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Pole-zero assignment by the receptance method: multi-input active vibration control

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ABSTRACT

The famous receptance method for the active vibration control has been mainly applied for pole placement. In this paper, it is exploited to solve the multi-input antiresonance assignment and then it is extended to handle the simultaneous pole-zero assignment. The design of the controllers is achieved through the measured receptances. The chief advantage is that system model is not needed, and the controller gains are synthetized through the data collected through experimental measurements. Two different approaches are proposed to compute the gains: a single-step method and a multi-step method. Both the techniques are developed for either state or state-derivative control. Two techniques to handle the non-uniqueness of the solution are proposed as well: the first one allows including specification on the eigenvectors, and hence on the spatial response of the system when excited at the antiresonance frequency; the second one places approximately all the poles through an optimization-based formulation. The proposed methods are validated through some numerical examples taken from common benchmarks in this field of research.

1. Introduction

1.1. State of the art

Vibration suppression is a goal intensively pursued over the decades to improve the performances of lightweight flexible systems. Both passive approaches exploiting optimal system modifications (see e.g., [1–4] and the references therein), and active control techniques have been proposed. The latter ones can be basically grouped into model based [5,6], model free [7] and receptance-based approaches [8]. Model based approaches use an accurate system model, either formulated as a first-order state space or a second-order one, to design and tune the controller; model updating [9,10] is therefore required to identify the model parameters. Model free approaches discard the model and therefore require trial-and-error controller tuning. The third approach merges the benefits of the other two ones, and therefore has been attracting an ever-growing attention over the last decades. On the one hand, using receptances does not require model updating to estimate the mass, damping and stiffness matrices and enables to avoid model reduction of models obtained from finite elements. On the other hand, receptances can be used in analytical controller design formulations, as in model-based approaches. These advantages have led to several studies on the application of the receptance method for active vibration control, that have been proposed by the most eminent researchers in this field (see the reviews [11,12]). The milestone study for the single-input pole, zero and pole-zero placement by the receptance method through state-feedback has been proposed by Mottershead

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and Ram in [8]. The receptance-based method to pole placement has been then formulated for asymmetric systems by Ouyang in [13], and applied to the stabilization of friction-induced vibrations. Extension of the receptance method by considering also the robustness of the assignment has been proposed in [14]. The formulation for asymmetric systems has been applied for pole assignment in flexible link mechanisms, whose damping and stiffness matrices are asymmetric due to the coupling between rigid-body motion and elastic motion, in [15].

The extension of the receptance-method to multi-input systems has been again proposed by Mottershead and Ram in [16] for partial pole placement, i.e. some closed-loop poles are assigned and the other are kept unchanged with respect to the open-loop system. Since the solution of the multi-input partial pole placement is not unique, the method has been then extended to achieve minimum norm control gains: an optimization-based approach has been proposed in [17], while an analytical solution is proposed in [18]. The simultaneous minimization of the control gains, together with the increase of the closed-loop system robustness through the minimization of the eigenvectors condition number has been studied in [19] where an optimization algorithm is proposed.

The state-feedback multi-input receptance method has been experimentally applied to the aeroelastic control for suppressing the flutter instabilities and for augmenting the flutter boundaries in [20]. An output-feedback approach, exploiting the redundancy introduced by acceleration measurements, has been proposed in [21] to tackle similar problems related to aeroelastic vibration suppression.

1.2. *Contributions of this paper*

The literature review highlights that the multi-input receptance based methods proposed up to now in the literature just considers pole assignment. In contrast, the assignment of antiresonances (i.e. of complex conjugate pairs of zeros) is not discussed, although imposing antiresonances is beneficial for vibration absorption (see e.g., [2,22]).

In the light of these needs, this paper proposes two solutions to the problem of active vibration control through the assignment of the closed-loop zeros for multi-input linear systems, by just exploiting the measured receptances. Both the methods are developed for either state feedback of state-derivative controllers and the proposed methods can be applied to symmetric and asymmetric systems. The methods are extended for the simultaneous pole-zero placement. Two heuristic approaches are also proposed to solve the non-uniqueness of the solution of the assignment problem.

The effectiveness of the proposed techniques is assessed through two numerical test-cases employed in the literature as benchmark examples.

2. Theoretical overview and problem statement

2.1. *Definitions*

Let us consider a linear, time-invariant vibrating system with an arbitrary number $N > 1$ of DOFs. Its dynamic model in term of mass, damping and stiffness matrices, denoted $\mathbf{M} \in \mathbb{R}^{N\times N}$, $\mathbf{C} \in \mathbb{R}^{N\times N}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ respectively is:

$$\mathbf{M}\ddot{\boldsymbol{x}}(t) + \mathbf{C}\dot{\boldsymbol{x}}(t) + \mathbf{K}\boldsymbol{x}(t) = \boldsymbol{f}(t) \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^N$ is the displacement vector ($\dot{\boldsymbol{x}}(t)$, $\ddot{\boldsymbol{x}}(t) \in \mathbb{R}^N$ are its derivatives) and $\boldsymbol{f}(t) \in \mathbb{R}^N$ is the vector of the external forces, that is composed by the vector of the external disturbance forces $\boldsymbol{f}_{\boldsymbol{e}}(t) \in \mathbb{R}^N$ and the contributions of the control forces $\boldsymbol{u}(t) \in \mathbb{R}^m$

$$\boldsymbol{f}(t) = \boldsymbol{f}_{\boldsymbol{e}}(t) + \mathbf{B}\boldsymbol{u}(t) \tag{2}$$

where $\mathbf{B} \in \mathbb{R}^{N\times m}$ is the actuator input matrix and $m > 1$ is the number of independent actuators.

In the case of uncontrolled systems, $\boldsymbol{u}(t) = \mathbf{0}$ and the dynamic model of the open-loop system in the Laplace domain $s = \mathrm{j}\omega$ ($\mathrm{j} = \sqrt{-1}$ and ω is the frequency) is:

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\boldsymbol{x}(s) = \boldsymbol{f}_{\boldsymbol{e}}(s) \tag{3}$$

Eq. (3) is recast into the following form by introducing the receptance matrix $\mathbf{H}(s) = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1}$

$$\boldsymbol{x}(s) = \mathbf{H}(s)\boldsymbol{f}_{\boldsymbol{e}}(s) \tag{4}$$

The pq-th entry of $\mathbf{H}(s)$, henceforth denoted $h_{pq}(s)$, is the transfer function from the force applied to the q-th coordinate to the displacement of the p-th coordinate, and can be computed by applying the Cramer's rule as follows [2,23]:

$$h_{pq}(s) = \frac{x_p(s)}{f_q(s)} = (-1)^{p+q}\frac{det\left(s^2\mathbf{M}_{\overline{\mathbf{qp}}} + s\mathbf{C}_{\overline{\mathbf{qp}}} + \mathbf{K}_{\overline{\mathbf{qp}}}\right)}{det\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)} \tag{5}$$

$\mathbf{M}_{\overline{\mathbf{qp}}}$, $\mathbf{C}_{\overline{\mathbf{qp}}}$, $\mathbf{K}_{\overline{\mathbf{qp}}} \in \mathbb{R}^{(N-1)\times(N-1)}$ are the adjunct system matrices, which are obtained by removing the q-th row and the p-th column from matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ respectively. By following the notation proposed in [24], the following relations hold

$$\begin{aligned}\mathbf{M}_{\overline{\mathbf{qp}}} &= \mathbf{B}_{\mathbf{n}}^{T}\mathbf{M}\mathbf{C}_{\mathbf{n}}\\ \mathbf{C}_{\overline{\mathbf{qp}}} &= \mathbf{B}_{\mathbf{n}}^{T}\mathbf{C}\mathbf{C}_{\mathbf{n}}\\ \mathbf{K}_{\overline{\mathbf{qp}}} &= \mathbf{B}_{\mathbf{n}}^{T}\mathbf{K}\mathbf{C}_{\mathbf{n}}\end{aligned} \tag{6}$$

where $\mathbf{B}_{\mathbf{n}}$ and $\mathbf{C}_{\mathbf{n}}$ are defined through the null space operator of vector $\boldsymbol{e}_q$ (column vector whose entries are all equal to zero except for the q-th entry which is equal to one) and $\boldsymbol{e}_p$ (column vector whose entries are all equal to zero except for the p-th entry which is equal to one)

$$\begin{aligned}\mathbf{B}_{\mathbf{n}} &= \mathrm{null}\left(\boldsymbol{e}_q^T\right)\\ \mathbf{C}_{\mathbf{n}} &= \mathrm{null}\left(\boldsymbol{e}_p\right)\end{aligned} \tag{7}$$

Eq. (5) reveals that the system poles are the complex numbers $p_i,\ \ i = 1, \dots, 2N$ for whose the denominator of Eq. (5) vanishes. Alternatively, the poles $p_i,\ \ i = 1, \dots, 2N$ of the system can be computed by solving the following eigenproblem [16]:

$$\left(p_i^2\mathbf{M} + p_i\mathbf{C} + \mathbf{K}\right)\boldsymbol{u}_{p,i} = \boldsymbol{0}, \quad i = 1, \dots, 2N \tag{8}$$

where $\boldsymbol{u}_{p,i} \in \mathbb{C}^N$ is the i-th eigenvector related to the i-th pole p_i.

The zeros of h_{pq} are the complex numbers $z_i,\ \ 1 \leqslant i \leqslant 2(N-1)$ such that the numerator in Eq. (5) vanishes. Similarly, the zeros $z_i,\ \ 1 \leqslant i \leqslant 2(N-1)$ of h_{pq} are obtained through the solution of the adjunct system eigenproblem [2]:

$$\left(z_i^2\mathbf{M}_{\overline{\mathbf{qp}}} + z_i\mathbf{C}_{\overline{\mathbf{qp}}} + \mathbf{K}_{\overline{\mathbf{qp}}}\right)\boldsymbol{u}_{z,i} = \boldsymbol{0}, \quad 1 \leqslant i \leqslant 2(N-1) \tag{9}$$

where $\boldsymbol{u}_{z,i} \in \mathbb{C}^{N-1}$ is the i-th right-eigenvector of the adjunct system related to the i-th zero z_i. The physical meaning of the adjunct system right-eigenvector is provided in [2,24].

2.2. Control Synthesis: Problem statement

Let us now consider the dynamic model of the controlled system. If state-feedback is assumed, the following control law is defined:

$$\boldsymbol{u}(t) = \mathbf{F}^T\dot{\boldsymbol{x}}(t) + \mathbf{G}^T\boldsymbol{x}(t) \tag{10}$$

where $\mathbf{F}, \mathbf{G} \in \mathbb{R}^{N \times m}$ are the velocity and displacement gain matrices, respectively.

If state-derivative feedback is assumed, the following control law is defined:

$$\boldsymbol{u}(t) = \mathbf{D}^T\ddot{\boldsymbol{x}}(t) + \mathbf{F}^T\dot{\boldsymbol{x}}(t) \tag{11}$$

where $\mathbf{D}, \mathbf{F} \in \mathbb{R}^{N \times m}$ are the acceleration and velocity gain matrices, respectively. In both the definition of the feedback law, the state reference is discarded since it does not affect the system poles and zeros.

2.3. Problem statements

Given the open-loop system receptance matrix $\mathbf{H}(s)$ and the actuator input matrix $\mathbf{B}$, four problems are addressed.

Zero assignment through state feedback (Section 3.1): compute the control gain matrices $\mathbf{F}$ and $\mathbf{G}$ such that the set of n_z prescribed zeros $\sum_z = \{\bar{z}_1, \dots, \bar{z}_{n_z}\}$ is assigned for the closed-loop receptance $\bar{h}_{pq}$.

Zero assignment through state-derivative feedback (Section 3.2): compute the control gain matrices $\mathbf{D}$ and $\mathbf{F}$ such that the prescribed set of n_z zeros $\sum_z$ is assigned for the closed-loop receptance $\bar{h}_{pq}$.

Pole-zero assignment through state feedback (Section 4.1): compute the control gain matrices $\mathbf{F}$ and $\mathbf{G}$ such that the prescribed set of $n_p < N$ poles $\sum_p = \left\{\bar{p}_1, \dots, \bar{p}_{n_p}\right\}$ is assigned for the closed-loop system and the prescribed set of n_z zeros $\sum_z$ is assigned for the closed-loop receptance $\bar{h}_{pq}$.

Pole-zero assignment through state-derivative feedback (Section 4.2): compute the control gain matrices $\mathbf{D}$ and $\mathbf{F}$ such that the prescribed set of $n_p < N$ poles $\sum_p$ is assigned for the closed-loop system and the prescribed set of n_z zeros $\sum_z$ is assigned for the closed-loop receptance $\bar{h}_{pq}$.

Two different approaches are proposed to solve the problems. The first one is referred to as “single-step” since it assigns the desired zeros (or zeros and poles) by solving a linear system. The second one is referred to as “multi-step” since it assigns the desired zeros (or zeros and poles) by solving a sequence of m linear systems.

