Radical theory of Scott-open filters

Daniel Misselbeck-Wessel and Peter Schuster
Università di Verona, Dipartimento di Informatica
Strada le Grazie 15, 37134 Verona
daniel.wessel@univr.it
peter.schuster@univr.it

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Abstract

Following the Kronecker–Duval or D5 philosophy of dynamic evaluation in computer algebra, the dynamical proof method was brought into constructive algebra in order to obtain computational interpretations of the individual, concrete instances of the Kuratowski–Zorn Lemma modern algebra abounds with. We now push this approach into complete lattices, and thus capture the computational content also of generic, abstract forms of the axiom of choice such as the Teichmüller–Tukey lemma.

To this end we first strengthen the (key lemma for) the Hofmann–Mislove theorem about Scott-open filters (SOF): to be a member of a SOF is equivalent, with the axiom of choice, to membership of all complete extensions. This then allows for a constructive treatment: membership of the given SOF means that from the potential element one can grow a finite labelled binary tree of an inductively defined type such that every branch of the tree eventually dips into the SOF.

The ideal objects characteristic for invocations of the Kuratowski–Zorn Lemma are thus approximated and actually replaced by paths in those trees, by which we shed light on the interaction of transfinite methods and their dynamical interpretation. The first test case includes abstract dependence, especially bases of vector spaces.

Key to the above lies in a corresponding theory of radical operators for Scott-open sets, which through a generalised inductive definition can even be pinned down in terms of universal properties. This radical theory reaches into classical pointfree topology, exemplified alongside by a succinct proof of Isbell's spatiality theorem.

Key words: dynamical proof, complete lattice, Scott-open filter, nucleus, Hofmann–Mislove theorem, Isbell's spatiality theorem, axiom of choice, Teichmüller–Tukey lemma, syntax from semantics, computational content, constructive algebra, finite binary tree, inductive definition.

1 Introduction

The *Jacobson radical* [43] $\mathrm{Jac}(I)$ of an ideal I of a commutative ring[1] $\mathbf{A}$ with 1 is usually defined as the intersection of the maximal ideals of $\mathbf{A}$ which contain I:

$$\mathrm{Jac}(I) = \bigcap \mathrm{Max}(A)/I = \{\, a \in \mathbf{A} \mid (\forall M \in \mathrm{Max}(A))(I \subseteq M \Rightarrow a \in M) \,\} \tag{1}$$

where an ideal M of A is *maximal* if it is such among the *proper* ideals ($\neq A$). With the *Axiom of Choice* (AC), the definition of $\mathrm{Jac}(I)$ is equivalent to

$$\mathrm{Jac}(I) = \{\, a \in \mathbf{A} \mid (\forall b \in \mathbf{A})(\, \langle a, b\rangle \ni 1 \rightarrow \langle I, b\rangle \ni 1) \,\} \tag{2}$$

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[1] The Jacobson radical will serve as a running example, where commutativity of $\mathbf{A}$ is assumed merely for sake of convenience. Within reach of our method, the Jacobson radical plays an important role in non-commmutative ring theory as well [50], and carries over to module theory [6].

where angle brackets $\langle \dots \rangle$ denote generated ideals. In constructive algebra [51], the *first-order* alternative (2) is taken as the definition of $\mathrm{Jac}(I)$, so as to provide for a computationally meaningful concept. With AC at hand the *second-order* original (1) then becomes a theorem, which we henceforth refer to as the *Jacobson Intersection Principle* (JIP). With classical logic, in fact, JIP is equivalent to Krull's *Maximal Ideal Theorem* (MIT) [49], tantamount to AC [40]: *every proper ideal is contained in a maximal ideal.*

Logically speaking, MIT is *model existence* and JIP is *semantic conservation.* Shifting focus from the former to the latter has helped us to pin down the computational content of MIT as the *syntactical conservation* underlying JIP [69]. Can we generalise, abstracting from the (relatively rich) setting of rings? *Do also maximality principles conceptually closer to* AC*, such as the* Teichmüller–Tukey Lemma (TTL) [77, 78, 80]*, have a syntactical underpinning?* We will first solve semantically and then interpret syntactically the problem

$$\textit{Which principle } \mathrm{X} \textit{ is to } \mathrm{TTL} \textit{ just as } \mathrm{JIP} \textit{ is to } \mathrm{MIT}? \tag{3}$$

Heuristics comes from further cases: the Jacobson radical can also be defined

1. for ideals J of distributive lattices D [21, 23]

$$\mathrm{Jac}(J) = \{\, a \in D \mid (\forall b \in D)(\, a \vee b = 1 \to J \vee b = 1) \,\} ;$$

2. for theories Γ of propositional languages P [32]

$$\mathrm{Jac}(\Gamma) = \{\, \alpha \in P \mid (\forall \beta \in P)(\, \alpha, \beta \vdash \bot \to \Gamma, \beta \vdash \bot) \,\} .$$

To achieve our goal, we abstract from ideals and theories to the elements of a complete lattice L, and from comaximality $\langle \dots \rangle \ni 1$ and inconsistency $\dots \vdash \bot$ to a Scott-open subset O of L: that is, a monotone predicate that splits directed joins. We assume that $1 \in O$, and sometimes that O be a Scott-open *filter* of L, i.e. meet-closed as well.

In this context we define the *generalised radical* $j : L \to L$ by setting $jx = \bigvee J_x$ where

$$J_x = \{\, a \in L \mid (\forall b \in L)(\, O(a \vee b) \to O(x \vee b)\,) \,\} .$$

This j is a closure operator of which the aforementioned Jacobson radicals are instances. Some key features of j include that it allows for a generalised inductive definition (Section 3.2); that j is the largest closure operator on L for which O consists of the j-dense elements of L (Proposition 20); and that if L is distributive, then O is a filter precisely when j is a nucleus (Proposition 16).

Related and special cases of j had appeared before on distributive lattices [23], quantales [8, 29], and frames [21, 39, 74]. Unlike those cases, we work with general complete lattices L, and normally do not impose any further condition on O other than to be Scott-open and inhabited, and perhaps to be a Scott-open filter. However, our radical theory finds applications in classical pointfree topology, e.g. by a swift proof of Isbell's spatiality theorem (Example 26).

We next briefly discuss our achievements. With classical logic and AC we can prove the following for every $x \in L$, where $y \in L$ is said to be *proper* if $\neg O(y)$ and *completeness* of a proper y amounts to maximality among proper elements (Section 2).

(a) *The radical jx is the meet of all proper complete $y \geqslant x$* (Corollary 25).

Since $O(x)$ is tantamount to $jx = 1$ (Proposition 11) we obtain from the former statement that

(b) *if x is proper, then there is a proper complete $y \geqslant x$.*

As the proper complete ideals of a ring are just the maximal ideals, (a) and (b) generalise JIP and MIT, respectively. If L is algebraic, then (b) generalises TTL (Section 5). With (a) we thus obtain the desired semantic solution X of (3), and can seek to interpret it syntactically. To this end we put (b) "upside up", which in fact will serve as our starting point:

(c) *any given $x \in L$ belongs to O if (and only if) every complete $y \geqslant x$ is in O* (Theorem 3).

Adapting our recent syntactical treatments of prime ideal theorems [68, 71] and of some specific maximality principles [67, 70], we inductively define a collection T of finite binary trees labelled by elements of L, with an appropriate termination concept for paths (Section 4). All this allows us to prove constructively—in particular, without AC—the following syntactical counterpart of (c) whenever O is a filter:

(d) *any given $x \in L$ belongs to O if (and only if) there is a tree in T with root labelled by x such that every branch of the tree terminates in O* (Corollary 36).

The feasibility of this syntactical characterisation is not obvious: unlike prime ideal theorems, which are of binary nature by the very form of the prime ideal axiom, abstract maximality principles such as (b) or (c) equivalent to full AC a priori fall short of lending themselves naturally to a computational simulation by finite binary trees.[2] Our key idea to overcome this barrier, and in fact to get by with binary branching also in cases of AC proper such as TTL, is to complement every $a \in L$ by the O-variant $\overline{a}$ of the pseudo-complement of a.

The remainder of this paper will be devoted to illustrating the practicability of our approach. On top of TTL proper (Section 5), we discuss how to include—via abstract dependence relations—a typical instance of TTL: the existence of a linear basis extending a given independent set of vectors (Section 6).

In retrospect, this paper pursues further a line of research on abstract radical theory. A first approach with symmetric irreflexive relations [67] has been generalised to abstract inconsistency predicates: that is, monotone families of finite subsets [70]. We widen scope by passing to complete lattices and Scott-open sets. This means to move from elements and subsets to a rather point-free algebraic framework, making necessary to adapt concepts and techniques accordingly. We thus can decode syntactically a transfinite method as versatile as TTL.

