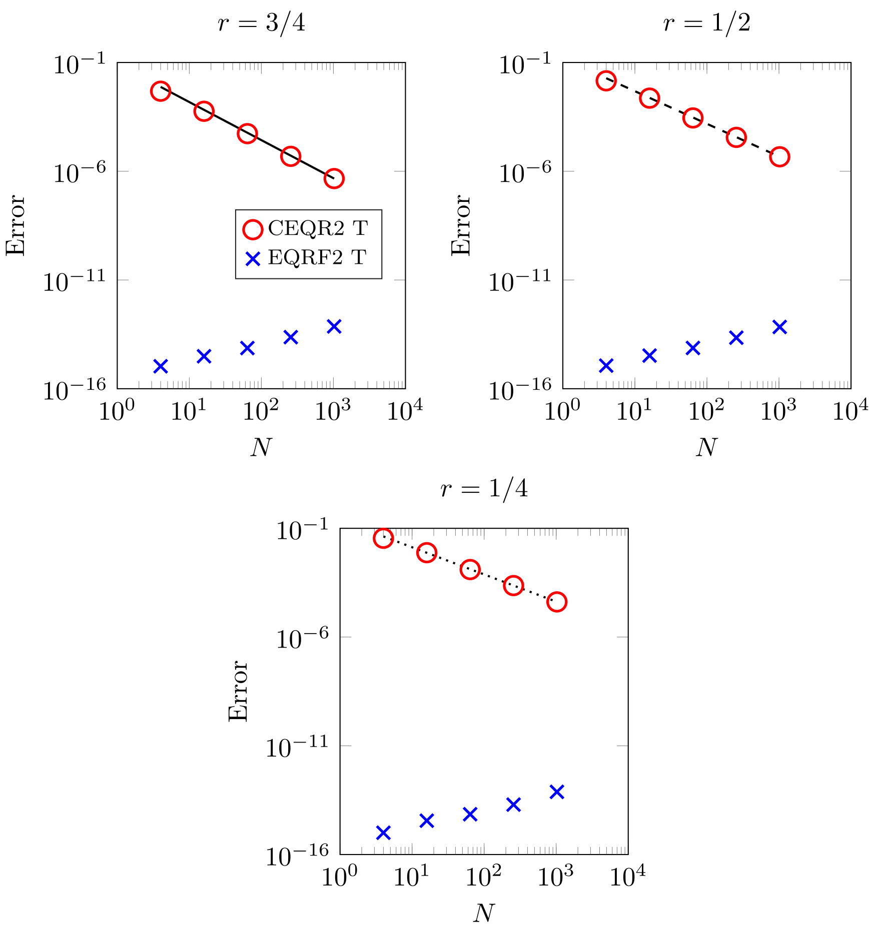

Fig. 3. Observed convergence rate of the error $\|y(T) - y_N\|_\infty$, with different values of r, for problem (19). For both the classical exponential quadrature rule CEQR2 T (17) and our proposed method EQRF2 T (11) we employ the points $c_1 = 0$ and $c_2 = 1$ (i.e., the trapezoidal quadrature nodes). The slope of the solid line is -1.75, that of the dashed lines is -1.5, and that of the dotted line is -1.25.

3.4. Results with two and three collocation points (EQRF2 and EQRF3)

We focus now our attention on the problem

$$\begin{cases} \partial_t y(t,x) = \mathrm{i}\partial_{xx} y(t,x) + \dfrac{\mathrm{e}^{t^r}}{2+\cos(2\pi x)}, \\ y(0,x) = \sin(2\pi x), \end{cases} \tag{20}$$

with periodic boundary conditions in the spatial domain $\Omega = (0,1)$ and different choices of r. The time interval is $[0,T]$, with final time set to $T = 2$. The spatial discretization is performed once again with a Fourier pseudospectral method using 500 modes. The results of the integrator EQRF2 (11), choosing as collocation points the trapezoidal nodes (labeled EQRF2 T) and the Gauss–Radau quadrature nodes (labeled EQRF2 GR), are summarized in Fig. 4. Remark that the trapezoidal nodes do not satisfy relation (13) in Theorem 3, while it is the case for the Gauss–Radau ones. The obtained outcome matches with the analysis carried out in Section 2.2.