3. Zero assignment

3.1. Zero assignment through state feedback

3.1.1. Method 1: Single-step solution

Let us consider, without loss of generality, $\boldsymbol{f}_e(t) = \mathbf{0}$ since the external disturbances do not affect the system eigenstructure. The dynamic model of the closed-loop system, with state-feedback control, is inferred by substitution of Eqs. (2) and (10) into Eq. (1):

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\boldsymbol{x}(s) = \mathbf{B}\left(s\mathbf{F}^T + \mathbf{G}^T\right)\boldsymbol{x}(s) \tag{12}$$

The zeros of the closed-loop receptance $\overline{h}_{pq}(s)$ can be obtained solving the adjunct system eigenproblem for the controlled system. By exploiting the formalism of Eq. (6), the eigenproblem of the closed-loop adjunct system is written in the following form (the proof is provided in Appendix A), where $\boldsymbol{w}_i \in \mathbb{C}^{N-1}$ is the i-th adjunct system right eigenvector related to the zero $\overline{z}_i$ to assign:

$$\mathbf{B}_\mathbf{n}^T\left(\overline{z}_i^2\mathbf{M} + \overline{z}_i\mathbf{C} + \mathbf{K}\right)\mathbf{C}_\mathbf{n}\boldsymbol{w}_i = \mathbf{B}_\mathbf{n}^T\mathbf{B}\left(\overline{z}_i\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{C}_\mathbf{n}\boldsymbol{w}_i, \quad i = 1, \dots, n_z \tag{13}$$

Let us define the adjunct system receptance matrix $\mathbf{H}_{\overline{\mathbf{qp}}}$, evaluated for $s = \overline{z}_i$, as follows:

$$\mathbf{H}_{\overline{\mathbf{qp}}}(\overline{z}_i) = \left(\mathbf{B}_\mathbf{n}^T\left(\overline{z}_i^2\mathbf{M} + \overline{z}_i\mathbf{C} + \mathbf{K}\right)\mathbf{C}_\mathbf{n}\right)^{-1} \tag{14}$$

$\mathbf{H}_{\overline{\mathbf{qp}}}$ can be conveniently expressed as follows [23,25]:

$$\mathbf{H}_{\overline{\mathbf{qp}}}(\overline{z}_i) = \left(\overline{z}_i^2\mathbf{M}_{\overline{\mathbf{qp}}} + \overline{z}_i\mathbf{C}_{\overline{\mathbf{qp}}} + \mathbf{K}_{\overline{\mathbf{qp}}}\right)^{-1} \tag{15}$$

The physical meaning of the adjunct system receptance matrix is often said to be “obscure” [23,25]; however $\mathbf{H}_{\overline{\mathbf{qp}}}$ can be determined numerically through the measured receptances of the open loop system $\mathbf{H}(\overline{z}_i)$, without involving the system matrices in the calculation, as suggested in [23]. Let us define $\mathbf{Z}(\overline{z}_i) = \mathbf{H}^{-1}(\overline{z}_i)$:

$$\mathbf{H}_{\overline{\mathbf{qp}}}(\overline{z}_i) = \mathbf{Z}_{\overline{\mathbf{qp}}}^{-1}(\overline{z}_i), \text{ with } \mathbf{Z}_{\overline{\mathbf{qp}}}(\overline{z}_i) = \mathbf{B}_\mathbf{n}^T\mathbf{Z}(\overline{z}_i)\mathbf{C}_\mathbf{n} \tag{16}$$

Finally, Eq. (13) is recast through Eq. (14), in terms of the adjunct system measured receptances:

$$\boldsymbol{w}_i = \mathbf{H}_{\overline{\mathbf{qp}}}(\overline{z}_i)\widetilde{\mathbf{B}}\left(\overline{z}_i\widetilde{\mathbf{F}}^T + \widetilde{\mathbf{G}}^T\right)\boldsymbol{w}_i, \quad i = 1, \dots, n_z \tag{17}$$

where:

$$\widetilde{\mathbf{B}} = \mathbf{B}_\mathbf{n}^T\mathbf{B} \in \mathbb{R}^{(N-1)\times m}\widetilde{\mathbf{F}}^T = \mathbf{F}^T\mathbf{C}_\mathbf{n} \in \mathbb{R}^{m\times(N-1)}\widetilde{\mathbf{G}}^T = \mathbf{G}^T\mathbf{C}_\mathbf{n} \in \mathbb{R}^{m\times(N-1)} \tag{18}$$

Let us partition the matrices in Eq. (18) through their column vectors $\widetilde{\boldsymbol{b}}_i, \widetilde{\boldsymbol{f}}_i, \widetilde{\boldsymbol{g}}_i$: $\widetilde{\mathbf{B}} = [\widetilde{\boldsymbol{b}}_1 \quad \dots \quad \widetilde{\boldsymbol{b}}_m]$, $\widetilde{\mathbf{F}} = [\widetilde{\boldsymbol{f}}_1 \quad \dots \quad \widetilde{\boldsymbol{f}}_m]$ and $\widetilde{\mathbf{G}} = [\widetilde{\boldsymbol{g}}_1 \quad \dots \quad \widetilde{\boldsymbol{g}}_m]$. Hence, Eq. (17) is equivalent to:

$$\boldsymbol{w}_i = \mathbf{H}_{\overline{\mathbf{qp}}}(\overline{z}_i)\left(\widetilde{\boldsymbol{b}}_1\left(\overline{z}_i\widetilde{\boldsymbol{f}}_1^T + \widetilde{\boldsymbol{g}}_1^T\right) + \dots + \widetilde{\boldsymbol{b}}_m\left(\overline{z}_i\widetilde{\boldsymbol{f}}_m^T + \widetilde{\boldsymbol{g}}_m^T\right)\right)\boldsymbol{w}_i, \quad i = 1, \dots, n_z \tag{19}$$

Let us define:

$$\boldsymbol{\psi}_{i,j} = \mathbf{H}_{\overline{\mathbf{qp}}}(\overline{z}_i)\widetilde{\boldsymbol{b}}_j, \quad i = 1, \dots, n_z \text{ and } j = 1, \dots, m \tag{20}$$

Since $\widetilde{\boldsymbol{f}}_i, \widetilde{\boldsymbol{g}}_i$ are arbitrary real numbers and no specifications on $\boldsymbol{w}_i$ are set, arbitrary values for the following scalar coefficients $\alpha_{i,j}$:

$$\alpha_{i,j} = \left(\overline{z}_i\widetilde{\boldsymbol{f}}_j^T + \widetilde{\boldsymbol{g}}_j^T\right)\boldsymbol{w}_i, \quad i = 1, \dots, n_z \text{ and } j = 1, \dots, m \tag{21}$$

This approach recalls the one proposed in [16] for partial pole placement.

By substituting Eqs. (20) and (21) into Eq. (19), the assignment problem is transformed into the following equality, which reveals that the attainable right-eigenvectors of the adjunct system are linear combinations of vectors $\boldsymbol{\psi}_{i,j}$, which in turn depends on the system parameters by means of $\mathbf{H}_{\overline{\mathbf{qp}}}(\overline{z}_i)$ and on the actuation matrix (partitioned through $\widetilde{\boldsymbol{b}}_j, j = 1, \dots, m$):

$$\boldsymbol{w}_i = \alpha_{i,1}\boldsymbol{\psi}_{i,1} + \dots + \alpha_{i,m}\boldsymbol{\psi}_{i,m}, \quad i = 1, \dots, n_z \tag{22}$$

The assignment of $\overline{z}_i$ in Eq. (21) can be transposed and recast into the following matrix form:

$$\begin{bmatrix} \overline{z}_i\boldsymbol{w}_i^T & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{w}_i^T & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \overline{z}_i\boldsymbol{w}_i^T & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{w}_i^T & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \overline{z}_i\boldsymbol{w}_i^T & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{w}_i^T \end{bmatrix} \begin{Bmatrix} \widetilde{\boldsymbol{f}}_1 \\ \vdots \\ \widetilde{\boldsymbol{f}}_m \\ \widetilde{\boldsymbol{g}}_1 \\ \vdots \\ \widetilde{\boldsymbol{g}}_m \end{Bmatrix} = \begin{Bmatrix} \alpha_{i,1} \\ \vdots \\ \alpha_{i,m} \end{Bmatrix} \tag{23}$$

that can be, in turn, summarized through the compact form of a linear system, with obvious meaning of the notation:

$$\boldsymbol{\Phi}_{sz,i}\widetilde{\boldsymbol{k}}_s = \boldsymbol{\alpha}_{z,i} \tag{24}$$

By collecting all the assignment equations of the desired n_z zeros belonging to the self-conjugate set $\sum_z$, the following linear problem is obtained:

$$\begin{bmatrix} \boldsymbol{\Phi}_{sz,1} \\ \vdots \\ \boldsymbol{\Phi}_{sz,n_z} \end{bmatrix} \widetilde{\boldsymbol{k}}_s = \begin{Bmatrix} \boldsymbol{\alpha}_{z,1} \\ \vdots \\ \boldsymbol{\alpha}_{z,n_z} \end{Bmatrix} \tag{25}$$

leading to

$$\boldsymbol{\Phi}_{sz}\widetilde{\boldsymbol{k}}_s = \boldsymbol{\alpha}_z \tag{26}$$

The state-feedback control gains are obtained by solving the linear system in Eq. (25). To ensure that matrices $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{G}}$ are real, whenever a complex zero $\bar{z}_i$ is assigned, also its complex conjugate must be assigned [16].

Once $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{G}}$ are computed by solving the linear system in Eq. (26), the gain matrices $\mathbf{F}$ and $\mathbf{G}$ are computed by inverting Eq. (18), by taking advantage that the columns of $\mathbf{C_n}$ are orthonormal and hence $\mathbf{C_n}^T\mathbf{C_n} = \mathbf{I}_{N-1}$.

$$\begin{aligned} \mathbf{F}^T &= \widetilde{\mathbf{F}}^T\mathbf{C}_\mathbf{n}^T \\ \mathbf{G}^T &= \widetilde{\mathbf{G}}^T\mathbf{C}_\mathbf{n}^T \end{aligned} \tag{27}$$

Indeed, the columns of $\mathbf{C_n}$ are linearly independent since $\mathbf{C_n} \in ker(\boldsymbol{e}_p)$ (see Eq. (7)), where $\boldsymbol{e}_p$ is a unit vector.

3.1.2. Method 2: Multi-step solution

The chief idea of the multi-step method for zero placement is inspired by a method for the state-feedback partial pole placement exploiting the system matrices in [5]. Such an idea is here exploited and adapted to perform zero assignment by means of the measured receptances. Basically, the m-rank assignment is solved as a sequence of m rank-one assignments, hence defining a placement path as sketched in Fig. 1 with reference to the assignment of a pair of complex conjugate zeros.

Let us partition the actuation matrix and the control gain matrices as $\mathbf{B} = [\boldsymbol{b}_1 \quad \dots \quad \boldsymbol{b}_m]$, $\mathbf{F} = [\boldsymbol{f}_1 \quad \dots \quad \boldsymbol{f}_m]$ and $\mathbf{G} = [\boldsymbol{g}_1 \quad \dots \quad \boldsymbol{g}_m]$. At the k-th step of the proposed method, the set of the prescribed zeros is assigned in the "temporary" locations defined the $\sum_{z,k} = \{\bar{z}_{1,k}, \dots, \bar{z}_{n_z,k}\}$ by means of the control gains $\boldsymbol{f}_k, \boldsymbol{g}_k$. Hence, after the m-steps the prescribed zeros in $\sum_z$ are assigned and the gain matrices $\mathbf{F}$ and $\mathbf{G}$ are completely determined.

The control gains that enable to assign the set of the prescribed zeros at the generic k-th step are computed by exploiting the single-input ($m = 1$) receptance method proposed in [8] and here briefly recalled.

In the single-input case, the receptance matrix of the closed-loop system $\overline{\mathbf{H}}_k(s)$ at the k-th step is defined as:

$$\overline{\mathbf{H}}_k(s) = \left(s^2\mathbf{M} + s\left(\mathbf{C}_{k-1} - \boldsymbol{b}_k\boldsymbol{f}_k^T\right) + \left(\mathbf{K}_{k-1} - \boldsymbol{b}_k\boldsymbol{g}_k^T\right)\right)^{-1} \tag{28}$$

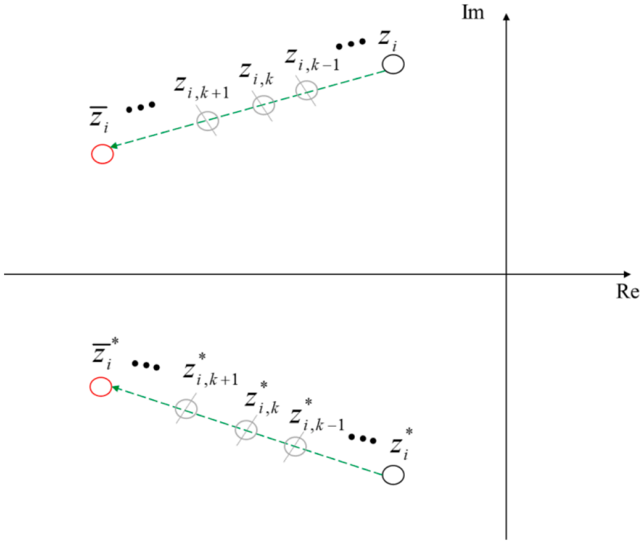

Fig. 1. Multi-step assignment from the i-th open-loop zero (black) to the i-th prescribed zero (red). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

where: $\mathbf{C}_{k-1} = \mathbf{C} - [\boldsymbol{b}_1 \quad \dots \quad \boldsymbol{b}_{k-1}][\boldsymbol{f}_1 \quad \dots \quad \boldsymbol{f}_{k-1}]^T$ and $\mathbf{K}_{k-1} = \mathbf{K} - [\boldsymbol{b}_1 \quad \dots \quad \boldsymbol{b}_{k-1}][\boldsymbol{g}_1 \quad \dots \quad \boldsymbol{g}_{k-1}]^T$.