Last but not least, our understanding of constructive versions of abstract classical theorems adopts the one of Coquand and Lombardi [22]:

> "When we say that we have a constructive version of an abstract algebraic theorem, this means that we have a theorem the proof of which is constructive, which has a clear computational content, and from which we can recover the usual version of the abstract theorem by an immediate application of a well classified non-constructive principle."

Structure of this paper

After laying out some preliminaries, we proceed as follows. In Section 2 we prove an induction principle for Scott-open predicates on complete lattices which subsumes a variety of choice principles in algebra, logic, and order theory. In Section 3 we develop a radical theory for Scott-open predicates, in order to put under constructive scrutiny the classical induction principle above. This will then be done in Section 4 by way of a suitable inductively generated class of labelled finite binary trees. We eventually capture TTL along with some prominent applications in Sections 5 and 6.

Method, foundations and preliminaries

The present paper is primarily set within the constructive but impredicative Intuitionistic Zermelo–Fraenkel Set Theory **IZF** [9, 10, 13, 14, 27, 34, 63]. In particular, as opposed to predicative [26] settings such as constructive Zermelo–Fraenkel set theory **CZF** [3, 4] or arithmetic universes[3] [52, 83], we have at our disposal both the power set axiom and the axiom scheme of full separation; the latter especially allows us to tacitly identify every predicate on a set with the subset it defines.

Due to the choice of this setting, sometimes certain assumptions have to be made explicit which otherwise would be trivial in classical set theory. Recall that a set S is said to be *discrete*

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[2] The Boolean Prime Ideal Theorem or Boolean Ultrafilter Theorem is strictly weaker than AC [41].

[3] We are grateful to one of the anonymous referees for pointing this out to us.

if equality on the set is decidable, which is to say that $(\forall a, b \in S)(a = b \vee a \neq b)$. A subset T of S is *detachable* if membership is decidable, i.e. $(\forall a \in S)(a \in T \vee a \notin T)$; and *stable* when $a \in T$ already if $\neg(a \notin T)$ for every $a \in S$. By a *finite* set we understand a set that can be written as $\{ a_1, \dots, a_n \}$ for some $n \geqslant 0$.[4] Every finite set is either empty or inhabited. We denote by $\mathrm{Fin}(S)$ the class of all finite subsets of a set S. If S is discrete, note that every finite subset U of S is detachable and thus stable.

To deduce and compare unseen forms of AC, which will later be studied from a constructive point of view, we often work first in **ZF** or **ZFC**. Any such deviation from the generally constructive framework will of course be highlighted appropriately.

We recall some order-theoretic notions. A *complete lattice* is a partially ordered set L in which every subset N has both a join $\bigvee N$ and a meet $\bigwedge N$. In particular, a complete lattice has top and bottom elements $1 = \bigwedge \emptyset$ and $0 = \bigvee \emptyset$. A subset D of L is *directed* if D is inhabited and such that, for every $x, y \in D$, there is $z \in D$ with $x \leqslant z$ and $y \leqslant z$. A subset O of L is *open* if, for all directed subsets D of L,

$$\bigvee D \in O \quad \text{implies} \quad D \between O,$$

where from formal topology [62] we borrow the *overlap* symbol: $D \between O$ means that $D \cap O$ is inhabited. The *Scott-open* subsets of L are the *monotone* open subsets, i.e. those open subsets O for which

$$x \in O \text{ and } x \leqslant y \quad \text{implies} \quad y \in O.$$

An element a of L is *compact* or *finite* [44,81] if, for all directed subsets D of L such that $a \leqslant \bigvee D$, there is $d \in D$ with $a \leqslant d$. Thus, a is compact precisely when the principal filter

$$\uparrow\! a = \{ b \in L \mid a \leqslant b \}$$

is Scott-open. The subset of compact elements of L is often denoted by KL. A complete lattice is *algebraic* if every element x of L is the directed join of the compact elements $a \leqslant x$, that is,

$$x = \bigvee \{ a \in KL \mid a \leqslant x \} .$$

The prime example of an algebraic lattice is the power set $L = \mathrm{Pow}(S)$ of a given set S, for which $KL = \mathrm{Fin}(S)$ consists of all finite subsets of S. There is the following generalisation.

Example 1. Let S be an arbitrary set. A *consequence relation* is a relation $\rhd \subseteq \mathrm{Fin}(S) \times S$ which is *reflexive, monotone* and *transitive* in the following sense, respectively:

$$\frac{U \ni a}{U \rhd a} \qquad \frac{U \rhd a}{U, V \rhd a} \qquad \frac{U \rhd b \quad U, b \rhd a}{U \rhd a}$$

where $U, V \equiv U \cup V$ and $U, b \equiv U \cup \{b\}$. As is well-known, every consequence relation $\rhd$ gives rise to a closure operator $\langle - \rangle : \mathrm{Pow}(S) \to \mathrm{Pow}(S)$, where

$$a \in \langle T \rangle \;\equiv\; (\exists U \in \mathrm{Fin}(T))\, U \rhd a.$$

which is *algebraic* in the sense that $\langle T \rangle$ is the directed union of the $\langle U \rangle$ where U ranges over the finite subsets of T. Conversely, from an algebraic closure operator $\langle - \rangle$ one gains back a consequence relation $\rhd$:

$$U \rhd a \;\equiv\; a \in \langle U \rangle .$$

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[4] As in [59,60], for the sake of a slicker wording we thus deviate from the prevalent terminology of constructive mathematics and set theory [3,4,15,16,51,55]: (1) to call 'subfinite' or 'finitely enumerable' a finite set in the sense above, i.e. a set S for which there is a surjection from $\{1, \dots, n\}$ to S for some $n \geqslant 0$; and (2) to reserve the term 'finite' to sets which are in bijection with $\{1, \dots, n\}$ for a necessarily unique $n \geqslant 0$. Also, finite sets in this stricter sense do not play a role in this paper.

The *ideals* or *saturated sets* of a consequence relation $\rhd$ are the subsets I of S which are *closed* with respect to the corresponding closure operator $\langle - \rangle$, which is to say that $I = \langle I \rangle$. Hence the saturated sets of $\rhd$ are precisely the subsets I of S such that if $I \supseteq U$ and $U \rhd a$, then $a \in I$. An ideal I is *proper* if $I \neq S$. The ideals of $\rhd$ form an algebraic lattice $\mathrm{Sat}(\rhd)$ with $\subseteq$ as $\leqslant$ for which

$$\bigvee_{i \in I} I_i = \Big\langle \bigcup_{i \in I} I_i \Big\rangle \quad \text{and} \quad \bigwedge_{i \in I} I_i = \bigcap_{i \in I} I_i \ ;$$

in particular, $0 = \langle \emptyset \rangle$ and $1 = S$. The compact elements of $\mathrm{Sat}(\rhd)$ are the *finitely generated* ideals, i.e. the ones of the form $\langle F \rangle$ with $F \in \mathrm{Fin}(S)$. For any given $F \in \mathrm{Fin}(S)$, the predicate

$$O(I) \ \equiv \ F \subseteq I$$

yields a Scott-open filter on $\mathrm{Sat}(\rhd)$; and so does especially, for every $a \in S$, the predicate

$$O(I) \ \equiv \ a \in I\,.$$

Of particular interest is the case in which S itself is compact, i.e. finitely generated. Under this assumption, in **ZFC** every proper ideal can be extended to a maximal one, which Schmidt [64, Satz 8, Korollar 1] has ascribed to Lindenbaum (through Tarski [76]). A concrete instance is the well-known theorem that in a finitely generated module every proper submodule is contained in a maximal submodule, for which see, e.g. [72, Proposition 1.6]. For ring theory the more specific case can be traced back to Krull [49] in which S is generated by a singleton set [64, Satz 8, Korollar 2], i.e. by a single element or *unit* [64], also named *convincing element* [58]. With the present paper we cover both instances through a form of TTL; see also Example 43 below.

Remark 2. Why do we work with complete lattices rather than with the more elementary directed-complete partial orders? Many applications of the Kuratowski–Zorn Lemma (KZL) take place in settings in which most likely there is a bottom element (which is to be extended) and there are binary joins (that allow for step-wise extensions) as well as directed joins (which ensure for KZL to be applicable). However, with finite and directed joins we also have arbitrary joins [44, I.4.1] which in turn yield arbitrary meets [44, I.4.3]. In all we find ourselves in the setting of complete lattices anyway, if only at the expense of some impredicativity.

A more predicative approach might still be feasible, e.g. through basic covers to present the lattices at hand [20], or by employing set-generated structures within **CZF** [2]. However, we aim at a universal toolbox for replacing, in situations of sufficiently concrete nature, the common indirect arguments based on forms of AC by direct elementary reasoning with, e.g. finite labelled binary trees, which are predicative at the outset. We believe that the present approach, some intermediate impredicativity notwithstanding, is well suited to this end.

