On the number of weakly Noetherian constants of motion of nonholonomic systems

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Abstract

We develop a method to give an estimate on the number of functionally independent constants of motion of a nonholonomic system with symmetry which have the so called 'weakly Noetherian' property [22]. We show that this number is bounded from above by the corank of the involutive closure of a certain distribution on the constraint manifold. The effectiveness of the method is illustrated on several examples.

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1 Introduction

How many (functionally independent) constants of motion does a nonholonomic system with symmetry have? In these terms, this question is certainly too vague to be answered. For holonomic systems with symmetry, a natural answer is provided by the conservation of the energy–momentum map, which implies a lower bound on the number of constants of motion. In the nonholonomic case, instead, the situation is not as clear even in the simplest case of systems with linear constraints, natural Lagrangians (= kinetic minus potential energies) and lifted actions, which is the case that we consider in this paper. There are two opposite reasons for this:

- On the one hand, only certain components of the momentum map are constants of motion, and their number may be difficult to assess. In fact, a component of the momentum map is conserved if and only if its infinitesimal generator is a section of a certain distribution, the so–called *reaction–annihilator* distribution $\mathcal{R}^\circ$, see [23].
- On the other hand, the class of constants of motion of nonholonomic systems with symmetry may be larger than just the components of the momentum map. In particular, this class may include functions linear in the momenta known as "gauge momenta". These are, roughly speaking, constants of motion generated by certain vector fields which are tangent to the

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group orbits but are not infinitesimal generators of the group action, see [5, 33, 22]. There are different ways of characterizing the vector fields which produce conserved gauge momenta [28, 16, 39, 22], but here too, no general way of assessing their number is known.

In this situation, it is very difficult to answer in full generality the question raised above. Therefore, in this paper we try to provide a first, very partial answer to it, by producing a bound *from above* on the number of all functionally independent constants of motion with a certain property—the so–called *weak Noetherian* property identified in [22].

In order to appreciate the origin of this class of constants of motion recall that, in Hamiltonian mechanics, one of the fundamental properties of the link symmetries–conservation laws is the fact that the momentum map of a Hamiltonian action depends only on the group action. Therefore, the conservation of the momentum map provides conserved quantities for all systems with invariant Hamiltonian. This property is sometimes called the "Noetherian" property (or condition) of the momentum map [35].

In nonholonomic mechanics, however, which components of the momentum map are conserved quantities depends on the Hamiltonian (and on the constraint manifold), and the Noetherian property is broken. In fact, it is not even completely clear what a 'Noetherian constant of motion' should be in the nonholonomic context (see the Remark at the end of Section 3.A). Among functions linear in the momenta, some conserved components of the momentum map (particularly those with 'horizontal' infinitesimal generators, if they exist) do have the Noetherian property, but in general gauge momenta do not. However, certain gauge momenta—particularly those with horizontal generators—have a weaker property: they are conserved quantities of all nonholonomic systems with fixed constraint manifold and fixed kinetic energy, but any invariant potential energy [22]. This is the "weak Noetherian" property.

In [22], this property was defined only for constants of motion which are linear in the momenta, which is the case of the momentum maps of lifted actions and is most commonly met in examples. In this article, we consider instead *all* weakly Noetherian constants of motion (even the nonlinear ones, if they exist) with the aim of establishing a technique which can give some information on the number of them which are functionally independent. The reason why our method does not apply to the linear weakly Noetherian constants of motion alone will become clear later.

Specifically, we will show that, under a certain smoothness hypothesis, the number of functionally independent smooth *local* weakly Noetherian constants of motion of a nonholonomic system whose Hamiltonian is invariant under a lifted action is given by the corank of the involutive closure Δ^∞ of a certain distribution Δ on the constraint manifold. We will give a complete description of Δ in the case of free and proper actions, and we will recall a standard technique to compute the corank of Δ^∞ if Δ is real analytic. In order to show the feasibility of the procedure, we will apply it to a few sample cases in which this corank can be determined analytically (a vertical disk with various symmetry groups) and to a more complex case where the analysis has to be carried over using symbolic manipulation software (a heavy ball which rolls on a surface of revolution).

The number of *local* weakly Noetherian constants of motion gives of course only an *upper bound* on the number of *globally* defined constants of motion, which are those significant for the dynamics. Nevertheless, knowing this bound can give some informations on the system, but also, can confirm whether all constants of motion with this property have been determined. We will analyze this situation on the examples.

Our study uses in an essential way the Hamiltonian formulation of nonholonomic mechanics, which is shortly reviewed in Section 2. In Section 3 we introduce the notion of weakly Noetherian constants of motion and we shortly review the case of gauge momenta. After this background material, in Section 4 we relate the existence of local weakly Noetherian constants of motion to the corank of the distribution Δ^∞, and in Section 5 we give explicit expressions for the distribution Δ in the case of a free and proper lifted action. Sections 6 and 7 are devoted to the examples. A section of conclusions and perspectives follows; in particular, we outline there the use of our method in a different context—that of non symplectic symmetry groups of Hamiltonian systems.

2 Nonholomic systems

A. Lagrangian formulation. As already mentioned, we shall consider only the case of systems subject to linear noholonomic constraints. For general references, see [34, 12, 14, 7, 9] and references therein.