The Sherman-Morrison formula for rank-one modifications, together with some algebraic manipulations cast the receptance $\overline{h}_{pq,k}$ at the k-th step as:

$$\overline{h}_{pq,k}(s) = \frac{\boldsymbol{e}_p^T\left[\left(1-(s\boldsymbol{f}_k+\boldsymbol{g}_k)^T\overline{\mathbf{H}}_{k-1}(s)\boldsymbol{b}_k\right)\overline{\mathbf{H}}_{k-1}(s)+\overline{\mathbf{H}}_{k-1}(s)\boldsymbol{b}_k(s\boldsymbol{f}_k+\boldsymbol{g}_k)^T\overline{\mathbf{H}}_{k-1}(s)\right]\boldsymbol{e}_q}{1-(s\boldsymbol{f}_k+\boldsymbol{g}_k)^T\overline{\mathbf{H}}_{k-1}(s)\boldsymbol{b}_k} \tag{29}$$

The zeros of $\overline{h}_{pq,k}$ are the roots of the numerator of Eq. (29).

By introducing the auxiliary complex vector $\boldsymbol{t}_{i,k} = \overline{h}_{pq,k-1}(\overline{z}_{i,k})\overline{\mathbf{H}}_{k-1}(\overline{z}_{i,k})\boldsymbol{b}_k - \left[\boldsymbol{e}_p^T\overline{\mathbf{H}}_{k-1}(\overline{z}_{i,k})\boldsymbol{b}_k\right]\overline{\mathbf{H}}_{k-1}(\overline{z}_{i,k})\boldsymbol{e}_q$ evaluated at the i-th prescribed zero $\overline{z}_{i,k}$, the n_z assignment problems at the k-th step are clustered leading to the following linear system (see Appendix B):

$$\begin{bmatrix} \overline{z}_{1,k}\boldsymbol{t}_{1,k}^T & \boldsymbol{t}_{1,k}^T \\ \overline{z}_{2,k}\boldsymbol{t}_{2,k}^T & \boldsymbol{t}_{2,k}^T \\ \vdots & \vdots \\ \overline{z}_{n_z,k}\boldsymbol{t}_{n_z,k}^T & \boldsymbol{t}_{n_z,k}^T \end{bmatrix} \begin{Bmatrix} \boldsymbol{f}_k \\ \boldsymbol{g}_k \end{Bmatrix} = \begin{Bmatrix} h_{rc,k-1}(\overline{z}_{1,k}) \\ h_{rc,k-1}(\overline{z}_{2,k}) \\ \vdots \\ h_{rc,k-1}(\overline{z}_{n_z,k}) \end{Bmatrix} \tag{30}$$

that can be summarized in the usual compact form:

$$\boldsymbol{\Phi}_{sz,k}\boldsymbol{k}_{s,k} = \boldsymbol{y}_{z,k} \tag{31}$$

Hence, by solving Eq. (30) for $k = 1, \dots, m$ it is possible to obtain the gain matrices **F** and **G**.

3.2. Zero assignment through State-Derivative feedback

The single-step and multi-step methods proposed respectively in Sections 3.1.1 and 3.1.2 can be easily adopted in the case of state-derivative feedback, as defined in Eq. (11), to solve Problem 2.

3.2.1. Method 1: Single-step solution

Let us first consider the single-step solution. The counterpart of Eq. (17) is:

$$\boldsymbol{w}_i = \mathbf{H}_{\overline{\mathbf{qp}}}(\overline{z}_i)\widetilde{\mathbf{B}}\left(\overline{z}_i^2\widetilde{\mathbf{D}}^T + \overline{z}_i\widetilde{\mathbf{F}}^T\right)\boldsymbol{w}_i, \quad i = 1, \dots, n_z \tag{32}$$

where: $\widetilde{\mathbf{D}}^T = \mathbf{D}^T\mathbf{C}_{\mathbf{n}} \in \mathbb{R}^{m\times(N-1)}$ is the acceleration gains matrix in the adjunct system.

Exploiting mathematical manipulations recalling those provided in Section 3.1.1 it is possible to obtain the following linear system for the assignment of the i-th zero:

$$\begin{bmatrix} \overline{z}_i^2\boldsymbol{w}_i^T & \boldsymbol{0} & \cdots & \boldsymbol{0} & \overline{z}_i\boldsymbol{w}_i^T & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \overline{z}_i^2\boldsymbol{w}_i^T & \cdots & \boldsymbol{0} & \boldsymbol{0} & \overline{z}_i\boldsymbol{w}_i^T & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \overline{z}_i^2\boldsymbol{w}_i^T & \boldsymbol{0} & \boldsymbol{0} & \cdots & \overline{z}_i\boldsymbol{w}_i^T \end{bmatrix} \begin{Bmatrix} \widetilde{\boldsymbol{d}}_1 \\ \vdots \\ \widetilde{\boldsymbol{d}}_m \\ \widetilde{\boldsymbol{f}}_1 \\ \vdots \\ \widetilde{\boldsymbol{f}}_m \end{Bmatrix} = \begin{Bmatrix} \alpha_{i,1} \\ \vdots \\ \alpha_{i,m} \end{Bmatrix} \tag{33}$$

that is, again, written in the following compact form:

$$\boldsymbol{\Phi}_{dz,i}\widetilde{\boldsymbol{k}}_d = \boldsymbol{\alpha}_{z,i} \tag{34}$$

Finally, the counterpart of Eq. (25) for the assignment of n_z zeros through multi-input state-derivative control is:

$$\begin{bmatrix} \boldsymbol{\Phi}_{dz,1} \\ \vdots \\ \boldsymbol{\Phi}_{dz,n_z} \end{bmatrix} \widetilde{\boldsymbol{k}}_d = \begin{Bmatrix} \boldsymbol{\alpha}_{z,1} \\ \vdots \\ \boldsymbol{\alpha}_{z,n_z} \end{Bmatrix} \tag{35}$$

that can be written again in the following compact form;

$$\boldsymbol{\Phi}_{dz}\widetilde{\boldsymbol{k}}_d = \boldsymbol{\alpha}_z \tag{36}$$

The control gain matrices **D** and **F** can be computed through the same procedure provided in Eq. (18).

3.2.2. Method 2: Multi-step solution

In the case of multi-step solution the mathematical formulation proposed in Section 3.1.2 is slightly modified and the linear system in Eq. (30) in the state-derivative case becomes:

$$\begin{bmatrix} \bar{z}_{1,k}^2 \boldsymbol{t}_{1,k}^T & \bar{z}_{1,k} \boldsymbol{t}_{1,k}^T \\ \bar{z}_{2,k}^2 \boldsymbol{t}_{2,k}^T & \bar{z}_{2,k} \boldsymbol{t}_{2,k}^T \\ \vdots & \vdots \\ \bar{z}_{n_z,k}^2 \boldsymbol{t}_{n_z,k}^T & \bar{z}_{n_z,k} \boldsymbol{t}_{n_z,k}^T \end{bmatrix} \begin{Bmatrix} \boldsymbol{d}_k \\ \boldsymbol{f}_k \end{Bmatrix} = \begin{Bmatrix} h_{rc,k-1}\left(\bar{z}_{1,k}\right) \\ h_{rc,k-1}\left(\bar{z}_{2,k}\right) \\ \vdots \\ h_{rc,k-1}\left(\bar{z}_{n_z,k}\right) \end{Bmatrix} \tag{37}$$

or with a more compact notation:

$$\boldsymbol{\Phi}_{dz,k} \boldsymbol{k}_{d,k} = \boldsymbol{y}_{z,k} \tag{38}$$

Obviously, the solution of Eq. (38) for $k = 1, \ldots, m$ yields to the gain matrices $\mathbf{D}$ and $\mathbf{F}$.

4. Pole-Zero assignment

4.1. Pole-Zero assignment through state feedback

4.1.1. Method 1: Single-step solution

Multi-input pole placement by the receptance method through a single-step approach has been widely studied by several authors (see e.g. [16,18,21] and the references therein). For the detailed formulation of the assignment method please refer to the milestone paper [16].

Let us consider the assignment of the i-th pole $\bar{p}_i$, that belongs to set of n_p desired poles $\sum_p = \left\{\bar{p}_1, \ldots, \bar{p}_{n_p}\right\}$(as for the zeros, see Section 3.1.1):

$$\begin{bmatrix} \bar{p}_i \boldsymbol{v}_i^T & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{v}_i^T & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \bar{p}_i \boldsymbol{v}_i^T & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{v}_i^T & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \bar{p}_i \boldsymbol{v}_i^T & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{v}_i^T \end{bmatrix} \begin{Bmatrix} \boldsymbol{f}_1 \\ \vdots \\ \boldsymbol{f}_m \\ \boldsymbol{g}_1 \\ \vdots \\ \boldsymbol{g}_m \end{Bmatrix} = \begin{Bmatrix} \beta_{i,1} \\ \vdots \\ \beta_{i,m} \end{Bmatrix} \tag{39}$$

where:

$$\beta_{i,j} = \left(\bar{p}_i \boldsymbol{f}_j^T + \boldsymbol{g}_j^T\right) \boldsymbol{v}_i, \quad i = 1, \ldots, n_p \text{ and } j = 1, \ldots, m \tag{40}$$

and:

$$\boldsymbol{v}_i = \beta_{i,1} \boldsymbol{r}_{i,1} + \ldots + \beta_{i,m} \boldsymbol{r}_{i,m}, \quad j = 1, \ldots, m \tag{41}$$

with:

$$\boldsymbol{r}_{i,j} = \mathbf{H}(\bar{p}_i) \boldsymbol{b}_j, \quad i = 1, \ldots, n_p \text{ and } j = 1, \ldots, m \tag{42}$$

The linear system in Eq. (39) can be recast with obvious meaning of the notation in a more compact form: $\boldsymbol{\Phi}_{sp,i} \boldsymbol{k}_s = \boldsymbol{\beta}_{p,i}$, leading to the linear system for the assignment of the n_p poles:

$$\begin{bmatrix} \boldsymbol{\Phi}_{sp,1} \\ \vdots \\ \boldsymbol{\Phi}_{sp,n_p} \end{bmatrix} \boldsymbol{k}_s = \begin{Bmatrix} \boldsymbol{\beta}_{p,1} \\ \vdots \\ \boldsymbol{\beta}_{p,n_p} \end{Bmatrix} \tag{43}$$

In a more compact form Eq. (43) can be written as $\boldsymbol{\Phi}_{sp} \boldsymbol{k}_s = \boldsymbol{\beta}_p$.

Then, the solution of the pole-zero placement Problem 3 is straightforward since the linear systems in Eqs. (25) and (43) are clustered together leading to:

$$\begin{bmatrix} \boldsymbol{\Phi}_{sz} \boldsymbol{T} \\ \boldsymbol{\Phi}_{sp} \end{bmatrix} \boldsymbol{k}_s = \begin{Bmatrix} \boldsymbol{\alpha}_z \\ \boldsymbol{\beta}_p \end{Bmatrix} \tag{44}$$

where $\mathbf{T} = \mathrm{blkdiag}\left(\mathbf{C}_{\mathbf{n}}^T, \ldots, \mathbf{C}_{\mathbf{n}}^T\right)$ (*blkdiag* returns the block diagonal matrix created by aligning the input matrices along the main diagonal) and matrix $\mathbf{C}_{\mathbf{n}}^T$ is repeated *2 m* times since the control gains $\tilde{\boldsymbol{k}}_s$ in Eq. (25) are defined in the adjunct system, i.e. the transformations in Eq. (18) are applied to obtain $\boldsymbol{k}_s$.

4.1.2. Method 2: Multi-step solution

The multi-step pole-zero assignment for the solution of Problem 3 is obtained by considering the same framework provided in Section 3.1.2.

First, the path of the prescribed n_p poles that belongs to $\sum_{p,k} = \left\{\bar{p}_{1,k}, \ldots, \bar{p}_{n_p,k}\right\}$ is defined.