2 Complete elements

Throughout, let L be a complete lattice, and let O be a fixed Scott-open subset of L. We assume that $1 \in O$, which by monotonicity is the case precisely when O is inhabited. We often write $O(x)$ to say that $x \in O$, moving back and forth between these notations. This helps to distinguish joins in L from disjunctions later on, but, more importantly, is to suggest that O rather play the role of a (comaximality or inconsistency) predicate.

We say that $x \in L$ is *(O-)complete* if, for every $y \in L$,

$$y \leqslant x \text{ or } O(y \vee x).^{5} \tag{4}$$

This amounts to saying that x is maximal in the sense that, for every $y \in L$,

$$\text{if } y \geqslant x, \text{ then } x = y \text{ or } O(y).$$

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[5] Maximality in disjunctive form has also been advocated in domain theory by Smyth [75], following Martin-Löf [53]. In constructive algebra, the corresponding axiom of a field is nowadays often put in this geometric form; for a discussion see [18].

We further say that $y \in L$ is *proper* if $\neg O(y)$. Hence with classical logic any proper $x \in L$ is complete precisely when x is maximal among the proper elements of L, whereas the elements of O are complete by monotonicity . Moreover, $x \in L$ is O-complete if and only if, for every $y \in L$,

$$y \leqslant x \text{ or } (\exists z \leqslant x)\, O(y \vee z). \tag{5}$$

For the set of O-complete elements of L we write

$$\mathrm{Comp}$$

and adopt the slice category notation for the set of O-complete elements above x, viz.

$$\mathrm{Comp}/x = \mathrm{Comp} \cap {\uparrow} x\,.$$

The following is reminiscent of commutative rings, for which comaximality can be checked (via MIT) modulo a generic maximal ideal [87].

Theorem 3 (**ZFC**)**.** *For every $x \in L$, the following are equivalent.*

1. $x \in O$.
2. $\mathrm{Comp}/x \subseteq O$.

Proof. If $x \notin O$, then the directed-complete complement of O in L is inhabited; so KZL yields a complete element extending x but avoiding O. ☐

For $O(x)$ it therefore is enough that $O(y)$ for every complete element y above x, which is a classical semantic conservation theorem. The purpose of the present note is to put this principle under constructive syntactical scrutiny, while giving evidence for its universality.

Remark 4. Through contrapositive reasoning, our proof of Theorem 3 employs classical logic on top of KZL. The latter principle is found to be *constructively neutral* [11] (see also [46, D4.5.14]),[6] which AC of course is not [28, 37]. In particular, with Theorem 3 we do not intend to offer an intuitionistic equivalent of KZL, of which there supposedly are rather few [12, 66].

Example 5. Let $\mathbf{A}$ be a commutative ring with 1. Let L be the complete lattice of ideals I of $\mathbf{A}$ for which KL consists of the finitely generated ideals of $\mathbf{A}$. Comaximality

$$O(I) \;\equiv\; 1 \in I$$

is a Scott-open predicate on L. The corresponding instance of Theorem 3 asserts that $1 \in I$ if and only if $1 \in M$ for every complete $M \in L$ which contains I. In **ZF**, the proper complete elements of L are precisely the maximal ideals of $\mathbf{A}$. In **ZFC**, Theorem 3 by contraposition thus gives MIT: *every proper ideal I of $\mathbf{A}$ is contained in a maximal one.*

Example 6. Suppose that L is a complete *multiplicative* lattice, i.e. a complete lattice equipped with a multiplication $\cdot$ which is compatible with the ordering on L [47]. An inhabited subset D of L is a *distributor* [29] if, for every $a, b, c \in L$,

$$a \vee c \in D \text{ and } b \vee c \in D \quad \text{if and only if} \quad (a \cdot b) \vee c \in D.$$

Suppose now that D is a Scott-open distributor. We claim that every D-complete element $p \notin D$ is *prime*, i.e. $p \neq 1$ and such that $a \cdot b \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$ for every $a, b \in L$. In fact, by D-completeness, the only case requiring attention is when $a \cdot b \leqslant p$ but both $a \vee p \in D$ and $b \vee p \in D$. But in this case we have $p = (a \cdot b) \vee p \in D$ after all, since D is a distributor. Therefore, by Theorem 3 in **ZFC**, every element outside a Scott-open distributor D is dominated by a prime element outside D. This is Erné's *Separation Lemma for Complete Multiplicative Lattices*, from which a variety of prime ideal principles follow [29].

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[6] One of the anonymous referees suggested to us that we bring this circumstance to the reader's attention.

Example 7. Suppose that L is a frame and let O be a Scott-open filter of L. The filter property renders O a distributor for the meet operation. The corresponding prime elements are precisely the *meet-irreducible* ones, i.e. those $p \neq 1$ such that for every $a, b \in L$,

$$a \wedge b \leqslant p \quad \text{implies} \quad a \leqslant p \text{ or } b \leqslant p.$$

Recall that the meet-irreducible elements correspond in **ZF** with the *completely prime filters* of L [56, II.3.3], the latter of which commonly serve to define the set $\mathrm{pt}(L)$ of *points* of L. As seen in Example 6, every proper complete element is prime and thus meet-irreducible. With this observation we gain back in **ZFC** from Theorem 3 that

$$O = \bigcap \{\, F \in \mathrm{pt}(L) \mid O \subseteq F \,\},$$

which plays a seminal role in the proof of the *Hofmann–Mislove Theorem* (also known as *Scott-Open Filter Theorem*) [5, 35, 81, 82].[7] To see this, let $x \in L$ belong to every completely prime filter F which contains O. In order to see that $x \in O$, it suffices by Theorem 3 to check that $\mathrm{Comp}/x \subseteq O$. Indeed, suppose that there is an O-complete $p \geqslant x$ with $p \notin O$. In particular, this p is meet-irreducible, whence

$$F_p = \{\, x \in L \mid x \nleqslant p \,\}$$

is a completely prime filter of L. Since $O \subseteq F_p$, by assumption on x it follows that $x \in F_p$, which yields a contradiction.

3 Radical theory

Alongside the analogy with rings and ideals, we next develop an abstract radical theory for Scott-open predicates O on complete lattices L, to which end we first introduce a closure operator $j : L \to L$. Specific cases of this j have appeared in point-free topology [8, 23, 74], leading, e.g. to an abstract treatment of Krull's Separation Lemma through quantales [8]. Unlike those cases, we work with general complete lattices L, and in general do not impose any further condition on O other than to be Scott-open and inhabited, and perhaps to be meet-closed as well, that is, a Scott-open filter.

3.1 Jacobson radical

The following definition stands to reason once one passes from lattices of ideals to complete lattices in general.

For every $x \in L$, set

$$J_x = \{\, a \in L \mid (\forall b \in L)(\, O(a \vee b) \to O(x \vee b)\,) \,\}.$$

Lemma 8. *For every $x, y \in L$,*

1. *$x \in J_x$, and*
2. *$J_x \subseteq J_y$ if $x \leqslant y$.*
3. *J_x is directed, and*
4. *J_x is closed under all joins.*

Proof. That $x \in J_x$ is trivial, while no. 2 follows from monotonicity of O, which by the way further implies that $0 \in J_x$. Next, note that if $a, a' \in J_x$, then $a \vee a' \in J_x$. In fact, if $O(a \vee a' \vee b)$, then $O(x \vee a' \vee b)$ since $a \in J_x$, and this implies $O(x \vee b)$ in view of $a' \in J_x$. In particular, J_x is directed. It remains to check that J_x is closed under directed joins, which is a consequence of O being Scott-open. In fact, let $D \subseteq J_x$ be directed and consider $b \in L$ such that $O(\bigvee D \vee b)$. Since O is open, there is $d \in D \subseteq J_x$ such that $O(d \vee b)$, and therefore $O(x \vee b)$. ☐

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[7] We are grateful to Steve Vickers for pointing this out to us.

Remark 9. By monotonicity of O, if $O(x)$, then $J_x = L$; and J_x is downward monotone, i.e. if $a' \leqslant a$ and $a \in J_x$, then $a' \in J_x$.

To get an element of L we take the (directed) join

$$jx = \bigvee J_x .$$

We thus obtain an expansive mapping on L which is characteristic for O, as will be seen below in Section 3.2.

Proposition 10. *$j : L \to L$ is a closure operator, i.e. expansive, monotone, and idempotent.*[8]

Proof. That j is expansive and monotone follows from Lemma 8.1 and 8.2, respectively. Note that $a \in J_x$ implies $J_a \subseteq J_x$, which in particular yields $J_{jx} \subseteq J_x$, since $jx \in J_x$ according to Lemma 8.4. It follows that $jjx \leqslant jx$, which is to say that j is idempotent. $\square$

Proposition 11. *For every $x \in L$, the following are equivalent.*

1. $jx = 1$.
2. $O(jx)$.
3. $O(x)$.
4. $J_x = L$.