We then repeat the same example using the proposed technique with three collocation points, i.e., using scheme (15) with $\nu = 3$. More in detail, we choose a set of points that does not satisfy condition (16) in Theorem 4 ($c_1 = 0$, $c_2 = \frac{1}{3}$, and $c_3 = 1$) and a set that satisfies it (the Gauss–Lobatto nodes, corresponding to Simpson's quadrature rule). The resulting integrators are labeled EQRF3 NC and EQRF3 GL, respectively. The results, presented in Fig. 5, are in line with the theoretical findings of Section 2.3.

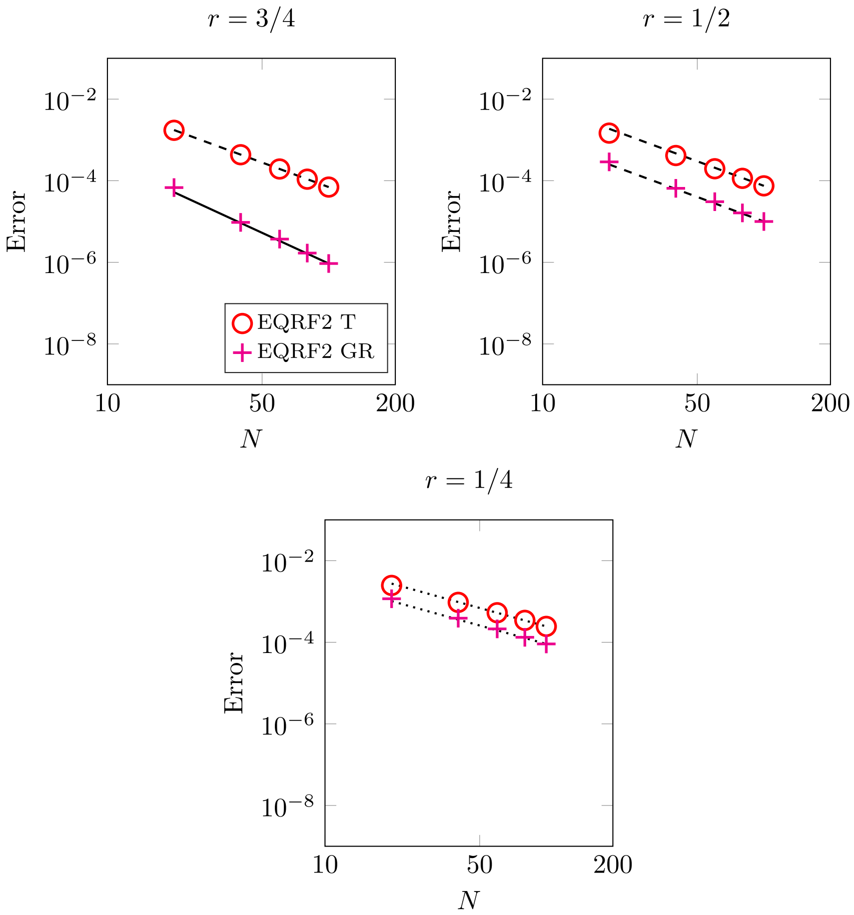

Fig. 4. Observed convergence rate of the error $\|y(T) - y_N\|_\infty$, with different values of r, for problem (20). The time marching is performed with EQRF2 (11). We employ the trapezoidal (EQRF2 T) and the Gauss–Radau (EQRF2 GR) collocation points. The slope of the solid line is −2.5, that of the dashed line is 2, and that of the dotted line is −1.5.