For the k-th step of the method, the n_p poles are assigned through the control gains solution of linear system with similar mathematical manipulations as those proposed for the single-input scenario in [8]:

$$\begin{bmatrix} \overline{p}_{1,k}\boldsymbol{r}^T_{1,k} & \boldsymbol{r}^T_{1,k} \\ \overline{p}_{2,k}\boldsymbol{r}^T_{2,k} & \boldsymbol{r}^T_{2,k} \\ \vdots & \vdots \\ \overline{p}_{n_p,k}\boldsymbol{r}^T_{n_p,k} & \boldsymbol{r}^T_{n_p,k} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f}_k \\ \boldsymbol{g}_k \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} \tag{45}$$

where $\boldsymbol{r}_{i,k} = \overline{\mathbf{H}}_{k-1}(\overline{p}_i)\boldsymbol{b}_k$ and $\overline{\mathbf{H}}_{k-1}(\overline{p}_i)$ is inferred from Eq. (28); a more detailed discussion is provided in Appendix B. Eq. (45) is recast, with a more simple notation, as: $\boldsymbol{\Phi}_{sp,k}\boldsymbol{k}_{s,k} = \boldsymbol{y}_{p,k}$. Finally, by clustering the compact notations of the linear systems in Eqs. (31) and (45) the pole-zero placement problem for the k-th step are synthesized by the solution of:

$$\begin{bmatrix} \boldsymbol{\Phi}_{sz,k} \\ \boldsymbol{\Phi}_{sp,k} \end{bmatrix} \boldsymbol{k}_{s,k} = \begin{Bmatrix} \boldsymbol{y}_{z,k} \\ \boldsymbol{y}_{p,k} \end{Bmatrix} \tag{46}$$

The solution of the k assignment problems in Eq. (46) for $k = 1,\ldots,m$, i.e. one foreach actuator, leads to the control gain matrices **F** and **G** solution of Problem 3.

4.2. Pole-Zero assignment through State-Derivative feedback

The multi-input pole-zero placement single and multi-step methods respectively proposed in Sections 4.1.1 and 4.1.2 for the tuning of state feedback controllers are adapted in this Section to tackle Problem 4, i.e. the multi-input pole-zero placement in the case of state-derivative control.

4.2.1. Method 1: Single-step solution

Let us first consider the single-step pole-zero assignment method proposed in Section 4, in the case of state-derivative feedback the linear system in Eq. (39) for the placement of the poles in $\sum_p$ becomes:

$$\begin{bmatrix} \overline{p}_i^2\boldsymbol{v}_i^T & \boldsymbol{0} & \cdots & \boldsymbol{0} & \overline{p}_i\boldsymbol{v}_i^T & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \overline{p}_i^2\boldsymbol{v}_i^T & \cdots & \boldsymbol{0} & \boldsymbol{0} & \overline{p}_i\boldsymbol{v}_i^T & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \overline{p}_i^2\boldsymbol{v}_i^T & \boldsymbol{0} & \boldsymbol{0} & \cdots & \overline{p}_i\boldsymbol{v}_i^T \end{bmatrix} \begin{Bmatrix} \boldsymbol{d}_1 \\ \vdots \\ \boldsymbol{d}_m \\ \boldsymbol{f}_1 \\ \vdots \\ \boldsymbol{f}_m \end{Bmatrix} = \begin{Bmatrix} \beta_{i,1} \\ \vdots \\ \beta_{i,m} \end{Bmatrix} \tag{47}$$

where: $\beta_{i,j} = \left(\overline{p}_i^2\boldsymbol{d}_j^T + \overline{p}_i\boldsymbol{f}_j^T\right)\boldsymbol{v}_i, \quad i = 1,\ldots,n_p$ and $j = 1,\ldots,m$. Through similar mathematical manipulations as in Section 4 a more synthetic notation for the assignment of n_p poles is: $\boldsymbol{\Phi}_{dp}\boldsymbol{k}_d = \boldsymbol{\beta}_p$. Hence, Problem 4 is solved by the control gains solution of the linear system obtained by clustering the compact notations of Eqs. (35) and (47), that yields:

$$\begin{bmatrix} \boldsymbol{\Phi}_{dz}\mathbf{T} \\ \boldsymbol{\Phi}_{dp} \end{bmatrix} \boldsymbol{k}_d = \begin{Bmatrix} \boldsymbol{\alpha}_z \\ \boldsymbol{\beta}_p \end{Bmatrix} \tag{48}$$

the meaning of matrix **T** has been discussed in Section 4.1.1.

4.2.2. Method 2: Multi-step solution

Let us now consider the case multi-step solution of Problem 4, in this scenario the linear system in Eq. (45) for the pole-placement at the k-th step becomes:

$$\begin{bmatrix} \overline{p}^2_{1,k}\boldsymbol{r}^T_{1,k} & \overline{p}_{1,k}\boldsymbol{r}^T_{1,k} \\ \overline{p}^2_{2,k}\boldsymbol{r}^T_{2,k} & \overline{p}_{2,k}\boldsymbol{r}^T_{2,k} \\ \vdots & \vdots \\ \overline{p}^2_{n_p,k}\boldsymbol{r}^T_{n_p,k} & \overline{p}_{n_p,k}\boldsymbol{r}^T_{n_p,k} \end{bmatrix} \begin{Bmatrix} \boldsymbol{d}_k \\ \boldsymbol{f}_k \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} \tag{49}$$

Or simply: $\boldsymbol{\Phi}_{dp,k}\boldsymbol{k}_{d,k} = \boldsymbol{y}_{p,k}$. Hence, the pole-zero placement is achieved by clustering Eqs. (38) and (49), leading to the linear system:

$$\begin{bmatrix} \boldsymbol{\Phi}_{dz,k} \\ \boldsymbol{\Phi}_{dp,k} \end{bmatrix} \boldsymbol{k}_{d,k} = \begin{Bmatrix} \boldsymbol{y}_{z,k} \\ \boldsymbol{y}_{p,k} \end{Bmatrix} \tag{50}$$

The control gain matrices **D** and **F** that solve Problem 4 are obtained by solving Eq. (50) for $k = 1,\ldots,m$.

5. Some issues on the method application

5.1. Discussion on the choice of the desider eigenvectors

In the single-step algorithm the coefficients $\alpha_{i,j}$ and $\beta_{i,j}$ in Eqs. (22) and (41) defines the desired closed-loop eigenvectors of the whole system and of the adjunct system, respectively. Such a choice is non-trivial; on the other hand, it has a relevant impact on the control gains and on the closed-loop system stability.

The literature proposes two solutions in the case of pole placement. In [16] the coefficients $\beta_{i,j}$ are computed to assign m-1 modal ratios foreach assigned pole, i.e. to impose m-1 ratios between the entries of the eigenvector. This method is named in [16] as method of the “modal constraints”, and its extension to zero assignment is not yet discussed in the literature.

A different approach is proposed in [18], where the coefficients $\beta_{i,j}$ are computed to minimize the norm of the control gains matrix in the case of partial pole placement. Such a solution is effective for the solution of the problem of partial pole placement, i.e. some closed-loop poles are assigned while the remaining are retained to the closed-loop configuration. Conversely, in the case of zero assignment (or pole-zero) it is much more attractive to ensure that the closed-loop system is stable, rather than minimizing the controller matrix norm, since assigning pairs of underdamped zeros often leads to instability, or to small stability margins [26].

Two heuristic solutions for a wise choice of $\alpha_{i,j}$ and $\beta_{i,j}$ are proposed in this paper, by following two different approaches.

5.1.1. Method 1: Imposition of the modal ratios

In this Section, by exploiting a formulation similar to the one in [16], the suitable values of $\alpha_{i,j}$ are obtained by imposing the modal ratios for the antiresonant response [24,27]. The modal ratio $\rho_{i,rc}$ between two arbitrary entries r and c of the i-th right-eigenvector $\boldsymbol{w}_i$ $(i = 1, \dots, n_z)$ of the controlled adjunct system $\boldsymbol{w}_i$ is defined as

$$\rho_{i,rc} = \frac{w_{i,r}}{w_{i,c}} \tag{51}$$

The goal is to shape the response of the system coordinates when excited at the frequency of the imposed zeros.

It should be stressed that the achievable eigenvectors are not arbitrary since they are constrained to belong to the allowable subspace. This issue, that has been often discussed in the case of eigenvectors of the whole system [28,29], i.e. those related to the mode shape, has an obvious counterpart in the case of the adjunct system. Let us rearrange Eq. (17) as:

$$\boldsymbol{w}_i = \mathbf{H}_{\overline{\mathbf{qp}}}(\bar{z}_i)\widetilde{\mathbf{B}}\boldsymbol{\varphi}_i, \quad i = 1, \dots, n_z \tag{52}$$

with $\boldsymbol{\varphi}_i = \left(\bar{z}_i\widetilde{\mathbf{F}}^T + \widetilde{\mathbf{G}}^T\right)\boldsymbol{w}_i$. Since $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{G}}$ are arbitrary real matrices, the right-eigenvector of the adjunct system $\boldsymbol{w}_i$ is assignable for a given zero $\bar{z}_i$ if it belongs to the m-dimensional subspace spanned by: $\boldsymbol{\Psi}_z = \mathbf{H}_{\overline{\mathbf{qp}}}(\bar{z}_i)\widetilde{\mathbf{B}} \in \mathbb{C}^{(N-1)\times m}$. The same subspace holds for the state-derivative controller (see Eq. (32)).

To develop the method for imposing the modal ratios, let us set $\alpha_{i,m} = 1$ without loss of generality [16], since eigenvectors are defined regardless to their normalization. Let us denote $\boldsymbol{\psi}_i = [\boldsymbol{\psi}_{i,1} \quad \cdots \quad \boldsymbol{\psi}_{i,m}]$ then $\boldsymbol{w}_i = \sum_{j=1}^{m}\alpha_{i,j}\boldsymbol{\psi}_{i,j}$ (see Eq. (22)) can be expressed as:

$$\boldsymbol{w}_i = \boldsymbol{\psi}_i[\alpha_{i,1} \quad \cdots \quad \alpha_{i,m-1} \quad 1]^T \tag{53}$$

The r-th and c-th entries of $\boldsymbol{w}_i$ are therefore:

$$\begin{aligned} w_{i,r} &= \boldsymbol{e}_r^T\boldsymbol{\psi}_i[\alpha_{i,1} \quad \cdots \quad \alpha_{i,m-1} \quad 1]^T \\ w_{i,c} &= \boldsymbol{e}_c^T\boldsymbol{\psi}_i[\alpha_{i,1} \quad \cdots \quad \alpha_{i,m-1} \quad 1]^T \end{aligned} \tag{54}$$

where $\boldsymbol{e}_r$ and $\boldsymbol{e}_c$ are the r-th and c-th unit vectors.

By exploiting the definition of modal ratio in Eq. (51) , the following relation holds:

$$\left(\boldsymbol{e}_r^T - \rho_{i,rc}\boldsymbol{e}_c^T\right)\boldsymbol{\psi}_i[\alpha_{i,1} \quad \cdots \quad \alpha_{i,m-1} \quad 1]^T = 0 \tag{55}$$

Once the *m-1* modal ratios are chosen together with the n_z prescribed zeros, $\alpha_{i,j}, j = 1, \dots, m-1$, are obtained by solving Eq. (55). The solution of the linear system is trivial since $[\alpha_{i,1} \quad \cdots \quad \alpha_{i,m-1} \quad 1]$ belongs to the null-space of $\left(\boldsymbol{e}_r^T - \rho_{i,rc}\boldsymbol{e}_c^T\right)\boldsymbol{\psi}_i$. The solvability of Eq. (55) is ensured if and only if the desired eigenvector belongs to the allowable subspace. It should be noted that in the single-input scenario ($m = 1$), that is not of interest for this paper since it has already been solved in [8] and [22], there is no chance to assign the desired eigenvector. In contrast, in the case of fully actuated system, and under the obvious assumption of controllable system, any arbitrary eigenvector can be obtained through active control.

5.1.2. Method 2: Optimization-based formulation with free eigenvectors

A different approach is here proposed, to overcome the need of choosing the eigenvectors through the modal ratio method. The idea is to solve the linear systems governing the assignment (i.e. Eqs. (23)–(25), (33)–(45)) without imposing $\alpha_{i,j}$ and $\beta_{i,j}$, and hence by treating them as design variables to accomplish the assignment task while ensuring a secondary task, and hence more specifications on the assignment. As already discussed, reducing the gains is not always effective in the case of zero assignment.

An attractive approach is boosting stability by placing more poles to desired locations, compared to those assigned through the pole-zero assignment performed in Section 4. A reasonable choice is to assign all the $2N$ system poles, together with the n_z desired zeros. The problem can be solved by relaxing the requirements on $\alpha_{i,j}$ and $\beta_{i,j}$, that are instead treated as problem unknowns, together with the $2N$ control gains of the state (or state derivative) feedback, leading to $4N + n_z(N\text{-}1)$ unknowns with $2N + 2n_z$ equations. The resulting problem is nonlinear, and therefore exact solution is not ensured for any arbitrary choice of the poles, meaning that some poles might not be exactly assigned. This is not an issue, since usually just few dominant poles are required to assume precise locations in the complex plane, while the remaining ones are required to lie in larger regions with some tolerance. A minimization-based formulation is therefore suggested, where the different control tasks (i.e. the assignment of the zeros, of the dominant poles and of the residual poles) are differently weighed (through the weight matrix **W**) leading to the following problem:

$$\min_{k_s,\alpha_z,\beta_p} \left\| \mathbf{W}\left(\begin{bmatrix} \mathbf{G}_{sz}(\boldsymbol{\alpha}_z)\mathbf{T} \\ \mathbf{G}_{sp}(\boldsymbol{\beta}_p) \end{bmatrix} \boldsymbol{k}_s - \begin{Bmatrix} \boldsymbol{\alpha}_z \\ \boldsymbol{\beta}_p \end{Bmatrix} \right) \right\|_2^2 \tag{56}$$

It should be noted that a non-convex minimization problem is obtained. Hence, a wise definition of the initial guess should be done to boost convergence towards an optimal solution.