Proof. Since $O(1)$, the first item entails the second. Next, if $O(jx)$, then by Lemma 8.3 there is $a \in J_x$ such that $O(a)$, which (with $b = 0$ in the definition of J_x) yields $O(x)$. Last but not least, if $O(x)$, then $J_x = L$ (Remark 9) and thus $jx = 1$. $\square$

Definition 12. We say that $x \in L$ is *radical* if x is a fixed point of j, i.e. if $jx = x$. Moreover, we say that x is *proper* if $\neg O(x)$.

Corollary 13. *For every complete $x \in L$,*

$$O(x) \text{ or } jx = x.$$

In particular, every proper complete element is radical.

Proof. By completeness either $jx \leqslant x$, whence $jx = x$ since j is expansive, or $O(jx \vee x)$. The latter means $O(jx)$ or, equivalently, $O(x)$ (Proposition 11). $\square$

To conclude this section, we discuss some special cases and observations.
For each $a \in L$ we write $\overline{a}$ for the *pseudo-complement* of a with respect to O:

$$\overline{a} = \{\, b \in L \mid O(a \vee b) \,\} . \tag{6}$$

With this, any given element $x \in L$ is complete precisely when, for every $y \in L$,

$$y \leqslant x \text{ or } y \in \overline{x}. \tag{7}$$

Lemma 14. *For every $x, y \in L$, the following are equivalent.*

1. $x \in J_y$.
2. $x \leqslant jy$.
3. $\overline{x} \subseteq \overline{y}$.

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[8] This is to say that $x \leqslant jx$, and $jx \leqslant jy$ if $x \leqslant y$, and $jjx \leqslant jx$, respectively, for every $x, y \in L$. Recall that the closure properties boil down to demanding that, for every $x, y \in L$, $x \leqslant jy$ if and only if $jx \leqslant jy$.

Proof. 2) $\to$ 3): Suppose that $x \leqslant jy$, and let $b \in \overline{x}$, i.e. $O(x \vee b)$. Then $O(jy \vee b)$ and thus $O(j(y \vee b))$ by monotonicity of O and the closure properties of j. Proposition 11 implies that $O(y \vee b)$, which is to say that $b \in \overline{y}$. Both remaining implications 1) $\to$ 2) and 3) $\to$ 1) are trivial. □

The following will be useful for applications in abstract ideal theory later on.

Proposition 15. *If L is algebraic, then, for every $x, y \in L$, the following are equivalent.*

1. $x \leqslant jy$.
2. $(\forall b \in KL)(\, O(x \vee b) \to O(y \vee b)\,)$.

Proof. It suffices to check that the second item implies the first, to which end we show (Lemma 14) that $\overline{x} \subseteq \overline{y}$. Accordingly, let $d \in L$ and suppose that $O(x \vee d)$. Write $d = \bigvee D$ as directed join of compact elements $b \leqslant d$. Since O is Scott-open, there is $b \in D$ such that $O(x \vee b)$. It follows that $O(y \vee b)$, thus $O(y \vee d)$ by monotonicity. □

Proposition 16. *Suppose that L is distributive. The following are equivalent.*

1. *O is a filter.*
2. *j is a nucleus, i.e. for every $x, y \in L$,*

$$jx \wedge jy \leqslant j(x \wedge y).$$

Proof. Let O be a filter, i.e. meet-closed. To see that j respects meets, we show that $jx \wedge jy \in J_{x \wedge y}$. Accordingly, consider $b \in L$ such that $O((jx \wedge jy) \vee b)$. Monotonicity of O and the closure properties of j imply that $O(j(x \vee b))$ and $O(j(y \vee b))$, and so $O(x \vee b)$ and $O(y \vee b)$ by Proposition 11. Since O is a filter it follows that $O((x \vee b) \wedge (y \vee b))$ which by distributivity is $O((x \wedge y) \vee b)$, as required. Regarding the converse, suppose that $O(x)$ and $O(y)$. By Proposition 11, this is to say that $jx = 1 = jy$. Therefore $j(x \wedge y) = jx \wedge jy = 1$, whence $O(x \wedge y)$ again by Proposition 11. □

In particular, if L is a frame, then every Scott-open filter O gives rise to a nucleus j.

Example 17. Suppose that L is a *compact* frame, which is to say that 1 is a compact element [44]. This amounts to the smallest filter

$$O(x) \;\equiv\; x = 1$$

being Scott-open, and in which case the corresponding j is a nucleus according to Proposition 16. We hasten to add that this j, which by definition reads

$$jx = \bigvee \{\, a \in L \mid (\forall b \in L)(\, a \vee b = 1 \to x \vee b = 1\,)\,\},$$

is well-known to be a nucleus irrespectively of L being compact [39, 74]. The Jacobson radical for rings is one such instance, as will be discussed below in Example 22.

3.2 Inductive generation

Inductive generation is to be understood in the generalised sense of [1].

Definition 18. We generate the binary relation $\sqsubseteq$ on L inductively by the following three rules:

$$\frac{x \leqslant y}{x \sqsubseteq y} \qquad \frac{O(y)}{x \sqsubseteq y} \qquad \frac{x \sqsubseteq y \vee a \quad (\forall b \in \overline{a})\, x \sqsubseteq y \vee b}{x \sqsubseteq y}$$

This helps to describe the principal ideals generated by radical elements.

Proposition 19. *For every $x, y \in L$, the following are equivalent.*

1. $x \sqsubseteq y$.
2. $x \leqslant jy$.

Thus, for every $y \in L$,

$$jy = \bigvee \{\, x \in L \mid x \sqsubseteq y \,\}.$$

Proof. We use Lemma 14 and show by induction on $x \sqsubseteq y$ that $x \in J_y$, which is trivial except perhaps for the third generating rule. Acccordingly, suppose that (i) $x \in J_{y \vee a}$ as well as (ii) $x \in J_{y \vee b}$ for every $b \in \overline{a}$. To see that $x \in J_y$, consider $b \in L$ with $O(x \vee b)$. Then $O(y \vee a \vee b)$ by (i). So $y \vee b \in \overline{a}$, thus (ii) yields $x \in J_{y \vee b}$. Since $O(x \vee b)$, this implies $O(y \vee b)$, as required.

Regarding the converse, if $x \leqslant jy$, then $\overline{x} \subseteq \overline{y}$ by Lemma 14 again. Thus $x \sqsubseteq y \vee b$ for every $b \in \overline{x}$ in view of the second generating rule for $\sqsubseteq$. The first rule gives $x \sqsubseteq y \vee x$, which with the third implies $x \sqsubseteq y$. □

Given a closure operator $f : L \to L$, we say that $x \in L$ is *f-dense* if $fx = 1$. According to the following Proposition 20, O consists precisely of the j-dense elements. In other words, O is the *top kernel* of j. Borrowing terminology from pointfree topology, we denote the latter

$$\nabla(j) = j^{-1}(1) = \{\, x \in L \mid jx = 1 \,\}$$

which in the case of a nucleus j is the *admissible filter* of j [73]. With this in place, every Scott-open filter in a frame is the admissible filter of the corresponding Jacobson radical. Moreover, we can pin down radicals up to universal properties, as follows.

Proposition 20. *Among closure operators on L, the radical j is uniquely determined by the following properties:*

1. $O = \nabla(j)$,
2. j folds up branchings[9] *in the sense that for every $x, a \in L$,*

$$jx = j(x \vee a) \wedge \bigwedge_{b \in \overline{a}} j(x \vee b).$$

More precisely, if f is a closure operator, then $O = \nabla(f)$ implies $f \leqslant j$, or equivalently $fj = j$; if in addition f folds up branchings, then $f = j$.

Proof. To begin with, we show that j enjoys the stated properties. The first one is contained in Proposition 11. Regarding the second, let $x, a \in L$. If $y \leqslant j(x \vee a)$ and $y \leqslant j(x \vee b)$ for every $b \in \overline{a}$, then $y \sqsubseteq x \vee a$ and $y \sqsubseteq x \vee b$ for every $b \in \overline{a}$. This implies $y \sqsubseteq x$, which is to say that $y \leqslant jx$, whence jx is indeed the greatest lower bound.

Consider now a closure operator $f : L \to L$ such that $O = \nabla(f)$. For every $x, b \in L$, if $O(fx \vee b)$, then $1 \leqslant f(fx \vee b) \leqslant f(f(x \vee b)) \leqslant f(x \vee b)$ since f is a closure operator. This in turn yields $O(x \vee b)$; so $fx \in J_x$ which means $fx \leqslant jx$, and therefore $f \leqslant j$. For the latter's equivalence to $fj = j$ see, e.g. [30, Lemma 3.1].