As a starting point, consider a holonomic mechanical system with n–dimensional configuration manifold Q and kinetic energy $T(v_q) = \frac{1}{2}a_q(v_q, v_q)$, where a is a Riemannian metric on TQ, subject to forces described by a potential energy which is the lift of a function V on Q. Then, the Lagrangian of the holonomic system is $L = T - V \circ \pi_Q$, where $\pi_Q : TQ \to Q$ is the tangent bundle projection.

For greater clarity, we resort to a coordinate description whenever appropriate. In doing so, we denote by $(q, \dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$ (local) bundle coordinates on TQ and by a dot the inner product in $\mathbb{R}^n$ and we use the symbol $\overset{\text{loc}}{=}$ to stress that the right–hand–side of an equality is an expression for a local representative of the left–hand–side. Thus, the kinetic energy is locally written

$$T(q, \dot{q}) \overset{\text{loc}}{=} \frac{1}{2}\dot{q} \cdot A(q)\dot{q}$$

for a symmetric and positive definite $n \times n$ matrix $A(q)$.

A *linear nonholonomic constraint* is given by a non–integrable distribution $\mathcal{D}$ on Q, that we assume to have constant rank r, $2 \leq r < n$. The distribution $\mathcal{D}$ is called the *constraint distribution* and can be locally defined as the kernel of $k := n - r$ linearly independent differential 1–forms $\tau_1, \ldots, \tau_k$ on Q, so that its fibers are

$$\mathcal{D}_q \;=\; \ker\big\{\tau_1(q), \ldots, \tau_k(q)\big\}\,. \tag{1}$$

In bundle coordinates $(q, \dot{q})$ in TQ, the fibers of $\mathcal{D}$ can be written as

$$\mathcal{D}_q \overset{\text{loc}}{=} \{\dot{q} \in \mathbb{R}^n \;:\; S(q)\dot{q} \;=\; 0\}$$

for a $k \times n$ matrix $S(q)$ with rank k, which depends smoothly on q. The constraint distribution can also be thought of as a submanifold D of TQ of dimension $2n - k$, which in bundle coordinates is given by

$$D \overset{\text{loc}}{=} \{(q, \dot{q}) \in \mathbb{R}^{2n} \;:\; q \in Q\,,\; \dot{q} \in \mathcal{D}_q\}$$

and will be called the (Lagrangian) constraint manifold.

D'Alembert's principle assumes that the reaction force exerted by the nonholonomic constraint annihilates the fibers of (an appropriate jet extension of) the distribution $\mathcal{D}$ and leads to a dynamical system on the submanifold D, which is described by Lagrange equations with multipliers. Eliminating the multipliers gives the reaction force as a function of the kinematical state $v_q \in D$ and leads to a vector field on the constraint manifold D, which gives the equations of motion. In bundle coordinates, the equations of motion have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = R \tag{2}$$

where $R = R(q, \dot{q})$ is the reaction force exerted by the nonholonomic constraint.

The expression of R in bundle coordinates can be found, e.g., in [1, 23, 6]. We recall this expression because we will use it in the examples of Section 6. Let $\Pi(q)$ be the $A(q)^{-1}$–orthogonal projector onto the orthogonal subspace of $\mathcal{D}_q = \ker S(q)$, that is

$$\Pi \;=\; S^T\big[SA^{-1}S^T\big]^{-1}SA^{-1}\,,$$

and define the vectors $\beta(q,\dot q)\in\mathbb{R}^n$, $V'(q)\in\mathbb{R}^n$ and $\gamma(q,\dot q)\in\mathbb{R}^k$ as having components

$$\beta_i \;=\; \sum_{j,h}\left(\frac{\partial A_{ij}}{\partial q_h}-\frac{1}{2}\frac{\partial A_{jh}}{\partial q_i}\right)\dot q_j\dot q_h\,,\quad V'_i \;=\; \frac{\partial V}{\partial q_i}\,,\quad \gamma_a \;=\; \sum_{j,h}\frac{\partial S_{aj}}{\partial q_h}\dot q_j\dot q_h$$

$(i,j,h=1,\dots,n,\ a=1,\dots,k)$. Then,

$$R \;=\; \Pi\beta - S^T\big[SA^{-1}S^T\big]^{-1}\gamma \;+\; \Pi V'\,. \tag{3}$$

B. Local Hamiltonian formulation. We pass now to the Hamiltonian formulation of nonholonomically constraint systems, see [40, 4, 32] for general references. We begin by giving a local description of this formulation, in Darboux coordinates (q,p) on T^*Q. We will use such local formulation for the computation of the distribution Δ in Section 5 and in the examples of Sections 6 and 7.

The Legendre transformation $\Lambda: TQ\to T^*Q$ relative to the Lagrangian $L=T-V\circ\pi_Q$ is a diffeomorphism and the image $M:=\Lambda(D)$ of the submanifold D of TQ is a submanifold (and a subbundle) of T^*Q of dimension $2n-k$, which will be called the (Hamiltonian) constraint manifold. In Darboux coordinates

$$\Lambda(q,\dot q)\ \stackrel{\rm loc}{=}\ (q,A(q)\dot q)$$

and

$$M \stackrel{\rm loc}{=} \{(q,p)\in\mathbb{R}^{2n}\,:\,q\in Q\,,\ A(q)^{-1}p\in\mathcal{D}_q\}\,.$$

The equations of motion of the nonholonomic system are Hamilton equations with the reaction force. The Hamiltonian is $H=T\circ\Lambda^{-1}+V\circ\pi_Q^*$, where $\pi_Q^*: T^*Q\to Q$ is the cotangent bundle projection. In Darboux coordinates, $H(q,p)\stackrel{\rm loc}{=}\frac12\,p\cdot A(q)^{-1}p+V(q)$ and the equations of motion are

$$\dot q \;=\; \frac{\partial H}{\partial p}(q,p)\,,\qquad \dot p \;=\; -\frac{\partial H}{\partial q}(q,p)+R\big(q,A(q)^{-1}p\big)\,.$$

These equations are the local representative of a dynamical system on the constraint manifold M, that we call the (Hamiltonian) *nonholonomic system* (H,M).