The extension of Eq. (56) for the state-derivative controller is trivial and here omitted for brevity.

5.2. Details on the method implementation

The single-step and multi-step algorithms for the gain computation are summarized in Tables 1 and 2.

As for the solution of the linear systems, let us summarize Eqs. (26), (31), (35), (38), (44), (46), (48), and (50) through the following general form of a linear system, with obvious meaning of the variables:

$$\boldsymbol{\Phi k} = \boldsymbol{y} \tag{57}$$

The solution $\boldsymbol{k}$ is defined as the sum of the particular solution of the non-homogeneous equation, denoted $\boldsymbol{k}_0$, and a solution $\boldsymbol{k}_h$ of the homogeneous system $\boldsymbol{\Phi k}_\mathrm{h} = \mathbf{0}$:

$$\boldsymbol{k} = \boldsymbol{k}_0 + \boldsymbol{k}_h \tag{58}$$

A more convenient way to formulate the solution of the homogeneous system is through matrix **V** whose columns span the null-space of $\boldsymbol{\Phi}$, and through the vector of real coefficients $\boldsymbol{k}_r$:

$$\boldsymbol{k} = \boldsymbol{k}_0 + \mathbf{V}\boldsymbol{k}_r \tag{59}$$

$\boldsymbol{k}_0$ can be computed through the pseudoinverse matrix to minimize the gain vector norm, or by forcing some entries to zero, or exploiting QR-decompositions (such as using the MATLAB command *mldivide*) to obtain the control gains with the fewest possible non-zero components.

If the system matrices are adopted, $\boldsymbol{k}_r$ could be computed through the powerful theory of Linear Matrix Inequalities, as shown in [22,30]. If just receptances are adopted, as this paper aims, the computation of the stabilizing $\boldsymbol{k}_r$ is not straightforward and this issue is

Table 1
The single-step multi-input pole-zero placement by the measured receptances algorithm.

Single-step algorithm

Inputs:
- The measured receptances of the original system **H***(s)* and the actuation matrix **B**.
- The sets of the desired closed-loop poles $\sum_p = \left\{\overline{p}_1, \ldots, \overline{p}_{n_p}\right\}$ and zeros $\sum_z = \{\overline{z}_1, \ldots, \overline{z}_{n_z}\}$ of receptance h_{pq}.
- The coefficients $\alpha_{i,j}$ for the zeros and $\beta_{i,j}$ for the poles.

Procedure:
Zero assignment:
1. Compute matrices $\mathbf{B_n}$, $\mathbf{C_n}$ through Eq. (7).
2. Compute the adjunct system receptances through Eq. (16) for all the prescribed zeros.
3. Compute the adjunct system actuation matrix through Eq. (18).
4. Compute the desired adjunct system eigenvectors through Eqs. (20) and (22).
5. Generate the linear system for the zero assignment: Eqs. (23)–(25), and for state feedback; Eqs. (33)–(35), and for state-derivative feedback.

Pole placement (if required):
6. Compute the desired eigenvectors through Eqs. (41) and (42).
7. Generate the linear system for the pole placement. For state feedback use Eqs. (39) and (43). For state-derivative feedback use Eq. (47).

Solution:
8. In the case of zero assignment solve the linear system obtained at Algorithm step 5 and transform the gains through Eq. (27).

In the case of simultaneous pole-zero placement, merge the linear systems obtained in the Algorithm steps 5 and 7 and solve Eq. (44), in the case of state feedback, Eq. (48) in the case of state-derivative feedback.

Outputs:
- State feedback control: velocity gains **F** and displacement gains **G**.
- State-derivative feedback control: acceleration gains **D** and velocity gains **F**.

Table 2
The multi-step multi-input pole-zero placement by the measured receptances algorithm.

```
Multi-step algorithm

Inputs:
• The measured receptances of the original system H(s) and the actuation matrix B.
• The sets of the desired closed-loop poles $\sum_p = \left\{\overline{p}_1, ..., \overline{p}_{n_p}\right\}$ and zeros $\sum_z = \{\overline{z}_1, ..., \overline{z}_{n_z}\}$ for receptance $h_{pq}$.
• The pole-zero path for the multi-step assignment and the open-loop poles and zeros to shift in the assignment.
Procedure:
for k = 1:m
    1. Compute the poles and the zeros to assign at the k-th step through Eq. (61).
    2. Calculate the closed-loop receptances at the previous step k-1 through Eq. (28), not needed for k = 1.
    Zero assignment:
    3. Generate the linear system for the zero assignment at the k-th step through Eqs. (30) or (37), for the state feedback or state-derivative feedback controller, respectively.
    Pole placement (if required):
    4. Generate the linear system for the pole placement at the k-th step through Eqs. (45) or (49), for the state feedback or state-derivative feedback controller, respectively.
    Solution:
    5. In case of zero assignment solve the linear system obtained at Algorithm step 3 to obtain the gain vectors for the k-th actuator.
    In the case of simultaneous pole-zero placement merge the linear systems obtained in the Algorithm steps 3 and 4 and solve Eq. (46), in the case of state feedback, or Eq. (50) in the case of state-derivative controller, to obtain the gain vectors for the k-th actuator.
end
Outputs:
• State feedback control: velocity gains F and displacement gains G.
• State-derivative feedback control: acceleration gains D and velocity gains F.
```

not investigated in this paper. However, the use of heuristic numerical methods such as genetic algorithms, random searches or method for global optimization is helpful (see e.g. [31]).

6. Method validation

6.1. Test-Case 1

6.1.1. Example 1.1: Zero assignment

Let us consider a benchmark 3-dofs system often employed in the literature on pole assignment (see e.g.[30,32]). The system is described by the following matrices:

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2.5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 10 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \tag{60}$$

It is wanted to assign, by means of example, the four zeros $\sum_z = \{-0.5 + 1i,\ -0.5 - 1i,\ -0.1 + 1.5i,\ -0.1 - 1.5i\}$ of point-receptance receptance h_{11}. Both the zero assignment methods have been applied in the state feedback and state-derivative feedback configurations leading to the closed-loop zeros and poles listed in Table 3.

The control gains are listed in Table 4. It is worth to notice that the first row of the control gain matrices, i.e. the one related to the output coordinate, is always equal to zero. Indeed, the contribution of the feedback of the p-th coordinate vanishes in the numerator of the receptance h_{pq} of the controlled system, since the p-th column is deleted by definition of the adjunct system.

In the single-step method with state feedback, it is chosen to define the desired closed-loop eigenvectors of the adjunct system introduced in Eq. (22) for the first complex conjugate pair of zeros through the coefficient vector $\boldsymbol{\alpha}_{z,1} = \boldsymbol{\alpha}_{z,2} = \{1, -0.3\}^T$ while for the

Table 3
Example 1.1: Open-loop and closed-loop poles and zeros of h_{11}.

	State feedback		State-derivative feedback	
Open-loop zeros	Closed-loop zeros single-step	Closed-loop zeros multi-step	Closed-loop zeros single-step	Closed-loop zeros multi-step
−0.8137 ± 0.9061i	−0.5000 ± 1.0000i	−0.5000 ± 1.0000i	−0.5000 ± 1.0000i	−0.5000 ± 1.0000i
−0.1029 ± 1.1567i	−0.1000 ± 1.5000i	−0.1000 ± 1.5000i	−0.1000 ± 1.5000i	−0.1000 ± 1.5000i
Open-loop poles	Closed-loop poles single-step	Closed-loop poles multi-step	Closed-loop poles single-step	Closed-loop poles multi-step
−0.0305 ± 0.5894i	−0.2869	−0.5025 ± 0.9263i	−0.1481	−0.5894 ± 0.8474i
−0.8503 ± 1.0119i	−0.1821 ± 1.1198i	−0.0293 ± 1.4705i	−0.2353 ± 1.0699i	0.8186 ± 0.7617i
−0.6609 ± 2.1200i	−1.4226	−0.6932 ± 2.3421i	−1.6967	−0.2867 ± 1.7603i
	−0.1881 ± 3.1176i		−0.1170 ± 3.7473i	

Table 4
Example 1.1: control gains.

State feedback		State-derivative feedback	
Single-step	Multi-step	Single-step	Multi-step
$\mathbf{F} = \begin{bmatrix} 0 & 0 \\ 1.9883 & -4.6372 \\ 0.8668 & -4.9316 \end{bmatrix}$	$\mathbf{F} = \begin{bmatrix} 0 & 0 \\ 0.3735 & -1.7498 \\ -0.1705 & 0.9500 \end{bmatrix}$	$\mathbf{F} = \begin{bmatrix} 0 & 0 \\ 2.5479 & -5.6224 \\ 2.6320 & -7.9804 \end{bmatrix}$	$\mathbf{F} = \begin{bmatrix} 0 & 0 \\ 0.2476 & -0.0246 \\ -0.0399 & 0.8257 \end{bmatrix}$
$\Vert\mathbf{F}\Vert_F = 7.1084$	$\Vert\mathbf{F}\Vert_F = 2.0329$	$\Vert\mathbf{F}\Vert_F = 10.4268$	$\Vert\mathbf{F}\Vert_F = 0.8633$
$\mathbf{G} = \begin{bmatrix} 0 & 0 \\ -1.0538 & 0.6760 \\ -2.4985 & 2.8103 \end{bmatrix}$	$\mathbf{G} = \begin{bmatrix} 0 & 0 \\ -0.0658 & -1.6827 \\ -0.5800 & -1.4802 \end{bmatrix}$	$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0.3472 & -0.1690 \\ 1.3267 & -1.9426 \end{bmatrix}$	$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0.0369 & 1.7423 \\ 0.3659 & -0.2471 \end{bmatrix}$
$\Vert\mathbf{G}\Vert_F = 3.9633$	$\Vert\mathbf{G}\Vert_F = 2.3159$	$\Vert\mathbf{D}\Vert_F = 2.3839$	$\Vert\mathbf{D}\Vert_F = 1.7977$

second pair through $\boldsymbol{\alpha}_{z,3} = \boldsymbol{\alpha}_{z,4} = \{0.5, -1\}^T$. The same values are adopted for tuning the state derivative controller. The linear systems defining the assignment problems have been solved through the *pinv* Matlab function, by setting$\mathbf{k}_r = \mathbf{0}$.

The multi-step method requires the definition of the path of the assigned zeros (and poles), i.e. the temporary locations at the k-th step: $\sum_{z,k} = \{\overline{z}_{1,k}, \dots, \overline{z}_{n_z,k}\}$ for the zeros and $\sum_{p,k} = \left\{\overline{p}_{1,k}, \dots, \overline{p}_{n_p,k}\right\}$ for the poles.

Let us introduce vector $\chi = \{\chi_1, \cdots, \chi_m\}^T$, with $\sum_{k=1}^{m} \chi_k = 1$, $1 > \chi_k \geqslant \chi_{k-1} > 0$, to define the spacings between poles and between zeros in the assignment path:

$$\begin{array}{ll} \overline{z}_{i,k} = z_i + \chi_k(\overline{z}_i - z_i) & \text{for } i = 1, \dots, n_z \\ \overline{p}_{i,k} = p_i + \chi_k(\overline{p}_i - p_i) & \text{for } i = 1, \dots, n_p \end{array} \tag{61}$$

In the application of the multi-step method for both the state and the state-derivative controllers it is chosen: $\chi = \{0.5, 0.5\}^T$. The path can be, however, chosen in different ways by evaluating a-posteriori the most convenient one in accordance with some parameters, such as control gains, closed-loop system robustness or stability.

The open-loop and closed loop poles and zeros of h_{11} are shown in Fig. 2 for the four controllers. The effectiveness of the proposed zero placement method is confirmed by the correct assignment of all the closed-loop zeros.