Suppose now in addition that f folds up branchings. To see that also $j \leqslant f$, it suffices to check $(\forall a \in J_x)\, a \leqslant fx$. By Proposition 19, we do so by induction on $a \sqsubseteq x$. If $a \leqslant x$, then $a \leqslant fx$ as f is expansive. If $O(x)$, then $fx = 1$ by assumption on f, and $a \leqslant fx$ is trivial. If there is $a_0 \in L$ such that $a \sqsubseteq x \vee a_0$ and $a \sqsubseteq x \vee b$ for every $b \in \overline{a_0}$, then by induction $a \leqslant f(x \vee a_0)$ and $a \leqslant f(x \vee b)$ for every $b \in \overline{a_0}$. Hence $a \leqslant f(x \vee a_0) \wedge \bigwedge_{b \in \overline{a_0}} f(x \vee b) = fx$, as required. □

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[9] The choice of this term will become clear in Section 4.

Remark 21. There is a constructive version of the Hofmann-Mislove Theorem, according to which in a frame the Scott-open filters are in order-reversing bijection with the compact fitted quotient frames [30, 45, 79, 82]. Given a Scott-open filter O, key part of Johnstone's argument [45, Lemma 3.4(ii)] lies in the construction of a nucleus j with $\nabla(j) = O$ as above, but with the further requirement that j be *fitted*, i.e. expressible as a join of open nuclei. Moreover, Johnstone's nucleus is the *smallest* one such that $\nabla(j) = O$ (discussion ibid.). Compare this with the Jacobson radical of O, which instead is the *largest* nucleus j with $\nabla(j) = O$ (Proposition 20). Every nucleus with said property lies in between those. Of course they need not coincide, as can be seen by means of the following.

Example 22. In the context of commutative rings $\mathbf{A}$ (Example 5), the radical jI of an ideal I of $\mathbf{A}$ is nothing but the Jacobson radical $\mathrm{Jac}(I)$. Recall that the *nilradical* $\sqrt{I}$ is the set of all ring elements a such that $a^n \in I$ for some $n \geqslant 0$. Assigning nilradicals is a closure operator on the lattice of ideals, the fixed points of which are called *radical ideals.* Moreover, $1 \in I$ if and only if $1 \in \sqrt{I}$. We now apply Proposition 20 to the case where $f = \sqrt{-}$. This yields the the well-known fact that Jacobson radicals are indeed radical ideals, i.e.

$$\sqrt{\mathrm{Jac}(I)} = \mathrm{Jac}(I). \tag{8}$$

For the corresponding instance of Proposition 20.1 see already [33].

The *Zariski frame* $\mathrm{Zar}(\mathbf{A})$ is the lattice of all radical ideals of a commutative ring $\mathbf{A}$ [65]. As opposed to the lattice of all ideals, this $\mathrm{Zar}(\mathbf{A})$ is indeed a frame, the compact elements of which are the radicals of finitely generated ideals. In particular, $\mathrm{Zar}(\mathbf{A})$ is compact. The discussion of Example 17 shows that Jac is a nucleus on $\mathrm{Zar}(\mathbf{A})$.

Example 23. Suppose that L is distributive and compact, and let the closure operator $f : L \to L$ be *Scott-continuous*, i.e. such that f preserves directed suprema. Notice that $f(1) = 1$ as f is expansive. Then the subset $O = f^{-1}(1)$ is inhabited and Scott-open, and the corresponding Jacobson radical j is an extension of f, i.e. $f \leqslant j$ (Proposition 20). If f commutes with meet, then O is a filter and j is a nucleus (Proposition 16). Needless to say, the nucleus of Example 17 corresponds to the special case $f = \mathrm{id}$.

3.3 Completeness

Closure is often understood in terms of logical consequence. We are thus led to read $x \sqsubseteq y$ as "x can be proved on the assumption y". Which are the corresponding models? This digression requires that we switch to **ZFC**.

For $x \in L$ let

$$\mathrm{Comp}^*/x = (\mathrm{Comp}/x) - O$$

be the set of proper complete elements above x.

Theorem 24 (**ZFC**)**.** *For every $x, y \in L$, the following are equivalent.*

1. $x \leqslant jy$.
2. $\mathrm{Comp}^*/y \subseteq {\uparrow}x$.
3. $\mathrm{Comp}^*/y \subseteq \mathrm{Comp}^*/x$.

Proof. Suppose that $x \leqslant jy$ and let c be proper complete above y. Thus either $x \leqslant c$ right away, or else $O(x \vee c)$. But since $x \leqslant jy$, the latter alternative would imply $O(y \vee c)$ or, equivalently, $O(c)$ in view of $y \leqslant c$, which is in conflict with c being proper.

Conversely, assume that every proper complete c above y lies above x. To see $x \leqslant jy$, we show $x \in J_y$. Accordingly, let $z \in L$ such that $O(x \vee z)$. To check $O(y \vee z)$, by Theorem 3 we need to verify $O(d)$ for every complete d above $y \vee z$. Now if d is as such, then by assumption (in particular, d is complete above y) either $O(d)$ right away, or else $x \leqslant d$ but which implies $O(d)$ in view of $O(x \vee z)$ and $z \leqslant d$. ☐

Corollary 25 (**ZFC**)**.** *For every* $y \in L$,

$$jy = \bigwedge \mathrm{Comp}^*/y.$$

In particular, if $jy \neq 1$*, then there is a proper complete* $c \geqslant y$.

Proof. By Theorem 24, both sides of the equation have the same elements below. ☐

Example 26. In the setting of Example 17, let L be a compact frame. By Proposition 11 we have $jy = 1$ if and only if $y = 1$. Every proper complete element y is meet-irreducible and thus determines a point of L (Example 7). Therefore, we reobtain from the case $y = 0$ of Corollary 25 the well-known result that in **ZFC** every nontrivial compact frame has at least one point [44, Lemma III.1.9]. Recall from the latter reference that this circumstance does not in general suffice for compact frames L to have enough points. However, every *regular* compact frame is spatial [44, Proposition III.1.10], which is an instance of *Isbell's Spatiality Theorem* [42] through the implication *regular* ⇒ *subfit* [56, V.5.5]. To reobtain Isbell's result within the current setting, recall from [56, V.1.1] that subfitness is commonly defined by demanding that, for every $x, y \in L$,

$$x \nleqslant y \quad \text{implies} \quad (\exists z \in L)(\, x \vee z = 1 \,\&\, y \vee z \neq 1\,). \tag{9}$$

With classical logic and Lemma 14, this (9) is to say that, for every $x, y \in L$,

$$x \leqslant jy \quad \text{implies} \quad x \leqslant y,$$

which of course amounts to nothing but $j = \mathrm{id}$: that is, every element of L is radical. Thus, with Corollary 25, in **ZFC** every compact subfit frame L is T_1-*spatial*, i.e. every element of L is a meet of maximal elements [57, Corollary 4.3]. In particular, every such frame is spatial.

Example 27. For rings (Examples 5 and 22), Corollary 25 returns JIP: the Jacobson radical $\mathrm{Jac}(I)$ of an ideal I is the intersection of all maximal ideals containing I. By contraposition this yields MIT, as $1 \in I$ if and only if $1 \in \mathrm{Jac}(I)$.

Proposition 28 (**ZF**)**.** *For each of Theorem 3, Theorem 24 and Corollary 25, the corresponding universal statement (i.e. for every complete lattice and Scott-open subset) is equivalent to* AC.

Proof. Theorem 3 implies Theorem 24, which in turn implies Corollary 25. The latter yields MIT (Example 27), which is equivalent to AC [7, 40, 41, 49]. ☐

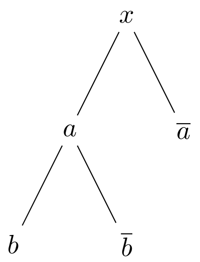
4 Trees

Let again L be a complete lattice, and let O be an inhabited Scott-open subset of L. Freely making use of the concepts defined in Section 3, we address the computational significance of (a characteristic case of) Theorem 3. We label trees (Definition 29) below by elements a and their pseudo-complements $\overline{a}$ with respect to O defined by (6). This allows us to get by with binary trees rather than infinitely branching ones, although the latter might seem necessary in view of the infinite character of the second disjunct of completeness. Throughout we understand paths to lead from the root of a tree to one of its leaves.

Definition 29. Let $x \in L$. We generate the collection $\mathcal{T}_x$ of labelled binary trees t inductively by the following rules:

1. The trivial tree (i.e. the root-only tree) labelled with x belongs to $\mathcal{T}_x$.
2. If $t \in \mathcal{T}_x$ has a path π, and $a \in L$, then add children labelled with a and $\overline{a}$, respectively, at the leaf of π. The resulting tree is added to $\mathcal{T}_x$.

For instance, if $x, a, b \in L$, then the following tree belongs to $\mathcal{T}_x$:



In the following, we represent paths as strings of elements of $L \cup \{\, \overline{a} \mid a \in L \,\}$.