Note that it is possible to express the constraint manifold M also in terms of the 1–forms τ. In fact, the submanifold D of TQ can be (locally) described as the zero–level set of the map $\tilde\tau=(\tilde\tau_1,\dots,\tilde\tau_k): TQ\to\mathbb{R}^k$, where $\tilde\tau_i: TQ\to\mathbb{R}$ is given by $\tilde\tau_i(v_q)\ :=\langle\tau_i(q),v_q\rangle$ for all $v_q\in\mathcal{D}_q$, $q\in Q$. Therefore

$$M=\hat\tau^{-1}(0)$$

for

$$\hat\tau(q,p)\ :=\ (\tilde\tau\circ\Lambda^{-1})(q,p)\ \stackrel{\rm loc}{=}\ \ S(q)A(q)^{-1}p\,,$$

see [39].

C. Global Hamiltonian formulation. A local, coordinate description will not be sufficient for our analysis and we need a geometric formulation of the equations of motion. In view of this, we preliminarily recall some facts from symplectic geometry; see e.g. [31, 35] for details on these topics.

Let ω be the canonical symplectic form of T^*Q. As usual, X_F denotes the Hamiltonian vector field of a smooth function F on T^*Q, that is, $i_{X_F}\omega=-dF$. Similarly, the Hamiltonian vector field X_τ of a closed 1–form τ is defined by $i_{X_\tau}\omega=-\tau$. If E_m is a subspace of the tangent space $T_m(T^*Q)$, $m\in T^*Q$, then its symplectic orthogonal is defined as

$$E_m^\omega\ :=\ \{v\in T_mT^*Q\,:\,\omega_m(v,u)=0\quad\forall\,u\in E_m\,\}\,.$$

Recall that $(E_m^\omega)^\omega = E_m$ and that E_m and E_m^ω have complementary dimensions. The *polar* of a distribution $\mathcal{T}$ on T^*Q is the distribution $\mathcal{T}^\omega$ on T^*Q whose fibers are the symplectic orthogonals to the fibers of $\mathcal{T}$.

The Hamiltonian formulation of nonholonomic mechanics requires the introduction of a certain distribution $\overline{\mathcal{D}}$ on T^*Q, whose polar is related to the reaction force, see [4, 32, 39, 13, 14]. Specifically, at each point $m \in T^*Q$, the fiber $\overline{\mathcal{D}}_m \subseteq T_m(T^*Q)$ of $\overline{\mathcal{D}}$ is the preimage of the fiber $\mathcal{D}_{\pi_Q(m)}$ of the constraint distribution $\mathcal{D}$ under the derivative $T\pi_Q^*$ of the cotangent bundle projection $\pi_Q^* : T^*Q \to Q$. With reference to the representation (1) of $\mathcal{D}$, the fibers of $\overline{\mathcal{D}}$ are thus given by

$$\overline{\mathcal{D}}_m \;=\; \ker\left\{(\pi_Q^*)^*\tau_1(m), \ldots, (\pi_Q^*)^*\tau_k(m)\right\} \overset{\text{loc}}{=} \left\{(u_q, u_p) \in \mathbb{R}^n \times \mathbb{R}^n \;:\; u_q \in \mathcal{D}_{\pi_Q(m)}\right\} \tag{4}$$

for all $m \in T^*Q$. If, as we assume in this work, $\mathcal{D}$ has constant rank $n-k$, then $\overline{\mathcal{D}}$ has constant rank $2n-k$.

The distribution $\overline{\mathcal{D}}$ is defined on all of T^*Q, but we will need only its restriction to M. On the points $m \in M$, the fiber $\overline{\mathcal{D}}_m$ is distinct from the tangent space T_mM, even though they have the same dimension. In fact, T_mM is the annihilator of the differentials of the k functions $\hat{\tau}_a$ which, in Darboux coordinates, are the functions $[S(q)A(q)^{-1}p]_a$. Note also that, while M depends both on $\mathcal{D}$ and Λ, and thus on the kinetic energy T, $\overline{\mathcal{D}}$ and its polar $\overline{\mathcal{D}}^\omega$ depend only on $\mathcal{D}$.

If, as we assume in this work, the constraints are linear and ideal, then the Hamiltonian counterpart of the reaction force is a section $\overline{R}$ of (the restriction to M of) the polar distribution $\overline{\mathcal{D}}^\omega$, see e.g. [4, 32, 39, 13, 14]. The distribution $\overline{\mathcal{D}}^\omega$ has constant rank k and is locally generated by the vector fields $X_{(\pi_Q^*)^*\tau_1}, \ldots, X_{(\pi_Q^*)^*\tau_k}$. Correspondingly, the polar distribution $[T_mM]^\omega$ has constant rank k, too, and is locally generated by $X_{\hat{\tau}_1}, \ldots, X_{\hat{\tau}_k}$.