Once the gains are computed, the closed-loop poles should be checked because no stability conditions are set in solving the zero-

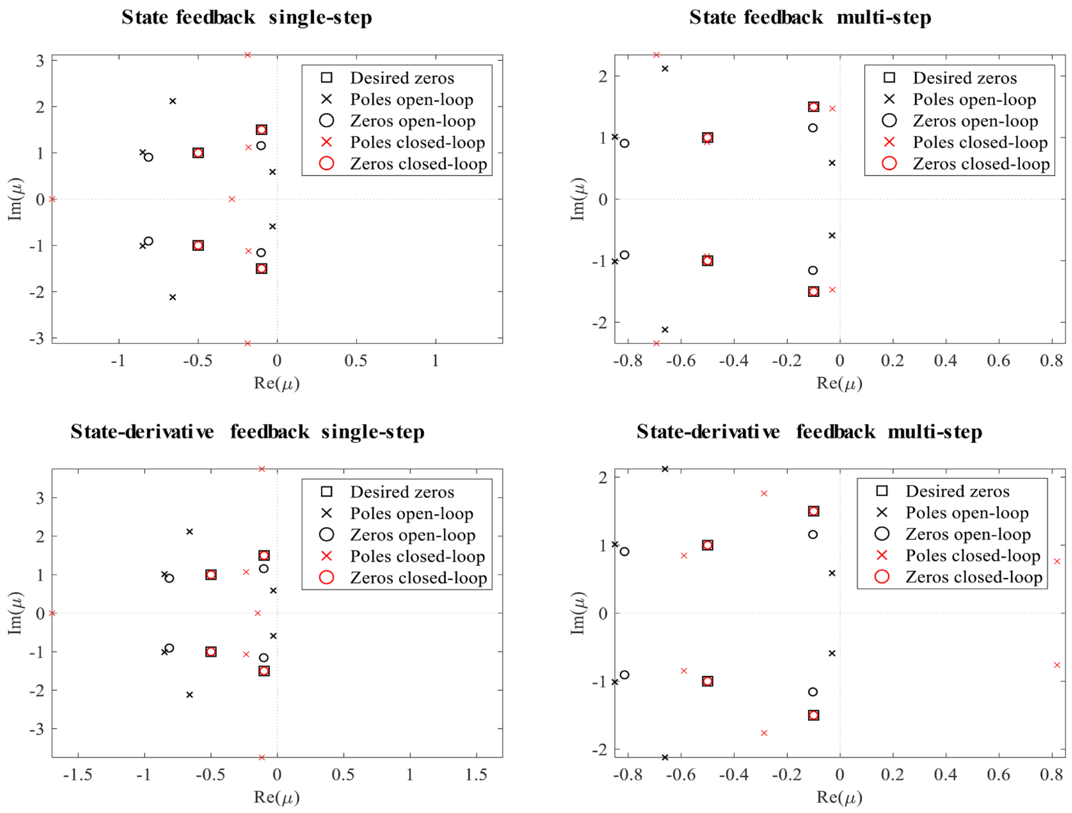

Fig. 2. Example 1.1: Poles and zeros of h_{11}.

assignment problem. For example, the designed state-derivative feedback controller synthetized through the multi-step method with $\chi = \{0.5, 0.5\}^T$ leads to unstable poles. The confinement of the closed-loop poles on the left-hand of the complex plane may be boosted by the simultaneous assignment of the dominant poles together with the prescribed zeros, as shown in Section 6.1.3.

6.1.2. Example 1.2: Zero assignment with imposition of the antiresonant response through the modal ratios

The same zeros of Section 6.1.1 are here assigned, by choosing the antiresonant response through imposition of the modal ratios, as discussed in Section 5.1.1. Just by way of example, to assess that the proposed method correctly assigns the eigenvector (if they belong to the allowable subspace) the following values have been chosen: $\rho_{1,23} = 0.75$ (and obviously $\rho_{2,23} = 0.75$), $\rho_{3,23} = -1$ (and $\rho_{4,23} = -1$).

The assignment task is satisfied through both the state-feedback controller and the state-derivative feedback controller. The closed-loop poles and zeros of receptance h_{11} are listed in Table 5, while the closed-loop right-eigenvectors of the adjunct system are reported in Table 6: all the requirements are exactly satisfied. The closed-loop poles analysis reveals that the closed-loop system is stable with both the controllers.

6.1.3. Example 1.3: Pole-zero assignment

In this example, the pole pair $\sum_p = \{-0.5 + 0.5i, \ -0.5 - 0.5i\}$ is assigned together with the zeros of the previous examples (Sections 6.1.1 and 6.1.2). No specifications on the eigenvectors are set. In tuning the controller through the single-step methods, vectors $\boldsymbol{\alpha}_{z,i}$ are the same adopted Section 6.1.1. As for the prescribed closed-loop pole pair, the eigenvectors are defined by imposing: $\boldsymbol{\beta}_{p,1} = \boldsymbol{\beta}_{p,2} = \{0.3, -0.8\}^T$ in the state feedback, while $\boldsymbol{\beta}_{p,1} = \boldsymbol{\beta}_{p,2} = \{1, -0.9\}^T$ in case of state-derivative. In the application of the multi-step method the path of the m assignments is the same adopted in Section 6.1.1, i.e. $\chi = \{0.5, 0.5\}^T$.

The open-loop and closed-loop poles and zeros are shown in Fig. 3 and listed in Table 7 and corroborate the correctness of the proposed method. The control gains are summarized in Table 8. It is worth to notice that just the first rows of the control gain matrices have changed, compared to the ones computed for just zero assignment, as listed in Table 4. Indeed, due to the features of the adjunct system, such rows are null in the case of assigning zeros for h_{11}.

6.2. Test-Case 2: Friction-Induced vibrations

6.2.1. Model of the system

Let us consider the model of an asymmetric mass-spring-damper system on a conveyor belt, proposed by Ouyang in [33], and often employed in the literature as a benchmark [34,35]. The system is sketched in Fig. 4.

The system is composed by 4 masses: m_1 moves in the along the horizontal axis x, m_4 dof is in the y direction while m_2 and m_4 moves along both directions. The displacement vector is: $\boldsymbol{x} = \{x_1, y_4, x_2, x_3, y_2, y_3\}^T$. The system matrices are:

$$
\begin{gathered}
\mathbf{M} = \mathrm{diag}(m_1,\ m_4,\ m_2,\ m_3,\ m_2,\ m_3) \\
\mathbf{C} = \begin{bmatrix} c_1 & 0 & -c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & 0 & c_1 + c_2 & -c_2 & 0 & 0 \\ 0 & 0 & -c_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\mathbf{K}_s = \begin{bmatrix} k_{11} & 0 & k_{13} & 0 & 0 & 0 \\ 0 & k_{22} & 0 & 0 & k_{25} & 0 \\ k_{31} & 0 & k_{33} & k_{34} & k_{35} & 0 \\ 0 & 0 & k_{43} & k_{44} & 0 & k_{46} \\ 0 & k_{52} & k_{53} & 0 & k_{55} & 0 \\ 0 & 0 & 0 & k_{64} & 0 & k_{66} \end{bmatrix} \quad \mathbf{K}_{as} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_c k_c & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_c k_c \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{gathered} \tag{62}
$$

where:

$$
\begin{gathered}
k_{11} = k_1 + k_2, \quad k_{13} = k_{31} = -k_2, \quad k_{22} = k_6 + k_8, \quad k_{25} = k_{52} = -k_6, \quad k_{33} = k_2 + k_3 + 0.5(k_5 + k_7), \\
k_{34} = k_{43} = -k_3, \quad k_{35} = k_{53} = 0.5(k_7 - k_5), \quad k_{44} = k_3 + k_4 + 0.5k_{10}, \quad k_{46} = k_{64} = -0.5k_{10} \\
k_{55} = k_c + k_6 + 0.5(k_5 + k_7), \quad k_{66} = k_c + k_9 + 0.5k_{10},
\end{gathered} \tag{63}
$$

The stiffness matrix is obtained by $\mathbf{K} = \mathbf{K}_s + \mathbf{K}_{as}$, where $\mathbf{K}_s$ and $\mathbf{K}_{as}$ are respectively the symmetric and asymmetric part of the overall stiffness matrix.

The values of the system parameters are listed in Table 9.

Table 5
Example 1.2: Open-loop and closed-loop poles and zeros of h_{11} with prescribed modal ratios.

Open-loop zeros	Closed-loop zeros state feedback	Closed-loop zeros state-derivative feedback
$-0.8137 \pm 0.9061i$	$-0.5000 \pm 1.0000i$	$-0.5000 \pm 1.0000i$
$-0.1029 \pm 1.1567i$	$-0.1000 \pm 1.5000i$	$-0.1000 \pm 1.5000i$
Open-loop poles	Closed-loop poles state feedback	Closed-loop poles state-derivative feedback
$-0.0305 \pm 0.5894i$	-0.0834	-0.3345
$-0.8503 \pm 1.0119i$	-0.8599	-0.7607
$-0.6609 \pm 2.1200i$	$-0.0946 \pm 1.3924i$	$-0.0948 \pm 1.3604i$
	$-0.6588 \pm 2.8171i$	$-1.2143 \pm 2.7353i$

Table 6
Example 1.2: closed-loop right-eigenvectors of the adjunct system related to h_{11}.

State feedback closed-loop eigenvectors		State-derivative feedback closed-loop eigenvectors	
w_1, w_2	w_3, w_4	w_1, w_2	w_3, w_4
$0.4898 \pm 0.3465i$	$0.0631 \mp 0.7043i$	$0.4922 \pm 0.3431i$	$0.6908 \pm 0.1510i$
$0.6531 \pm 0.4620i$	$-0.0631 \pm 0.7043i$	$0.6563 \pm 0.4575i$	$-0.6908 \mp 0.1510i$

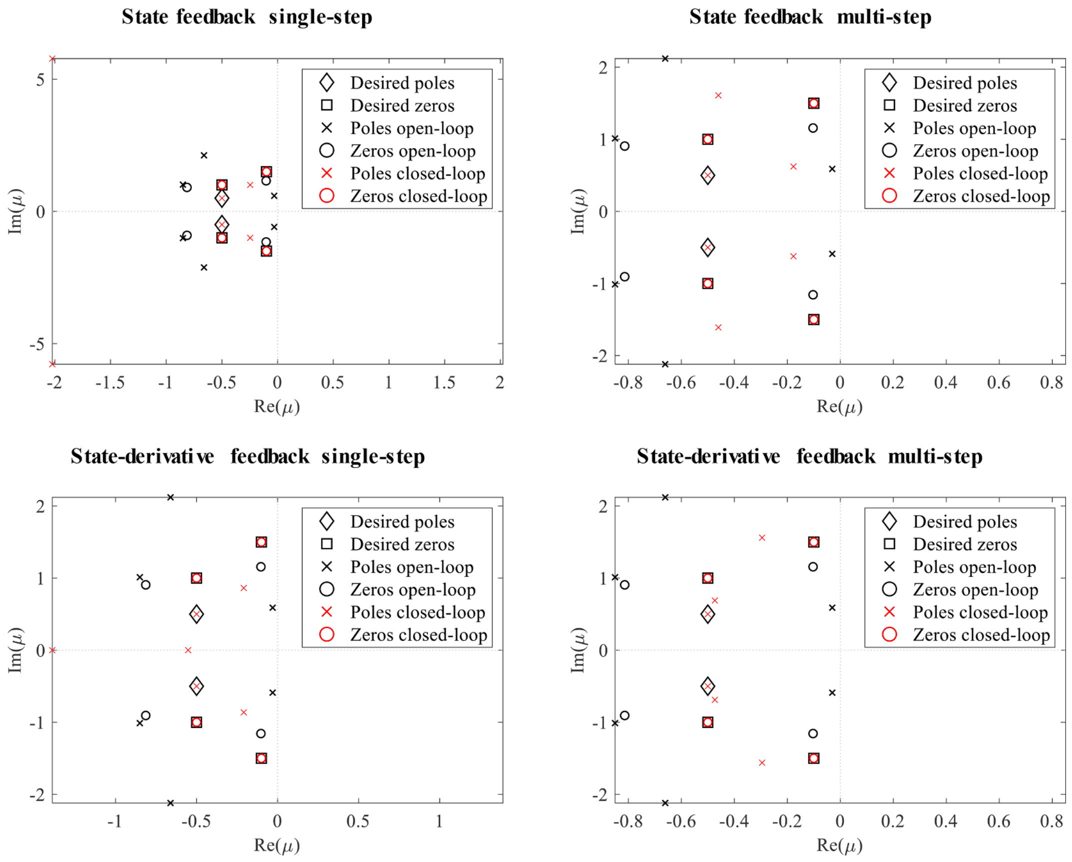

Fig. 3. Example 1.3: Poles and zeros of h_{11}.

The slider-belt interface is governed by the Coulomb friction and the belt moves at constant speed. Asymmetry arises due to the stiffness matrix and might cause instability when the stiffness and the friction coefficient reach the critical values k_c and μ_c. In the case under investigation, such values have been assumed beyond the critical threshold thus leading to an unstable open-loop system, as shown by the open-loop poles summarized in Table 10.

6.2.2. Pole-zero assignment through the methods with imposed α and β

The control specifications, chosen by way of example, are the following ones:

Table 7
Example 1.3: Open-loop and closed-loop poles and zeros of h_{11}.

	State feedback		State-derivative feedback	
Open-loop zeros	Closed-loop zeros single-step	Closed-loop zeros multi-step	Closed-loop zeros single-step	Closed-loop zeros multi-step
$-0.8137 \pm 0.9061i$	$-0.5000 \pm 1.0000i$	$-0.5000 \pm 1.0000i$	$-0.5000 \pm 1.0000i$	$-0.5000 \pm 1.0000i$
$-0.1029 \pm 1.1567i$	$-0.1000 \pm 1.5000i$	$-0.1000 \pm 1.5000i$	$-0.1000 \pm 1.5000i$	$-0.1000 \pm 1.5000i$
Open-loop poles	Closed-loop poles single-step	Closed-loop poles multi-step	Closed-loop poles single-step	Closed-loop poles multi-step
$-0.0305 \pm 0.5894i$	$-0.5000 \pm 0.5000i$	$-0.1771 \pm 0.6226i$	-0.5518	$-0.5000 \pm 0.5000i$
$-0.8503 \pm 1.0119i$	$-0.2461 \pm 1.0006i$	$-0.5000 \pm 0.5000i$	$-0.5000 \pm 0.5000i$	$-0.4734 \pm 0.6887i$
$-0.6609 \pm 2.1200i$	$-2.0229 \pm 5.7795i$	$-0.4602 \pm 1.6093i$	$-0.2088 \pm 0.8627i$	$-0.2952 \pm 1.5603i$
			-1.3903	

Table 8
Example 1.3: control gains.