Definition 30. We define $\pi \vdash m$ as a relation $\vdash$ between paths π and elements m of L recursively over π in the following way:

$$\begin{aligned} x \vdash m \;&\equiv\; m \sqsubseteq x \\ x\pi a \vdash m \;&\equiv\; (x \vee a)\pi \vdash m \\ x\pi\overline{a} \vdash m \;&\equiv\; (\forall b \in \overline{a})\, x\pi b \vdash m. \end{aligned}$$

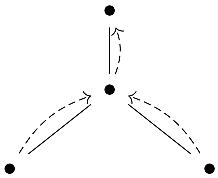
Note that the length of the paths does decrease in the recursive call.

It is suggestive to read $\pi \vdash m$ as “the data accumulated along path π yield information m”.

Remark 31. By commutativity of joins and quantifier swap, when evaluating instances of $\vdash$ along paths of $t \in \mathcal{T}_x$, we may rearrange the string entries after the root as occasion demands. In fact, if $\ell = x\ell'$ is the string of labels occurring along a path of $t \in \mathcal{T}_x$, then, for every permutation σ of appropriate length and every $m \in L$,

$$x\ell' \vdash m \quad \text{if and only if} \quad x\sigma(\ell') \vdash m\,.$$

By Lemma 32 below one can climb any tree from the leaves to the root in the following manner:



Lemma 32. *Let π be a path of some tree $t \in \mathcal{T}_x$, and let $a, m, m' \in L$. Then:*

$$\frac{\pi a \vdash m \qquad \pi\overline{a} \vdash m'}{\pi \vdash m \wedge m'}$$

Proof. As indicated in Remark 31, we may assume that π be given in the form

$$\pi = [x, a_1, \dots, a_k, \overline{b_1}, \dots, \overline{b_\ell}]$$

for certain $a_i, b_j \in L$. Suppose now that $\pi a \vdash m$ and $\pi\overline{a} \vdash m'$. To see that $\pi \vdash m \wedge m'$, let $c_1 \in \overline{b_1}, \dots, c_\ell \in \overline{b_\ell}$. We need to check that

$$m \wedge m' \sqsubseteq x \vee a_1 \vee \cdots \vee a_k \vee c_1 \vee \cdots \vee c_\ell.$$

By assumption $m \sqsubseteq x \vee \bigvee a_i \vee \bigvee c_j \vee a$ as well as $m' \sqsubseteq x \vee \bigvee a_i \vee \bigvee c_j \vee b$ for every $b \in \overline{a}$. Proposition 20.2 (with Proposition 19) implies that $m \wedge m' \sqsubseteq x \vee \bigvee a_i \vee \bigvee c_j$. Since $c_1, \dots, c_\ell$ had been considered arbitrary, this translates back as required. ☐

Definition 33. Let $M \subseteq L$. A tree $t \in \mathcal{T}_x$ *terminates in* M, for short

$$t \downarrow M,$$

if, for every path π of t, there is $m \in M$ such that $\pi \vdash m$. To indicate that t terminates in a singleton set $\{\, m \,\}$, we write $t \downarrow m$.

Considering trees $t \in \mathcal{T}_x$ to represent a form of nondeterministic computation [86], termination $t \downarrow M$ is to be read as "every possible run of the program yields one of the desired results", as which we view the elements of M.

Theorem 34. *Let $M \subseteq L$ be meet-closed, i.e. such that if $m, m' \in M$, then $m \wedge m' \in M$. For every $x \in L$, the following are equivalent.*

1. *There is $m \in M$ such that $m \leqslant jx$.*
2. *The trivial tree in $\mathcal{T}_x$ terminates in M.*
3. *There is a tree in $\mathcal{T}_x$ which terminates in M.*

Proof. By definition, the root-only tree labelled with x terminates in M precisely when there is $m \in M$ with $m \leqslant jx$. Reasoning inductively, consider the case in which a tree in $\mathcal{T}_x$ has been extended at the leaf of one of its paths π by children labelled with a and $\overline{a}$. If there are $m, m' \in L$ such that $\pi a \vdash m$ and $\pi\overline{a} \vdash m'$, then Lemma 32 yields $\pi \vdash m \wedge m' \in M$, and induction applies. $\square$

As singleton sets are trivially meet-closed, we obtain from Theorem 34 the following constructive counterpart to Corollary 25.

Corollary 35. *For every $m, x \in L$ we have $m \leqslant jx$ if and only if there is $t \in \mathcal{T}_x$ such that $t \downarrow m$.*

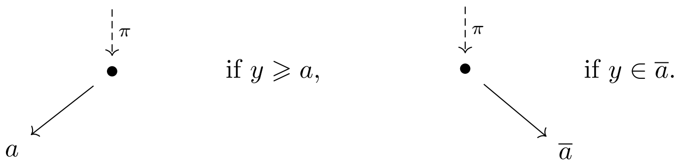
A slogan to capture Theorem 34 is that "membership in a radical element amounts to termination". The case of Scott-open filters is of particular interest, which leads to the following central result of the present paper, our constructive counterpart to Theorem 3.

Corollary 36. *Suppose that O is a filter. For every $x \in L$, the following are equivalent.*

1. *$x \in O$.*
2. *There is $t \in \mathcal{T}_x$ such that $t \downarrow O$.*

Proof. According to Theorem 34, there is $t \in \mathcal{T}_x$ terminating in O precisely when $m \leqslant jx$ for some $m \in O$, in which case monotonicity implies $jx \in O$, hence $x \in O$ by Proposition 11. $\square$

Remark 37. Much in the spirit of dynamical algebra [24, 51, 87, 88], every tree $t \in \mathcal{T}_x$ represents the course of a dynamic argument *as if* there were a complete y above x. Note that every element y of L above x which is indeed proper and complete gives rise to a path π through t. In fact, at each branching, corresponding to some $a \in L$, by way of completeness either $a \leqslant y$ or $O(a \vee y)$, according to which y leads in the respective direction to pursue, i.e. a or $\overline{a}$, accordingly:



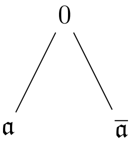
Suppose now that $\pi \vdash m$. We claim that $m \leqslant y$, which can be proved by induction on the definition of $\pi \vdash m$. In fact, $x \vdash m$ reads $m \sqsubseteq x$ which is to say that $m \leqslant jx \leqslant jy = y$. Consider next the case in which π has been extended along a branching with respect to some $a \in L$. Now either $\pi = x\pi' a$ and $a \leqslant y$, then $\pi \vdash m$ is $(x \vee a)\pi' \vdash m$, and since $x \vee a \leqslant y$ the induction hypothesis applies. If instead $\pi = x\pi'\overline{a}$ where $y \in \overline{a}$, then $(\forall b \in \overline{a})(x\pi' b \vdash m)$. In particular, $x\pi' y \vdash m$ which is to say that $(x \vee y)\pi' \vdash m$, to which again the induction hypothesis applies.

It follows that if $t \downarrow m$, then $m \leqslant \bigwedge \mathrm{Comp}^*/x$, and so $m \leqslant jx$ according to Corollary 25. However, in view of Corollary 35 we need not invoke this form of AC in order to conclude from termination that $m \leqslant jx$. It suffices that a suitable tree be available! Such trees can sometimes be read off corresponding classical arguments, provided they pass through finitely many case distinctions according to the completeness property. This strategy will next be illustrated by way of an example.

Example 38. We reconsider McCabe's short proof of Zariski's Lemma [54], following up on our discussion of McCabe's argument via inconsistency predicates [70].[10] Let $\mathbf{A}$ be a commutative ring *without zero-divisors* [51], i.e. such that

$$(\forall a, b \in \mathbf{A})(ab = 0 \to a = 0 \lor b = 0).$$

Essential part of the proof in question asserts that *if* $a \in \mathbf{A}$ *is not invertible, and such that the localization* $\mathbf{A}[a^{-1}]$ *is a field, then* $a \in \operatorname{Jac}(0)$, which is shown by an appeal to JIP/MIT: every maximal ideal contains a. Instead, folding up the corresponding tree, we obtain a constructive argument featuring the same computation:



where $\mathfrak{a} = \langle a \rangle$ denotes the principal ideal generated by a. Aiming at $\mathfrak{a} \leqslant j0$, which by Definition 30 is $0 \vdash \mathfrak{a}$, in view of Lemma 32 we need to check that $0\mathfrak{a} \vdash \mathfrak{a}$ (which is trivial), as well as that $0\overline{\mathfrak{a}} \vdash \mathfrak{a}$. The latter reduces to $\mathfrak{a} \sqsubseteq \mathfrak{b}$ for every ideal $\mathfrak{b}$ such that $\mathfrak{a} + \mathfrak{b} = \mathbf{A}$, which in turn amounts to $\mathfrak{a} \leqslant j\mathfrak{b}$, i.e. $a \in \operatorname{Jac}(\mathfrak{b})$. To see this, consider $b \in \mathfrak{b}$ and $r \in \mathbf{A}$ such that $ra + b = 1$. Since a is not invertible, $b \neq 0$ in $\mathbf{A}$. Next, because $\mathbf{A}[a^{-1}]$ is a field, either $b = 0$ in $\mathbf{A}[a^{-1}]$, which is to say that $a^n b = 0$ for some $n \geqslant 0$, and thus $a = 0 \in \operatorname{Jac}(\mathfrak{b})$ since $b \neq 0$ and $\mathbf{A}$ is without zero-divisors; or b is invertible in $\mathbf{A}[a^{-1}]$, whence there are $x \in \mathbf{A}$ and $n \geqslant 0$ such that $xb = a^n$. Now $a^n \in \operatorname{Jac}(\langle xb \rangle) \subseteq \operatorname{Jac}(\mathfrak{b})$, and so $a \in \operatorname{Jac}(\mathfrak{b})$ (Example 22).