As proven in [4, 32] at each point $m \in M$ the subspaces T_mM and $\overline{\mathcal{D}}_m^\omega$ of T_mT^*Q are transversal and complementary, so that

$$T_m(T^*Q) \;=\; T_mM \oplus \overline{\mathcal{D}}_m^\omega \qquad \forall\, m \in M\,. \tag{5}$$

This implies that, at each point $m \in M$, there is a unique splitting $u = u^{TM} + u^{\overline{\mathcal{D}}^\omega}$ of vectors $u \in T_m(T^*Q)$, where for each subspace E of $T_m(T^*Q)$, $u^E \in E$. Correspondingly, any vector field X on T^*Q can be uniquely decomposed on the points of M as

$$X \;=\; X^{TM} + X^{\overline{\mathcal{D}}^\omega} \tag{6}$$

with X^{TM} a section of TM and $X^{\overline{\mathcal{D}}^\omega}$ a section of $\overline{\mathcal{D}}^\omega$.

Splitting (6) is used to write globally the equations of motion of nonholonomic systems [4, 32]. In fact, a constrained motion $t \mapsto m_t \in M$ satisfies $\dot{m} = X_H(m) + \overline{R}(m) = X_H^{TM}(m) + X_H^{\overline{\mathcal{D}}^\omega}(m) + \overline{R}(m)$. Hence, $\dot{m} \in T_mM$ and $\overline{R} \in \overline{\mathcal{D}}^\omega$ if and only if the reaction force satisfies $\overline{R} = -X_H^{\overline{\mathcal{D}}^\omega}{}_{|M}$ and $t \mapsto m_t$ satisfies the equation of motion

$$\dot{m} \;=\; X_H^{TM}(m)\,, \qquad m \in M\,.$$

The dynamical system on M defined by the vector field X_H^{TM} is the (Hamiltonian) nonholonomic system (H, M) we mentioned before.

Splitting (5) plays a central role also in the study of constants of motion [18, 39]; we will come back on this in Section 4.

Remark: For completeness, we mention here two facts that we will not use. (1) In the case of linear constraints, the vector field X_H^{TM} belongs to $\overline{\mathcal{D}}$ as well and is therefore a section of the distribution $\mathcal{H} = TM \cap \overline{\mathcal{D}}$ along M, see [4, 39]. As proven in [4, 16], the symplectic form restricted to $\mathcal{H}$ is nondegenerate and $T_m(T^*Q) = \mathcal{H}_m \oplus \mathcal{H}_m^\omega$ for $m \in M$. These facts play a central role in the reduction procedure developed in [4] and in the distributional Hamiltonian aproach to nonholonomic systems developed in [17, 15]. (2) The nonholonomic dynamics conserves the energy, that is, $X_H^{TM}(H) = 0$ in all of M.

3 Weakly Noetherian constants of motion

A. Definition. We introduce now the notion of weakly Noetherian constants of motion and review the case of gauge momenta. From now on, it is tacitly understood that (H, M) stands for a Hamiltonian nonholonomic system of the type specified in Section 2, which corresponds to a configuration manifold Q, a constraint distribution $\mathcal{D}$, a kinetic energy T and a potential energy V.

By a *constant of motion* of the nonholonomic system (H, M) we mean a smooth function $F : M \to \mathbb{R}$ such that the Lie derivative $X_H^{TM}(F) = 0$ in all of M. Given an action Ψ^Q of a Lie group G on Q, we denote by Ψ^{T^*Q} its lift to the cotangent bundle T^*Q.

Definition *Fix the configuration manifold Q, the constraint distribution $\mathcal{D}$ on Q, an action Ψ^Q of a Lie group G on Q, and a Ψ^{T^*Q}–invariant kinetic energy $T : T^*Q \to \mathbb{R}$. Then, a smooth function $F : M \to \mathbb{R}$ is said to be a* weakly Noetherian constant of motion relative to (T, M, Ψ^Q) *if it is a constant of motion of all nonholonomic systems $(T + V \circ \pi_Q^*, M)$ with Ψ^Q–invariant potential energy $V : Q \to \mathbb{R}$.*

The restriction to lifted actions in this Definition is not necessary, but this is the case we will consider in the sequel.

Remark: Notions of Noetherian and of weakly Noetherian constants of motion of nonholonomic systems were given in [22] in the special case of functions linear in the momenta. The treatment there was in the Lagrangian context but, since under our hypotheses the Legendre transformation is a diffeomorphism, everything transfers to the Hamiltonian context. However, there is a subtlety involved in extending the notion of Noetherianity to nonlinear functions, and this extension cannot be achieved by just dropping the constancy of the kinetic energy T in the previous Definition. The fact is that, in a mechanical context, one starts with a given *Lagrangian* constraint manifold D and (even assuming that this operation could be done without changing D, which might be unrealistic) changing the kinetic energy changes the Legendre transformation Λ and hence the Hamiltonian constraint manifold $M = \Lambda(D)$. Since M is the natural domain of definition of the constants of motion, it is not completely clear what should the appropriate definition of Noetherian constant of motion be, if any. If one considers only functions which are defined in all of T^*Q, then 'horizontal momenta' (in the sense of the next subsection) are obviously Noetherian [22]. Note that this difficulty cannot be overcome by simply working in the Lagrangian context, where the constraint manifold D is fixed, because changing the kinetic energy changes the pull–back of J to TQ.