State feedback		State-derivative feedback	
Single-step	Multi-step	Single-step	Multi-step
$\mathbf{F} = \begin{bmatrix} 7.5001 & -6.8380 \\ 1.9883 & -4.6372 \\ 0.8668 & -4.9316 \end{bmatrix}$	$\mathbf{F} = \begin{bmatrix} -1.1063 & 0.7286 \\ 0.3735 & -1.7498 \\ -0.1705 & 0.9500 \end{bmatrix}$	$\mathbf{F} = \begin{bmatrix} -1.4910 & 4.6780 \\ 2.5479 & -5.6224 \\ 2.6320 & -7.9804 \end{bmatrix}$	$\mathbf{F} = \begin{bmatrix} -1.1093 & -2.1337 \\ 0.2476 & -0.0246 \\ -0.0399 & 0.8257 \end{bmatrix}$
$\Vert\mathbf{F}\Vert_F = 12.3911$	$\Vert\mathbf{F}\Vert_F = 2.4265$	$\Vert\mathbf{F}\Vert_F = 11.5429$	$\Vert\mathbf{F}\Vert_F = 2.5551$
$\mathbf{G} = \begin{bmatrix} 9.1480 & -12.7851 \\ -1.0538 & 0.6760 \\ -2.4985 & 2.8103 \end{bmatrix}$	$\mathbf{G} = \begin{bmatrix} 0.7625 & 4.3460 \\ -0.0658 & -1.6827 \\ -0.5800 & -1.4802 \end{bmatrix}$	$\mathbf{D} = \begin{bmatrix} 12.2844 & -21.5628 \\ 0.3472 & -0.1690 \\ 1.3267 & -1.9426 \end{bmatrix}$	$\mathbf{D} = \begin{bmatrix} -0.2674 & -2.3393 \\ 0.0369 & 1.7423 \\ 0.3659 & -0.2471 \end{bmatrix}$
$\Vert\mathbf{G}\Vert_F = 16.2128$	$\Vert\mathbf{G}\Vert_F = 4.9832$	$\Vert\mathbf{D}\Vert_F = 24.9308$	$\Vert\mathbf{D}\Vert_F = 2.9624$

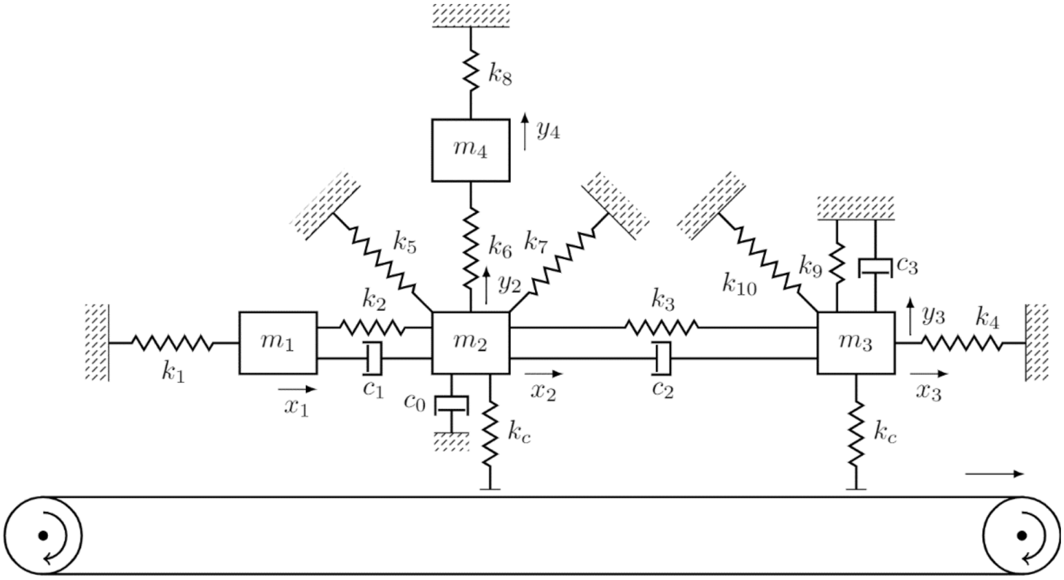

Fig. 4. Example 2: sketch of the asymmetric system for friction-induced vibrations.

- Assign the closed-loop zeros $\sum_z = \{\pm 11.5i,\ \pm 13.5i\}$ for cross-receptance h_{63};
- Assign the closed-loop poles $\sum_p = \{-1 \pm 10.5i,\ -1 \pm 12.5i,\ -1 \pm 15.0i,\ -1 \pm 17.5i\}$.

Four controllers are synthetized to cope with this task, by means of the single and the multi-step methods with state and state-derivative feedback.

In the case of the single-step method, the following coefficients have been assumed: $\boldsymbol{\alpha}_{z,1} = \boldsymbol{\alpha}_{z,2} = \{1, -0.3\}^T$, $\boldsymbol{\beta}_{p,1} = \boldsymbol{\beta}_{p,2} = \{1, -0.9\}^T$, $\boldsymbol{\beta}_{p,3} = \boldsymbol{\beta}_{p,4} = \{0.4, 0.5\}^T$, $\boldsymbol{\beta}_{p,5} = \boldsymbol{\beta}_{p,6} = \{0.6, -0.1\}^T$, $\boldsymbol{\beta}_{p,7} = \boldsymbol{\beta}_{p,8} = \{0.2, -0.3\}^T$, for state feedback; $\boldsymbol{\alpha}_{z,1} = \boldsymbol{\alpha}_{z,2} = \{0.5, -1\}^T$, $\boldsymbol{\beta}_{p,1} = \boldsymbol{\beta}_{p,2} = \{1, -0.9\}^T$, $\boldsymbol{\beta}_{p,3} = \boldsymbol{\beta}_{p,4} = \{0.4, 0.5\}^T$, $\boldsymbol{\beta}_{p,5} = \boldsymbol{\beta}_{p,6} = \{0.5, -0.1\}^T$, $\boldsymbol{\beta}_{p,7} = \boldsymbol{\beta}_{p,8} = \{0.2, -0.3\}^T$, for state-derivative feedback.

In the multi-step method for the state feedback controller, it has been chosen: $\chi = \{0.5, 0.5\}^T$ while for the state-derivative

Table 9
Example 2: system parameters.

Parameter	Magnitude
m_i, $i = 1, 2, 3, 4$, [kg]	1
c_i, $i = 0, 1, 2, 3$, $[\mathrm{Nsm}^{-1}]$	0.5
k_i, $i = 1, 2, 3, 4, 5, 6, 8, 9, 10$, $[\mathrm{Nm}^{-1}]$	100
k_7, $[\mathrm{Nm}^{-1}]$	50
k_c, $[\mathrm{Nm}^{-1}]$	110
μ_c, [−]	0.5

Table 10
Example 2: Open-loop and closed-loop poles and zeros of receptance h_{63}.

	State feedback		State-derivative feedback	
Open-loop zeros	Closed-loop zeros single-step	Closed-loop zeros multi-step	Closed-loop zeros single-step	Closed-loop zeros multi-step
−0.0761 ± 11.57i	± 11.5i	−0.885 ± 9.454i	−2.3467	± 11.5i
−0.25 ± 14.14i	11.907 ± 6.056i	± 11.5i	± 11.5i	−5.974 ± 10.529i
−0.1739 ± 18.736i	± 13.5i	± 13.5i	± 13.5i	4.949 ± 11.458i
−200	−1.232 ± 13.921i	0.272 ± 15.254i	−1.402 ± 13.906i	± 13.5i
	67.093	−142.05	−17.386	0.3413 ± 15.914i
Open-loop poles	Closed-loop poles single-step	Closed-loop poles multi-step	Closed-loop poles single-step	Closed-loop poles multi-step
0.0069 ± 10.384i	−1 ± 10.5i	−1 ± 10.5i	−1 ± 10.5i	−1 ± 10.5i
−0.0903 ± 11.45i	−0.7211 ± 11.062i	−0.0593 ± 11.353i	−0.9786 ± 11.256i	−0.12 ± 11.346i
−0.2517 ± 15.208i	−1 ± 12.5i	−1 ± 12.5i	−1 ± 12.5i	−1 ± 12.5i
−0.2465 ± 15.979i	−1 ± 15i	−1 ± 15i	−1 ± 15i	−1 ± 15i
−0.0838 ± 18.865i	−1.2081 ± 16.315i	−1 ± 17.5i	−1.2966 ± 15.363i	−1 ± 17.5i
−0.8346 ± 19.696i	−1 ± 17.5i	−0.1182 ± 18.305i	−1 ± 17.5i	−0.9808 ± 18.85i

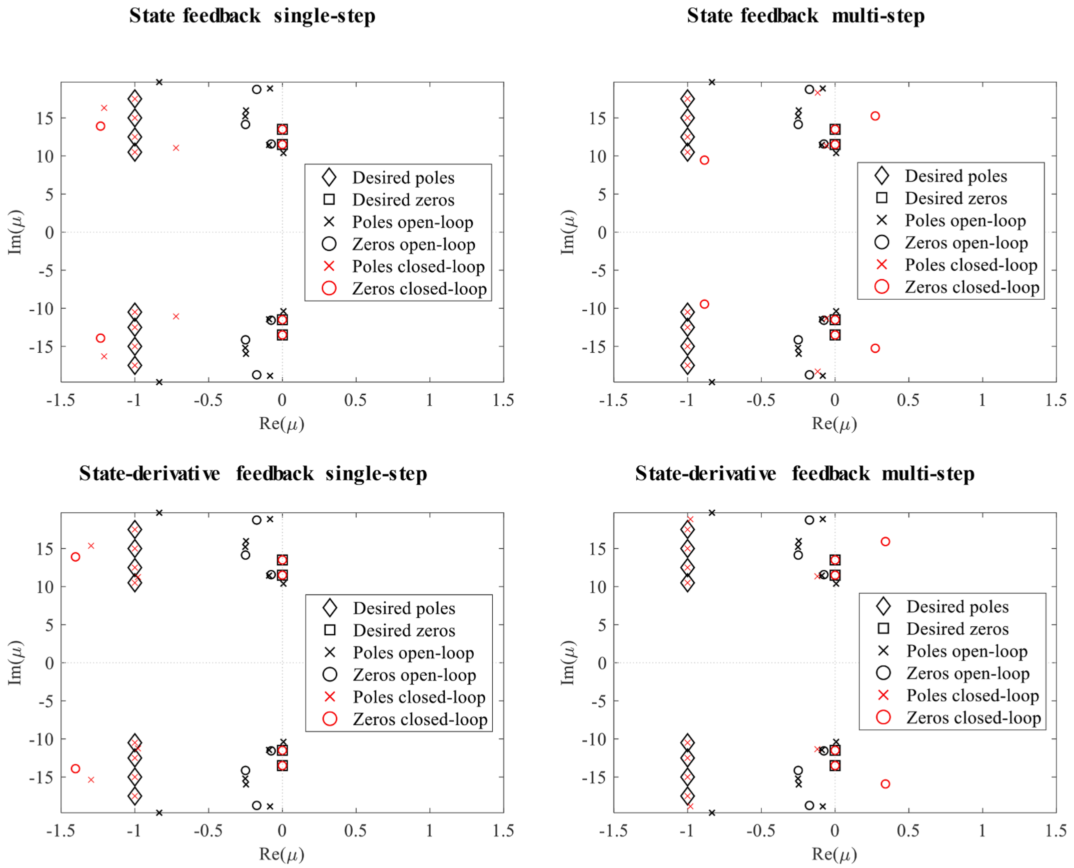

Fig. 5. Example 2: pole-zero maps of h_{63} (magnified views).

Table 11
Example 2: control gain norms.

State feedback		State-derivative feedback	
Single-step	Multi-step	Single-step	Multi-step
$\|\mathbf{F}\|_F = 25.819$	$\|\mathbf{F}\|_F = 11.358$	$\|\mathbf{F}\|_F = 55.085$	$\|\mathbf{F}\|_F = 33.318$
$\|\mathbf{G}\|_F = 421.362$	$\|\mathbf{G}\|_F = 259.48$	$\|\mathbf{D}\|_F = 5.077$	$\|\mathbf{D}\|_F = 1.687$

controller $\chi = \{0.7, 0.3\}^T$ is adopted, to ensure stable controller.

The fulfillment of the assignment task is corroborated by the closed-loop poles and zeros summarized in Table 10, and by the pole-zero map of the open-loop, h_{63}, and closed-loop, $\overline{h}_{63}$, receptance shown in Fig. 5 for all the controllers: exact placement of the desired poles and zeros and stabilization of the system is achieved with the proposed controllers. The Frobenius norms of the control gain matrices are listed in Table 11.