5 Teichmüller–Tukey

We apply our approach to time-honoured incarnations of AC. In each of these cases we enable the reader to identify as an instance of Corollary 25 the form of AC under consideration, from which perspective it will be immediate to formulate the constructive counterpart as an instance of Theorem 34. A large part of the discussion to follow is thus set within $\mathbf{ZF(C)}$.

To start with, we deal with the *Teichmüller-Tukey Lemma* (TTL) [77, 78, 80]. Let L again be a complete lattice. Now, with an eye towards TTL, we say that a subset O of L is *of cofinite character* if, for every $x \in L$,

$$x \in O \quad \text{if and only if} \quad Kx \between O, \tag{10}$$

where $Kx = \{\, a \in KL \mid a \leqslant x \,\}$.

Remark 39. There is an order-preserving one-to-one correspondence between sets $O \subseteq L$ of cofinite character and monotone sets $R \subseteq KL$, given by the assignments

$$R_O = O \cap KL\,, \quad O_R = \{\, x \in L \mid Kx \between R \,\}\ .$$

Proposition 40. *Every* $O \subseteq L$ *of cofinite character is Scott-open. Conversely, in algebraic lattices the Scott-open subsets are precisely those of cofinite character.*

Proof. Let $O \subseteq L$ be of cofinite character. Clearly $x \leqslant y$ implies $Kx \subseteq Ky$, whence monotonicity of O is immediate in view of (10). To see that O is open, suppose that $D \subseteq L$ is directed and such that $\bigvee D \in O$. By (10), there is a compact $a \in O$ with $a \leqslant \bigvee D$, for which $a \leqslant d$ for some $d \in D$, and thus $d \in O$ by monotonicity of O.

For the converse, suppose that L is algebraic, and let $O \subseteq L$ be Scott-open. By monotonicity, $Kx \between O$ implies $x \in O$. Conversely, if $x \in O$, then, since $x = \bigvee Kx$ and O is Scott-open, we get $Kx \between O$. □

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[10] An elementary, constructive proof of Zariski's Lemma has recently been found by Wiesnet [84, 85].

To compute radicals in *algebraic* lattices L is an affair of compact elements only:

Corollary 41. *Let L be algebraic and let $O \subseteq L$ be of cofinite character. For every $y \in L$, the radical jy is the join of all the $x \in L$ such that*

$$(\forall a \in KL)(\, K(x \vee a) \between O \to K(y \vee a) \between O\,).$$

Proof. Combine Proposition 40, Lemma 14, Proposition 15 and (10). ☐

Example 42. The present approach generalises an earlier point-set treatment of abstract inconsistency predicates [70]. To see this, let S be a set, and let $R \subseteq \mathrm{Fin}(S)$ be monotone, the elements of which are to be considered "inconsistent". This extends to arbitrary subsets X of S by

$$O(X) \;\equiv\; \mathrm{Fin}(X) \between R.$$

As $K\,\mathrm{Pow}(S) = \mathrm{Fin}(S)$, this O is of cofinite character (Remark 39) and thus Scott-open (Proposition 40).

Example 43. Here we follow [70]. Let S be an arbitrary set with consequence relation $\rhd$ (Example 1). The predicate O on ideals I defined by

$$O(I) \;\equiv\; (\exists U \in \mathrm{Fin}(I))\, \langle U \rangle = S, \tag{11}$$

is of cofinite character and thus Scott-open (Proposition 40). For this choice of O and with classical logic, a proper ideal is O-complete if and only if it is maximal among proper ideals.

We briefly recall from [70] the following two instances. In both cases an ideal I is proper with respect to O (i.e. $\neg O(I)$) precisely when I is a proper ideal (i.e. $I \neq S$).

1. For the consequence relation $\rhd$ of ideals of a commutative ring, the corresponding instance of Corollary 25 is JIP/MIT, with j being the Jacobson radical Jac.
2. Let $\rhd$ stand for (deducibility in) intuitionistic logic in a propositional language S. With classical metalogic, i.e. in **ZF**, the (complete and proper) ideals are precisely the (complete and consistent) theories. The radical of a theory Γ has a conspicuously simple form [32],

   $$j\Gamma = \{\, \varphi \in S \mid \neg\neg\varphi \in \Gamma \,\},$$

   which is the least *stable* theory in S that contains Γ, and Corollary 25 yields a variant of Lindenbaum's Lemma [31,32], which prompts a proof of Glivenko's Theorem [32,36].

Example 44 (ZFC). Let S be a set, and let $\mathcal{F} \subseteq \mathrm{Pow}(S)$ be of *finite* character, that is,

$$T \in \mathcal{F} \quad \text{if and only if} \quad \mathrm{Fin}(T) \subseteq \mathcal{F}$$

for every $T \subseteq S$. Under this condition upon $\mathcal{F}$, the original form of TTL says that

every element of $\mathcal{F}$ is contained in a maximal one.

To gain back this form of TTL, we notice that

$$O(T) \;\equiv\; T \notin \mathcal{F}$$

is of cofinite character, and thus a Scott-open predicate on $\mathrm{Pow}(S)$. According to Theorem 3, by contraposition, for every $T \in \mathcal{F}$ (which is to say that $T \notin O$), there is an O-complete subset $C \notin O$ which contains T. This C is a maximal element of $\mathcal{F}$ just as asserted by TTL.

By way of Proposition 15 and with classical logic, the radical jT of a subset T of S consists of all elements $a \in S$ such that, for every $U \in \mathrm{Fin}(S)$,

$$U \cup T \in \mathcal{F} \quad \text{implies} \quad U \cup \{\, a \,\} \in \mathcal{F}.$$

According to Proposition 20.1 in the present context,

$$T \in \mathcal{F} \quad \text{if and only if} \quad jT \neq S,$$

which by Corollary 25 yields back TTL in **ZFC**. Equivalently, and still with some classical logic, a subset T of S is *not* a member of $\mathcal{F}$ precisely when $jT = S$, i.e. $\{ a \} \subseteq jT$ for every $a \in S$, which by Corollary 35 is the case precisely when for every $a \in S$ there is $t \in \mathcal{T}_T$ such that $t \downarrow \{ a \}$. Equivalently, there is $t' \in \mathcal{T}_T$ such that $t' \downarrow 1$, which in the present setting is $t' \downarrow S$.

The presumably most prominent application of TTL concerns vector space bases, to which we now turn our attention.

6 Dependence

Let S be a set and consider a consequence relation $\rhd$ (Example 43) with the *MacLane-Steinitz exchange property* [61] that $U, a \rhd b$ implies $U \rhd b$ or $U, b \rhd a$. The exchange property renders the corresponding closure operator $\langle - \rangle$ a *pregeometry* [25], or *finitary matroid* [19].[11]

Following, e.g. [48] we say that a subset T of S is *dependent*, for short $\mathrm{dep}(T)$, if there is $a \in T$ along with $U \in \mathrm{Fin}(T - \{ a \})$ such that $U \rhd a$. A subset that is not dependent is said to be *independent*. We say that T is *spanning* if $\langle T \rangle = S$. Note that $\emptyset$ is independent and that S is spanning, as well as that

$$\mathrm{dep}(\{ a \}) \quad \text{if and only if} \quad \emptyset \rhd a.$$

A *basis* is an independent spanning set. We write

$$\mathrm{Bases}/T$$

for the set of bases for S containing an independent subset T.

Rather than with the ideals of $\rhd$, we now work with arbitrary subsets T of S, i.e. in the complete algebraic lattice $L = \mathrm{Pow}(S)$ with $KL = \mathrm{Fin}(S)$, on which we consider

$$O(T) \;\equiv\; \mathrm{dep}(T). \tag{12}$$

This property is of cofinite character (simply replace T by $U \cup \{ a \}$ as above), and thus (Proposition 40) a Scott-open predicate on $\mathrm{Pow}(S)$. The proper elements with respect to O as in (12) are nothing but the independent subsets of S.