B. Weakly Noetherian gauge momenta. In order to give some perspective for the subsequent treatment we review very quickly the properties of weak Noetherianity of the class of constants of motion which has been more extensively studied, that formed by momenta and gauge momenta. For motivations, details and examples see [22, 24] and references therein.

By a 'linear' function on M we mean the restriction to M of a function on T^*Q which is linear in the momenta, that is, which can be written as

$$F = p \cdot \xi^Q|_M$$

with some vector field ξ^Q on Q. The vector field ξ^Q is called a generator of F and, since $M \subset T^*Q$, it is not unique (see [28, 22, 23] for further details). Fix now an action Ψ^Q of a Lie group G on Q and let $\mathcal{G}$ denote the distribution on Q whose fibers $\mathcal{G}_q$ are the tangent spaces $T_q\mathcal{O}_q$ to the orbits $\mathcal{O}_q$ of Ψ^Q. A linear function on M is called a *gauge momentum* relative to the action Ψ^Q if it has a generator ξ^Q with the following two properties: (1) it is a section of $\mathcal{G}$, and (2) its cotangent lift ξ^{T^*Q} infinitesimally preserves the Hamiltonian in M, namely $\xi^{T^*Q}(H)|_M = 0$. Any such generator is called a *gauge simmetry*. (Note that a gauge momentum may have as well generators which do not satisfy either condition and are not gauge symmetries). If the gauge symmetry ξ^Q is an infinitesimal generator of the action Ψ^Q, then the corresponding gauge momentum is a component of the momentum map of the lifted action Ψ^{T^*Q} and will be called a *momentum*. Therefore, in the sequel we refer only to gauge momenta. A gauge momentum is said to be *horizontal* if it is generated by a horizontal gauge symmetry, that is, a gauge symmetry which is a section of the constraint distribution $\mathcal{D}$.

A gauge momentum is a constant of motion of the nonholonomic system (H, M) if and only if it is generated by a gauge symmetry which is a section of a certain distribution $\mathcal{R}^{\circ}_{\mathcal{D},T,V}$ on Q [23, 22]. This distribution is called the *reaction–annihilator* distribution, depends on $\mathcal{D}$, T and V and is an overdistribution of the constraint distibution $\mathcal{D}$, that is, its fibers contain those of $\mathcal{D}$. This implies, in particular, the very well known fact that all horizontal momenta and gauge momenta are constants of motion [30, 3, 16, 8, 13, 5, 33, 7, 19, 15]. This also implies that *a gauge momentum is weakly Noetherian if and only if it is generated by a gauge symmetry ξ^Q which is a section of the distribution $\cap_V \mathcal{R}^{\circ}_{\mathcal{D},T,V}$ which is obtained by taking the intersection of all the distributions $\mathcal{R}^{\circ}_{\mathcal{D},T,V}$ corresponding to Ψ^Q–invariant potential energies V.*

Since $\cap_V \mathcal{R}^{\circ}_{\mathcal{D},T,V}$ is an overdistribution of $\mathcal{D}$, this implies in particular that *horizontal momenta and horizontal gauge momenta are weakly Noetherian* [22]. Moreover, if the action is transitive on Q, or more generally if $\mathcal{G}$ is an overdistribution of $\mathcal{D}$, that is, $\mathcal{G}_q \supseteq \mathcal{D}_q$ for all $q \in Q$, then *any* conserved gauge momentum is horizontal ([22], Proposition 4) and hence weakly Notherian. However, the class of weakly Noetherian momenta and gauge momenta might be larger than the horizontal ones, a possibility which might be particularly important in cases such as the generalized Chaplygin systems, for which $\mathcal{D}_q \cap \mathcal{G}_q$ is trivial.

Remark: The number of functionally independent horizontal gauge momenta (or, equivalently, of functionally independent constants of motion identified by the momentum equation [22]) is not bounded from above by the rank of the distribution with fibers $\mathcal{G}_q \cap \mathcal{D}_q$, unless this rank is either zero or one. In fact, given two or more vector fields on Q, lifting their linear combinations with coefficient functions produce vector fields on TQ which need not belong to the span of the lifts of the given vector fields. For a (holonomic) example of a two–dimensional Lie group with three indipendent horizontal gauge momenta see Section 6 of [22].

4 On the number of weakly Noetherian constants of motion.

A. First integrals of distributions. As explained in the Introduction, our aim is to develop a method to compute an upper bound on the number of functionally independent weakly Noetherian constants of motion. The key will be regarding these functions as 'first integrals' of a certain distribution. We thus begin this analysis by introducing the notion of first integrals of a distribution and studying some of their properties.

For all notions and properties relative to distributions, and in particular for the definition of smooth and real analytic distributions, we refer to [31, 35] (which call however generalized distribution what we call here distribution) and to [29, 12].

Given two distributions $\mathcal{T}$ and $\mathcal{S}$ on a manifold M, we say that $\mathcal{T}$ is an *over–distribution* of $\mathcal{S}$, and write $\mathcal{T} \supseteq \mathcal{S}$, if their fibers satisfy $\mathcal{T}_m \supseteq \mathcal{S}_m$ for all $m \in M$. We use the term *under-distribution* in a similar way.