3.5. Work-precision diagram

Finally, we consider the variable-coefficient heat equation

$$\begin{cases} \partial_t y(t,x) = \dfrac{1+x^2}{10}\partial_{xx} y(t,x) + \mathrm{e}^{t^{\frac{3}{4}}} x(1-x), \\ y(0,x) = 4x(1-x), \end{cases} \tag{21}$$

coupled with homogeneous Dirichlet boundary conditions, in the spatial domain $\Omega = (0,1)$. The time interval is $[0,T]$, with $T = 2$. We perform the spatial discretization with standard second-order finite differences using 1000 inner points, which leads to a system of ODEs in the form (1). As time integrator we employ the EQRF2 method (11) with two Gauss quadrature nodes. We label the scheme EQRF2 G (F) when using the formulation based on fractional φ functions, while we label it EQRF2 G (I) when using the equivalent integral formulation (see the discussion in Section 3.1). As term of comparison we consider the classical exponential quadrature rule (17) with two Gauss quadrature nodes and denote it CEQR2 G. We remark that the matrix A of the ODEs system can be symmetrized and subsequently diagonalized as $A = (DV)\Lambda(V^\top D^{-1})$, being D the diagonal symmetrization matrix, V an orthogonal matrix, and Λ the diagonal matrix collecting the eigenvalues. Therefore, all the relevant matrix functions can be computed in a scalar fashion on Λ. The outcome of the simulations is summarized in Fig. 6.

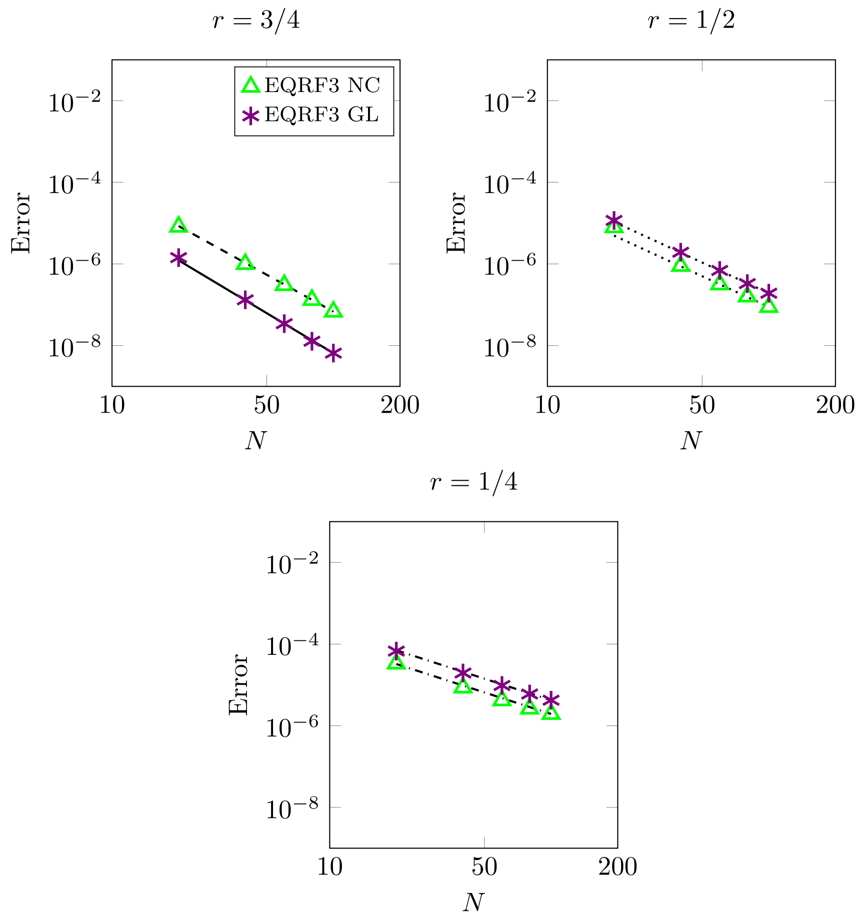

Fig. 5. Observed convergence rate of the error $\|y(T) - y_N\|_\infty$, with different values of r, for problem (20). The time marching is performed with EQRF3 (15). We employ $c_1 = 0$, $c_2 = \frac{1}{3}$, and $c_3 = 1$ (EQRF3 NC) and the Gauss–Lobatto (EQRF3 GL) collocation points. The slope of the solid line is -3.25, that of the dashed line is -3, that of the dotted line is -2.5, and that of the dashdotted line is -1.75.