6.2.3. Zero assignment with the optimization-based formulation

In this Section, no imposition of the values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is done, and stabilization is obtained by relying on the optimization-based formulation proposed in Section 5.1.2, Matrix **W** in Eq. (56) is set as a diagonal matrix where the weights are 10^3 for the four zeros and for the eight poles specified in the same assignment problem of Section 6.2.2, while for the remaining four poles the weights are set to 1. This choice is aimed at obtaining precise pole-zero assignment, while it approximates the placement of the remaining poles, just to ensure stabilization. Different choices of **W** would lead to different results; however, the optimal tuning of such a matrix is not the goal of this paper.

The closed-loop zeros and poles obtained through the solution of Eq. (56) are summarized in Table 12, with both state feedback and state-derivative feedback. The closed-loop system is stable, the prescribed zeros are correctly assigned, as well as the most important poles. The remaining poles are approximately assigned to the prescribed locations and stability is ensured. The closed-loop system poles and the zeros with the two controllers are shown in Fig. 6. The control gain norms are listed in Table 13.

7. Conclusions

This paper proposes two approaches to zero assignment and pole-zero placement by the measured receptances in multi-input systems. The proposed methods adopt the measured receptances, hence the system matrices are not necessary, thus simplifying the implementation on experimental systems.

In the first approach, a mathematical framework for the zero assignment through a single-step solving method is developed based on the solution of the closed-loop adjunct system eigenproblem. In the second approach, the zero assignment is tackled through a multi-step method where the multi-input control problem is solved as a sequence of single input assignment problems. Both the methods are formulated for the state feedback control law and for the state-derivative case, providing a general framework for the synthesis of the controllers. Control laws simultaneously exploiting acceleration, velocity and displacement feedbacks can be simply inferred from the proposed formulations.

The methods are extended to address the challenging problem of the simultaneous pole-zero placement which is beneficial in the suppression of the vibration arising due to harmonic excitations, together with stabilizing the closed-loop system and shaping its transient response.

Since the multi-input assignment has more solutions leading to the desired zeros (and poles as well, when required) two approaches to solve this redundancy are proposed for the case of single-step assignment. The first method assigns the antiresonant response by

Table 12
Receptance h_{63}: Open-loop and closed-loop zeros and system poles.

	State feedback	State-derivative feedback
Open-loop zeros	Closed-loop zeros Eq. (56)	Closed-loop zeros Eq. (56)
−0.0761 ± 11.57i	−0.460	−8.237 ± 3.828i
−0.25 ± 14.14i	± 11.5i	± 11.5i
−0.1739 ± 18.736i	± 13.5i	± 13.5i
−200	−1.510 ± 14.433i	−0.176 ± 13.815i
	19.455	6.651 ± 29.451i
	−145.220	
Open-loop poles	Closed-loop poles Eq. (56)	Closed-loop poles Eq. (56)
0.0069 ± 10.384i	−1 ± 10.5i	−1 ± 10.5i
−0.0903 ± 11.45i	−1.129 ± 11.318i	−0.437 ± 11.229i
−0.2517 ± 15.208i	−1 ± 12.5i	−1 ± 12.5i
−0.2465 ± 15.979i	−1 ± 15i	−1 ± 15i
−0.0838 ± 18.865i	−0.766 ± 16.332i	−1 ± 17.5i
−0.8346 ± 19.696i	−1 ± 17.5i	−0.384 ± 20.754i

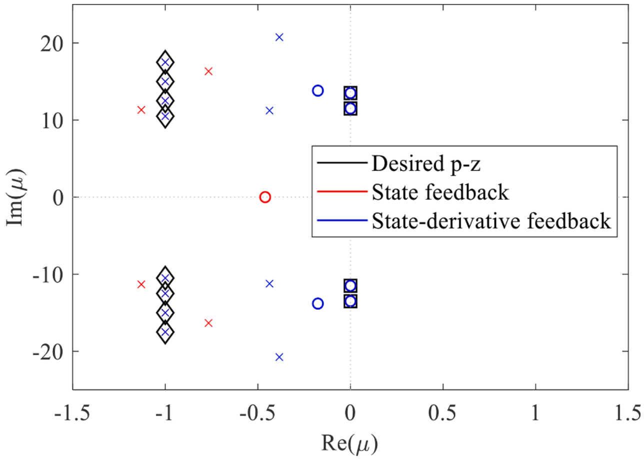

Fig. 6. Pole-zero map of receptance h_{63}: stabilizing state and state-derivative feedback controllers, magnified view.

Table 13
Control gain norms.

State feedback Eq. (56)	State-derivative feedback Eq. (56)
$\|\mathbf{F}\|_F = 27.295$	$\|\mathbf{F}\|_F = 20.767$
$\|\mathbf{G}\|_F = 464.14$	$\|\mathbf{D}\|_F = 1.2963$

imposing the modal ratios. The second method solves an optimization problem to obtain the approximate placement of all the system poles, to ensure stability while exactly placing the desired zeros.

The effectiveness of the proposed methods is validated through two numerical examples taken from the literature and commonly used as a benchmark in this field of research: a three-mass system with two actuation forces and an asymmetric system adopted for the design of controllers tackling the problem of instability due to friction-induced vibrations.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proof of equation

Let us consider the dynamic of the closed-loop system described in Eq. (12), it can be written as:

$$\left(s^2\mathbf{M}+s\mathbf{C}+\mathbf{K}-\mathbf{B}\left(s\mathbf{F}^T+\mathbf{G}^T\right)\right)\boldsymbol{x}(s)=\boldsymbol{0} \tag{64}$$

Applying the Cramer’s rule to Eq. (64) yields the pq-th closed-loop receptance $\overline{h}_{pq}(s)$:

$$\overline{h}_{pq}(s)=\frac{x_p(s)}{f_q(s)}=(-1)^{p+q}\frac{det\left(s^2\mathbf{M}_{\overline{\mathbf{qp}}}+s\mathbf{C}_{\overline{\mathbf{qp}}}+\mathbf{K}_{\overline{\mathbf{qp}}}-\left(\mathbf{B}\left(s\mathbf{F}^T+\mathbf{G}^T\right)\right)_{\overline{\mathbf{qp}}}\right)}{det\left(s^2\mathbf{M}+s\mathbf{C}+\mathbf{K}-\mathbf{B}\left(s\mathbf{F}^T+\mathbf{G}^T\right)\right)} \tag{65}$$

The zeros $\overline{z}_i, \quad i=1,\dots,n_z$ of the pq-th closed-loop adjunct system are those complex numbers for which the numerator in Eq. (65) vanishes. Hence, the counterpart of the pq-th open-loop adjunct system eigenproblem in Eq. (9) for the closed-loop adjunct system is:

$$\left(\overline{z}_i^2\mathbf{M}_{\overline{\mathbf{qp}}}+\overline{z}_i\mathbf{C}_{\overline{\mathbf{qp}}}+\mathbf{K}_{\overline{\mathbf{qp}}}-\left(\mathbf{B}\left(\overline{z}_i\mathbf{F}^T+\mathbf{G}^T\right)\right)_{\overline{\mathbf{qp}}}\right)\boldsymbol{w}_i=\boldsymbol{0}, \quad i=1,\dots,n_z \tag{66}$$

where $(\overline{z}_i, \boldsymbol{w}_i)$ is the i-th right eigenpair of the closed-loop adjunct system.

Let us exploit the formalism in Eq. (6), obviously the same formulation can be adopted for the controller hence $\left(\mathbf{B}\left(s\mathbf{F}^T+\mathbf{G}^T\right)\right)_{\overline{\mathbf{qp}}} = \mathbf{B}_{\mathbf{n}}^T\mathbf{B}\left(s\mathbf{F}^T+\mathbf{G}^T\right)\mathbf{C}_{\mathbf{n}}$. Eq. (66) is recast into:

$$\left(\overline{z}_i^2\mathbf{B}_\mathbf{n}^T\mathbf{M}\mathbf{C}_\mathbf{n} + \overline{z}_i\mathbf{B}_\mathbf{n}^T\mathbf{C}\mathbf{C}_\mathbf{n} + \mathbf{B}_\mathbf{n}^T\mathbf{K}\mathbf{C}_\mathbf{n} - \mathbf{B}_\mathbf{n}^T\mathbf{B}\left(\overline{z}_i\mathbf{F}^T + \mathbf{G}^T\right)\mathbf{C}_\mathbf{n}\right)\boldsymbol{w}_i = \boldsymbol{0}, \quad i = 1, \ldots, n_z \tag{67}$$

Collecting the common terms $\mathbf{B}_\mathbf{n}^T$ and $\mathbf{C}_\mathbf{n}$ the more compact form in Eq. (13) is obtained.

Appendix B. Proof of equations (30) and (45)

Let us consider the k-th stage of the multi-step method. The closed-loop receptance matrix at the k-th stage is $\overline{\mathbf{H}}_k(s) = \left(s^2\mathbf{M} + s\left(\mathbf{C}_{k-1} - \boldsymbol{b}_k\boldsymbol{f}_k^T\right) + \left(\mathbf{K}_{k-1} - \boldsymbol{b}_k\boldsymbol{g}_k^T\right)\right)^{-1}$. This equation is similar to the case of single-input control proposed in [8], with the difference that the stiffness and damping matrices ($\mathbf{K}_{k-1}$ and $\mathbf{C}_{k-1}$) are here modified through the displacement and velocity gains computed in the previous k-1 stages as described in Section 3.1.2.

The control task to be fulfilled at the k-th stage, is to compute the control gains, $\boldsymbol{f}_k$ and $\boldsymbol{g}_k$ (in the state-feedback scenario) that assigns the prescribed zeros or poles.

Let us first consider the scenario of zero assignment. The numerator in Eq. (29) vanishes once it is evaluated for the prescribed zeros $\overline{z}_{i,k}$, $i = 1, \ldots, n_z$ of $\overline{h}_{pq,k}$, i.e.:

$$\boldsymbol{e}_p^T\left[\left(1 - \left(\overline{z}_{i,k}\boldsymbol{f}_k + \boldsymbol{g}_k\right)^T\overline{\mathbf{H}}_{k-1}\left(\overline{z}_{i,k}\right)\boldsymbol{b}_k\right)\overline{\mathbf{H}}_{k-1}\left(\overline{z}_{i,k}\right) + \overline{\mathbf{H}}_{k-1}\left(\overline{z}_{i,k}\right)\boldsymbol{b}_k\left(\overline{z}_{i,k}\boldsymbol{f}_k + \boldsymbol{g}_k\right)^T\overline{\mathbf{H}}_{k-1}\left(\overline{z}_{i,k}\right)\right]\boldsymbol{e}_q = 0, \quad i = 1, \ldots, n_z \tag{68}$$

The substitution of $\boldsymbol{t}_{i,k} = \overline{h}_{pq,k-1}\left(\overline{z}_{i,k}\right)\overline{\mathbf{H}}_{k-1}\left(\overline{z}_{i,k}\right)\boldsymbol{b}_k - \left[\boldsymbol{e}_p^T\overline{\mathbf{H}}_{k-1}\left(\overline{z}_{i,k}\right)\boldsymbol{b}_k\right]\overline{\mathbf{H}}_{k-1}\left(\overline{z}_{i,k}\right)\boldsymbol{e}_q$ with $\overline{h}_{pq,k-1}\left(\overline{z}_{i,k}\right) = \boldsymbol{e}_p^T\overline{\mathbf{H}}_{k-1}\left(\overline{z}_{i,k}\right)\boldsymbol{e}_q$ yields:

$$\left(\overline{z}_{i,k}\boldsymbol{f}_k + \boldsymbol{g}_k\right)^T\boldsymbol{t}_{i,k} = \overline{h}_{pq,k-1}\left(\overline{z}_{i,k}\right), \quad i = 1, \ldots, n_z \tag{69}$$

Hence, the control gains $\boldsymbol{f}_k$ and $\boldsymbol{g}_k$ that assigns the prescribed zeros are those solving Eq. (69) for $i = 1,\ldots,n_z$. This in turn leads to the linear system in Eq. (30).

In the scenario of pole placement, the control gains $\boldsymbol{f}_k$ and $\boldsymbol{g}_k$ that assigns the prescribed poles $\overline{p}_{i,k}$, $i = 1, \ldots, n_p$ are those such that the denominator in Eq. (29) vanishes:

$$1 - \left(\overline{p}_{i,k}\boldsymbol{f}_k + \boldsymbol{g}_k\right)^T\overline{\mathbf{H}}_{k-1}\left(\overline{p}_{i,k}\right)\boldsymbol{b}_k = 0, \quad i = 1, \ldots, n_p \tag{70}$$

Defining $\boldsymbol{r}_{i,k} = \overline{\mathbf{H}}_{k-1}(\overline{p}_{i,k})\boldsymbol{b}_k$, Eq. (70) is recast into:

$$\left(\overline{p}_{i,k}\boldsymbol{f}_k + \boldsymbol{g}_k\right)^T\boldsymbol{r}_{i,k} = 1, \quad i = 1, \ldots, n_p \tag{71}$$

The solution of Eq. (71) for the n_p prescribed poles is achieved by the gains $\boldsymbol{f}_k$ and $\boldsymbol{g}_k$ that solve the linear system in Eq. (45).

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