We say that a smooth function $F : M \to \mathbb{R}$ is an *first integral* of a distribution $\mathcal{T}$ on a manifold M if the distribution with fibers $\ker dF(m)$ is an over–distribution of $\mathcal{T}$, that is,

$$\ker dF(m) \supseteq \mathcal{T}_m \qquad \forall\, m \in M\,. \tag{7}$$

A *local first integral* of $\mathcal{T}$ is an first integral of the restriction of $\mathcal{T}$ to some open subset of M.

The notion of first integrals of a distribution generalizes in an obvious way that of first integrals (or constants of motion) of a vector field. Even though this notion seems to be rather natural, we have not been able to find any reference to it in the literature; therefore, we collect here a few properties of these objects. Roughly speaking, the relevant fact is that the first integrals of a distribution are constant on the (Stefan–Sussmann) orbits of the distribution, which, for non integrable distributions, have dimensions larger than the rank of the distribution. Thus, first integrals of a distribution are first integrals of the foliation that it generates, and the dimension of the orbits puts a bound on the number of functionally independent first integrals which is stricter than that given by the rank of the distribution. We give now a formal, and computationally more convenient, 'infinitesimal' statement of this fact.

Preliminarily we recall that, given a smooth distribution $\mathcal{T}$ there exists a unique smooth over–distribution $\mathcal{T}^\infty$ of $\mathcal{T}$ which is involutive and is minimal among all smooth involutive over–distribution of $\mathcal{T}$. This distribution is called the *involutive closure* of $\mathcal{T}$. If $\mathcal{T}$ is real analytic, then $\mathcal{T}^\infty$ is real analytic and there is a standard way to compute $\mathcal{T}^\infty$, which is based on taking commutators, see Section 4.C. For some details, see [29, 12] and references therein.

Proposition 1. *Let $\mathcal{T}$ be a smooth distribution on M and $U \subset M$ an open set. A smooth function $F : U \to \mathbb{R}$ is a first integral of $\mathcal{T}|_U$ if and only if it is a first integral of $\mathcal{T}^\infty|_U$.*

Proof We use the following notation: if $\mathcal{D}$ is a (not necessarily smooth) distribution, then $\mathrm{smt}(\mathcal{D})$ denotes the largest smooth under–distribution of $\mathcal{D}$ [29].

Preliminarily, we prove that $(\mathcal{T}|_U)^\infty = \mathcal{T}^\infty|_U$. One inclusion is obvious: $\mathcal{T}^\infty|_U$ is a smooth involutive over–distribution of $\mathcal{T}|_U$ and hence is an over–distribution of $(\mathcal{T}|_U)^\infty$. Vice–versa, let $\mathcal{B}$ be the distribution on M which coincides with $(\mathcal{T}|_U)^\infty$ on U and has fibers T_mM for $m \notin U$. Therefore $\mathrm{smt}(\mathcal{B})|_U = (\mathcal{T}|_U)^\infty$ because the smoothing operation does not change the fibers of a smooth distribution. On the other hand, $\mathrm{smt}(\mathcal{B})$ is a smooth over–distribution of $\mathcal{T}$ which is also involutive. Hence, $\mathrm{smt}(\mathcal{B}) \supseteq \mathcal{T}^\infty$ and $(\mathcal{T}|_U)^\infty = \mathrm{smt}(\mathcal{B})|_U \supseteq \mathcal{T}^\infty|_U$.

We thus write $\mathcal{T}|_U^\infty$ for $(\mathcal{T}_U)^\infty = \mathcal{T}^\infty|_U$. If F is a first integral of $\mathcal{T}|_U^\infty$ then it is a first integral of $\mathcal{T}|_U \subseteq \mathcal{T}|_U^\infty$. Conversely, assume that F is a first integral of $\mathcal{T}|_U$, that is, $\ker dF \supseteq \mathcal{T}|_U$. The distribution $\ker dF$ need not be smooth. Nevertheless, since $\mathcal{T}|_U$ is smooth, $\ker dF \supseteq \mathrm{smt}(\ker dF) \supseteq \mathcal{T}|_U$. Since the commutator of any two smooth sections of $\ker dF$ is a smooth section of $\ker dF$, the distribution $\mathrm{smt}(\ker dF)$ is involutive. That $\ker dF \supseteq \mathcal{T}|_U^\infty$ follows now from the minimality of $\mathcal{T}|_U^\infty$ among the smooth involutive over–distributions of $\mathcal{T}|_U$, which implies $\mathrm{smt}(\ker dF) \supseteq \mathcal{T}|_U^\infty$. ■

For each $m \in M$, define

$$c_m(\mathcal{T}^\infty) \ := \ \mathrm{corank}\, \mathcal{T}^\infty_m \, .$$

If $\mathcal{T}$ is a smooth distribution, so is $\mathcal{T}^\infty$ and the set M^∞_{reg} of its regular points is open and dense in M [12] (a point is said to be regular if the the distribution has constant rank in a neighbourhood of it). Thus, Proposition 1 implies