For later use, we need to make explicit the following lemma, which is classically trivial.

Lemma 45. *Let $a, x \in S$ and $T \subseteq S$. For every $U \in \mathrm{Fin}((T \cup \{ a \}) - \{ x \})$ there is $V \in \mathrm{Fin}(T - \{ x \})$ such that $U \subseteq V \cup \{ a \}$.*

Proof. We argue by induction on the number n enumerating U. The case $n = 0$, i.e. $U = \emptyset$ is clear. Next consider the case in which $U = U' \cup \{ b \}$ and the inductive hypothesis applies to U'. Accordingly, there is $V' \in \mathrm{Fin}(T - \{ x \})$ such that $U' \subseteq V' \cup \{ a \}$. Moreover, either $b \in T - \{ x \}$, whence $V = V' \cup \{ b \}$ will do; else $b = a$, in which case $U = U' \cup \{ b \} \subseteq V' \cup \{ a \}$ and $V = V'$ is as required. ☐

With respect to dependence (12) we have the following.

Lemma 46. *Every O-complete independent subset of S is spanning. Conversely, in* **ZF** *every spanning independent subset is O-complete.*

---

[11] Davide Rinaldi suggested to us that the exchange property be put in this rather than the more conventional form that $U, a \rhd b$ and $U \not\rhd b$ imply $U, b \rhd a$. However, this is not essential to pin down the related consequence of Theorem 24, for which we need to switch to **ZFC**, anyway.

Proof. Suppose that C is an O-complete independent subset of S. For every $a \in S$ either $a \in C$ anyway, by completeness, or else $\operatorname{dep}(C, a)$. In the latter case, there is $x \in C \cup \{ a \}$ and $U \in \operatorname{Fin}((C \cup \{ a \}) - \{ x \})$ such that $U \rhd x$. Now either $x = a$, which implies $a \in \langle C \rangle$. Else $x \in C$, and by Lemma 45 we can find $V \in \operatorname{Fin}(C - \{ x \})$ such that $V, a \rhd x$ by monotonicity of $\rhd$. By exchange, either $V \rhd x$, but this is impossible since C is independent, or $V, x \rhd a$, which again implies $a \in \langle C \rangle$. Thus $\langle C \rangle = S$.

Conversely, suppose that C is spanning, and let $A \subsetneq S$. Suppose that there is $a \in A - C$. Since C is spanning, there is $U \in \operatorname{Fin}(C)$ such that $U \rhd a$, which shows $\operatorname{dep}(A, C)$. ☐

In particular, with respect to dependence (12) and in **ZF** the proper complete elements of $\operatorname{Pow}(S)$ are nothing but the bases of S.

Next we consider the Jacobson radical with respect to dependence. From Proposition 20 we infer that

$$jT = S \quad \text{if and only if} \quad \operatorname{dep}(T),$$

hence $jT \neq S$ precisely when T is independent. In the present context Corollary 25 then reads as follows.

Corollary 47 (ZFC). *For every subset T of S,*

$$jT = \bigcap \mathrm{Bases}/T.$$

In particular, every independent set can be extended to a basis.

Proposition 15, moreover, instantiates to the following:

Proposition 48. *Let $T \subseteq S$ and $a \in S$. The following are equivalent.*

1. $a \in jT$.
2. $(\forall U \in \operatorname{Fin}(S))(\operatorname{dep}(a, U) \to \operatorname{dep}(T, U))$.

Example 49. Let $\mathbf{K}$ be a field[12] of characteristic $\neq 2$ and let S be a *discrete* $\mathbf{K}$-vector space, which amounts to demanding that for every $a \in S$ either $a = 0$ or $a \neq 0$ (which of course is classically trivial). Let $\rhd$ on S be given by linear span, i.e.

$$a_1, \ldots, a_k \rhd b \equiv (\exists \lambda_1, \ldots, \lambda_k \in \mathbf{K}) \sum_{i=1}^{k} \lambda_i a_i = b,$$

In this prime example of a consequence relation with the exchange property, the terms (in)dependence, spanning and basis have their customary meanings. For the related O as in (12) we make the following observations.

1. For every $T \subseteq S$, if $0 \in T$, then $\operatorname{dep}(T)$. In particular, $\operatorname{dep}(T)$ already if $0 \in jT$, by Proposition 48.
2. For every $a \in S$ we have $\operatorname{dep}(a, -a)$. This is by no. 1 for $a = 0$, whereas for $a \neq 0$ uses $a \neq -a$ (recall that $\operatorname{char}(K) \neq 2$).

   It is indeed necessary for $\operatorname{dep}(a, -a)$ that S be discrete: if $\operatorname{dep}(a, -a)$ and, say, $U \rhd a$ with U a finite subset of $\{ a, -a \} - \{ a \} = \{ -a \} - \{ a \}$, then either $U = \emptyset$ and thus $a = 0$, or else U is inhabited and thus $a \neq 0$ (if $a = 0$, then $a = -a$, so $U = \emptyset$).

---

[12] We assume that K is a *discrete* [51] or *geometric* [17] field, which is to say that every $x \in K$ is zero or invertible.

3. *Every independent* $T \subseteq S$ *is radical*, i.e. $jT = T$.

   To see this, let $a \in jT$. By no. 2 we have $\mathrm{dep}(a, -a)$, so $\mathrm{dep}(T, -a)$; whence there is $x \in T \cup \{ -a \}$ and a finite subset U of $(T \cup \{ -a \}) - \{ x \}$ such that $U \rhd x$.

   We now show $a \in \langle T \rangle$ in each of the two possibilities for x. First, if $x = -a$, then $U \subseteq T$ and thus $a \in \langle T \rangle$ because $a = -x \in \langle U \rangle \subseteq T$. Secondly, if $x \in T$, then, by Lemma 45 there is $V \in \mathrm{Fin}(T - \{ x \})$ such that $U \subseteq V \cup \{ -a \}$ and thus $V, -a \rhd x$. By exchange either $V \rhd x$ and thus $\mathrm{dep}(T)$, but $\neg\mathrm{dep}(T)$ by hypothesis, or $V, x \rhd -a$ and thus $a \in \langle T \rangle$.

   Hence $a \in \langle T \rangle$ in any case, i.e. there is $U_0 \in \mathrm{Fin}(T)$ such that $a \in \langle U_0 \rangle$. Now if $a \notin U_0$, then $\mathrm{dep}(U_0, a)$, and thus $\mathrm{dep}(T)$ since $a \in jT$, but $\neg\mathrm{dep}(T)$ by hypothesis. So $a \in U_0 \subseteq T$ by the stability of U_0.

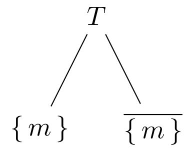
4. In all, for every $T \subseteq S$ the following are equivalent:
   (a) T is independent.
   (b) T is radical and $T \neq S$.
   (c) $jT \neq S$.

5. Let T and M be an independent and an arbitrary set of vectors in S. By Corollary 35 and no. 3 above, to check whether

   $$M \subseteq T = jT$$

   amounts to exhibiting a tree in $\mathcal{T}_T$ which terminates in $\{ M \}$, with the underlying heuristics that we can do *as if* T were a basis for S.

   In the specific case $M = \{ m \}$, to check whether $m \in T$ it suffices that the tree



   terminates in $\{ \{ m \} \}$, which is trivial for the left-hand branch, and for the right-hand branch amounts to have that $m \in j(T \cup X)$ for every $X \subseteq S$ with $\mathrm{dep}(X, m)$. So $m \in T$ means (Proposition 48) that

   $$\mathrm{dep}(X, m) \wedge \mathrm{dep}(U, m) \to \mathrm{dep}(U, T, X)$$

   holds for all $X \in \mathrm{Pow}(S)$ and $U \in \mathrm{Fin}(S)$.

By Example 49.3, in **ZFC** and by Corollary 47 *every independent set* T *is the intersection of all bases for* S *which contain* T:

$$T = \bigcap \mathrm{Bases}/T\,. \tag{13}$$

By contraposition, (13) yields the time-honoured form of AC going back to the perhaps first invocation after Zermelo by Hamel [38]: the special case of Corollary 47 that *every independent set* T *can be extended to a basis of* S, simply because $0 \notin T$. Recall how this follows from TTL: the independent subsets containing T form $\mathcal{F} \subseteq \mathrm{Pow}(S)$ of finite character; and every maximal independent subset of S is spanning.

With this paper we offer a universal toolbox for replacing, in concrete applications of maximality principles such as (13), the common indirect arguments based on forms of AC by direct elementary reasoning with, e.g. finite labelled binary trees, as indicated in Example 49.4.

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