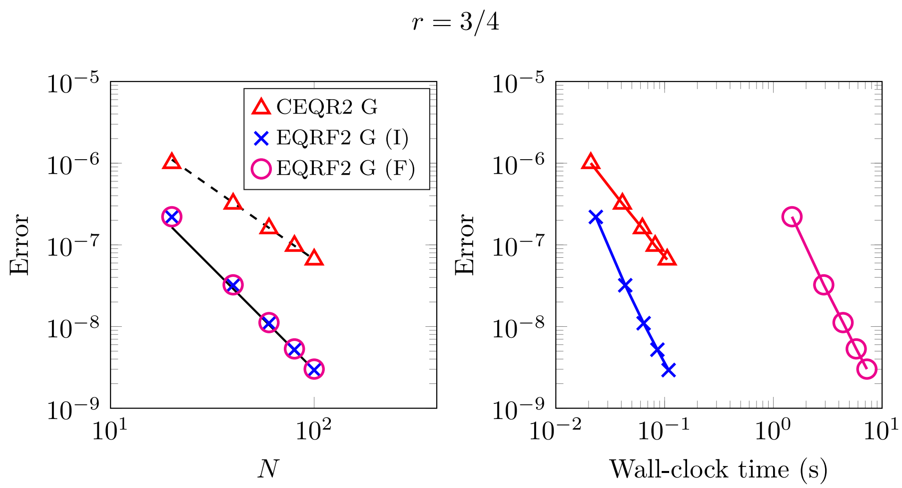

Fig. 6. Observed convergence rate of the error $\|y(T) - y_N\|_\infty$ (left) and work-precision diagram (right) for problem (21). The time marching schemes employ two Gauss quadrature nodes. We label CEQR2 G the classical exponential quadrature rule (17), EQRF2 G (I) the integral formulation of the proposed method (8), and EQRF2 G (F) the fractional φ functions formulation of the proposed method (11). The slope of the solid line is -2.5 and that of the dashed line is -1.75.

First of all, from the left plot we observe that the classical exponential quadrature rule CEQR2 G shows the expected order of convergence $1+r$. The proposed method EQRF2 clearly exhibits order of convergence $1+2r$, both when using the integral formulation EQRF2 G (I) and when employing the fractional φ functions formulation EQRF2 G (F). In particular, notice that the overall accuracy of the time marching method is not influenced by the underlying technique employed to compute the matrix functions. On the other hand, in terms of computational cost, the advantage of EQRF2 G (I) compared to EQRF2 G (F) is clear (see the right plot in Fig. 6). Therefore, we conclude that for this example the best performing approach is EQRF2 G (I).

4. Conclusions and future work

In this paper we presented a class of collocation-type exponential integrators tailored for linear problems with time-dependent fractional sources. The proposed schemes are analyzed in the context of strongly continuous semigroups, and may reach higher order compared to classical exponential quadrature rules. The performed numerical experiments match satisfactorily with the theoretical investigations and clearly show the superiority of the approach. As future work, we plan to study a more sophisticated technique to efficiently compute the special instances of matrix Mittag–Leffler functions which arise in the proposed integrators EQRFν. Also, we plan to generalize the approach for the time integration of stiff semilinear PDEs. This would be useful, for instance, in the context of diffusion equations with sublinear growth [24].

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Data availability

No data was used for the research described in the article.

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