Proposition 2. *If $\mathcal{T}$ is a smooth distribution on a manifold M, then each point $m \in M^\infty_{\mathrm{reg}}$ has a neighbourhood in which there exist $c_m(\mathcal{T}^\infty)$, but not $c_m(\mathcal{T}^\infty)+1$, functionally independent first integrals of $\mathcal{T}$.*

Proof In a neighbourhood U of a regular point m of $\mathcal{T}^\infty$ the smooth distribution $\mathcal{T}|_U^\infty$ has constant rank and, being involutive, defines a smooth regular foliation of U with leaves of codimension $c_m(\mathcal{T}^\infty)$; in any set of local coordinates adapted to this foliation, the coordinates transversal to the leaves are first integrals of the restriction of $\mathcal{T}^\infty$ to the coordinate domain. ■

B. On the number of weakly Noetherian constants of motion. We consider now a nonholonomic Hamiltonian system (H, M) of the type considered so far and, as a first step, we give a characterization of its constants of motion as first integrals of a certain distribution on the constraint manifold M. (From now on we shall consider only objects which are defined on M and we shall therefore work exclusively in M, not anymore in T^*Q).

We will use the following notation. Given a finite number of smooth vector fields $X_1, \ldots, X_d$ on M, $\langle X_1, \ldots, X_d\rangle_m$ or $\langle X_1(m), \ldots, X_d(m)\rangle$ denotes the subspace of T_mM spanned by $X_1(m), \ldots, X_d(m)$. Moreover, $\langle X_1, \ldots, X_d\rangle$ denotes the distribution with fibers $\langle X_1, \ldots, X_d\rangle_m$.

Proposition 3. *A smooth function $F : M \to \mathbb{R}$ is a constant of motion of a nonholonomic system (H, M) if and only if*

$$\ker dF(m) \supseteq \big(\ker dH(m) \cap \overline{\mathcal{D}}_m\big)^{\omega} \cap T_mM \qquad \forall\, m \in M\,. \tag{8}$$

Proof A function F on M is a constant of motion of (H, M) if and only if $\langle dF, X_H^{TM}\rangle = 0$ at all points of M, that is,

$$\ker dF(m) \supseteq \big\langle X_H^{TM}\big\rangle_m \qquad \forall\, m \in M\,.$$

Note that the TM–component u^{TM} of a vector $u \in T_m(T^*Q)$ is given by $(u+\overline{\mathcal{D}}_m^{\omega}) \cap T_mM$. Hence, $\langle X_H^{TM}\rangle_m = (\langle X_H\rangle_m + \overline{\mathcal{D}}_m^{\omega}) \cap T_mM$. Recalling that X_H is symplectically orthogonal to $\ker dH$ we thus have $\langle X_H^{TM}\rangle_m = [(\ker dH(m))^{\omega} + \overline{\mathcal{D}}_m^{\omega}] \cap T_mM = (\ker dH(m) \cap \overline{\mathcal{D}}_m)^{\omega} \cap T_mM$. ∎

Remarks: (i) Characterization (8) of constants of motion of nonholonomic systems is in a sense dual to characterizations given in [16, 18, 39]. Since (5) implies $T_m(T^*Q) = (T_mM)^{\omega} \oplus \overline{\mathcal{D}}_m$ for all $m \in M$, on the points of M there is a splitting $X = X^{TM^{\omega}} + X^{\overline{\mathcal{D}}}$ of vector fields on T^*Q which is dual to (6). As proven in [39], a function $F : M \to \mathbb{R}$ is a constant of motion of (H, M) if and only if for one, and therefore any, extension $\tilde{F}$ of F off M,

$$X_{\tilde{F}}^{\overline{\mathcal{D}}}(H)_{|M} \;=\; 0\,. \tag{9}$$

Note that this condition is equivalent to

$$X_{\tilde{F}}^{\overline{\mathcal{D}}}(m) \;\in\; \big(\ker dH(m) \cap \overline{\mathcal{D}}_m\big) + T_mM^{\omega} \qquad \forall\, m \in M\,,$$

which can be compared to (8). A reformulation of condition (9) which, like condition (8), avoids the use of extensions of functions off M can be given in terms of distributional Hamiltonian vector fields [15].

(ii) A function F is a constant of motion of a Hamiltonian vector field X_H if and only if either $X_H(F) = 0$ or, equivalently, $X_F(H) = 0$. Condition (9) is manifestly an analogue of the latter condition. Instead, (8) is an analogue of condition $X_H(F) = 0$, that is, $\ker dF \supseteq \langle X_H\rangle$. In fact, if there are no nonholonomic constraints, then $M = T^*Q$, $\overline{\mathcal{D}} = T(T^*Q)$ and therefore $\big(\ker dH \cap \overline{\mathcal{D}}\big)^{\omega} \cap TM = (\ker dH)^{\omega} = \langle X_H\rangle$.

Based on Proposition 3, we can now give the following characterization of weakly Noetherian constants of motion:

Proposition 4. *Fix the configuration manifold Q, the constraint distribution $\mathcal{D}$ on Q, an action Ψ^Q of a Lie group G on Q, and a Ψ^{T^*Q}–invariant kinetic energy $T : T^*Q \to \mathbb{R}$. Let I_G be the set of all Ψ^Q–invariant smooth real functions on Q. For each $m \in M$ define*

$$\mathcal{I}_m \;:=\; \bigcap_{V \in I_G} \ker\big(dT(m) + d(V \circ \pi_Q^*)(m)\big) \tag{10}$$

$$\Delta_m \;:=\; \big(\mathcal{I}_m \cap \overline{\mathcal{D}}_m\big)^{\omega} \cap T_mM\,. \tag{11}$$

Then, a smooth function $F : M \to \mathbb{R}$ is a weakly Noetherian constant of motion relative to (T, M, Ψ^Q) if and only if

$$\ker dF(m) \supseteq \Delta_m \qquad \forall\, m \in M\,. \tag{12}$$

Proof A function F is a weakly Noetherian constant of motion if and only if, at each point $m \in M$, inclusion (8) is satisfied for all $H = T + V \circ \pi_Q^*$ with $V \in I_G$. Omitting for shortness the indication of the point m, this is equivalent to $(\ker dF)^\omega \subseteq \big(\ker d(T + V \circ \pi_Q^*) \cap \overline{\mathcal{D}}\big) + TM^\omega$ for all $V \in I_G$, namely

$$(\ker dF)^\omega \subseteq \bigcap_{V \in I_G} \big[\big(\ker d(T+V) \cap \overline{\mathcal{D}}\big) + TM^\omega\big] \ .$$

Since the transversality of $\overline{\mathcal{D}}^\omega$ and TM implies that of $\overline{\mathcal{D}}$ and TM^ω, the subspace $\bigcap_{V \in I_G} \big[\big(\ker d(T+V) \cap \overline{\mathcal{D}}\big) + TM^\omega\big]$ equals $(\mathcal{I} \cap \overline{\mathcal{D}}) + TM^\omega$. ■

Let Δ be the distribution on M with fibers $\Delta_m \subseteq T_m M$ as in (12). In view of (7), Proposition 4 means that, if Δ is smooth, then a function $F : M \to \mathbb{R}$ is a weakly Noetherian constant of motion if and only if it is a first integral of the distribution Δ. Therefore, if $c_m(\Delta^\infty)$ denotes the codimension of the fiber Δ_m^∞ in $T_m M$, that is,

$$c_m(\Delta^\infty) \ = \ \dim M - \operatorname{rank} \Delta_m^\infty \ = \ (2n-k) - \operatorname{rank} \Delta_m^\infty \, ,$$

then by Proposition 2 *in a neighbourhood of each regular point m of Δ^∞ there exist $c_m(\Delta^\infty)$, but not $c_m(\Delta^\infty)+1$, functionally independent weakly Noetherian constants of motion relative to* (T, M, Ψ^Q). From a dynamical point of view, the existence of local constants of motion of a dynamical system has no particular significance—the rectification theorem ensures the existence of the maximal number of them. Thus, this fact cannot be considered as an existence result of weakly Noetherian constants of motion, but an upper bound on their number:

Theorem 1. *Under the hypotheses of Proposition 2, every set of functionally independent global weakly Noetherian smooth constants of motion relative to (T, M, Ψ^Q) contains at most $c(\Delta^\infty) := \min_{m \in M} c_m(\Delta^\infty)$ elements.*

In practice, however, the estimate provided by Theorem 1 may be used as a guide for the search of as many weakly Noetherian constants of motion as possible, including horizontal gauge momenta.

Of course, the effectiveness of this approach depends on the possibility of constructing the distribution Δ^∞, which in turn requires the knowledge of Δ. As we review in the next subsection, there is a standard procedure for the construction of Δ^∞ if the distribution Δ is *real analytic* (see e.g. Chapter 3 of [12] for details and references). The construction of the distribution Δ will be done in Section 5 under suitable assumptions on the actions Ψ^Q.

C. Construction of Δ^∞. We assume now that, as it happens in typical cases, the distribution Δ is real analytic. As we have already mentioned, there a standard way of constructing the involutive closure of a real analytic distribution [12]. Since any real analytic distribution is locally finitely generated, in any sufficiently small open set $U \subseteq M_{\rm reg}^\infty$ there are $r = \operatorname{rank} \Delta$ independent vector fields $X_1, \ldots, X_d$ such that $\Delta^1 := \Delta_{|U}$ has fibers $\Delta_m^1 = \big\langle X_1, \ldots, X_d \big\rangle_m$, that is,

$$\Delta^1 \ = \ \langle X_1, \ldots, X_d \rangle \ .$$

Define

$$\Delta^{s+1} \ := \ \Delta^s \cup [\Delta^s, \Delta^s] \, , \qquad s = 1, \ldots, n-1 \, ,$$

where $[\Delta^s, \Delta^s]$ is the distribution whose sections are all the commutators $[\eta, \xi]$ for η, ξ sections of Δ^s, that is,

$$\begin{aligned} \Delta^2 \ &= \ \langle X_1, \ldots, [X_1, X_2], \ldots \rangle \\ \Delta^3 \ &= \ \langle X_1, \ldots, [X_1, X_2], \ldots, [X_1, [X_1, X_2]], \ldots, [[X_1, X_2], [X_1, X_3]], \ldots \rangle \end{aligned}$$

etc. Thus, $\Delta^\infty = \Delta^s$ where s is the smallest positive integer such that Δ^s is integrable or, in other words, the smallest positive integer such that $\Delta^s = \Delta^{s+1}$.

Thus, in order to apply the criterion of Theorem 1 to a specific nonholonomic system with analytic distribution Δ, it suffices to find a system of generators of Δ in a neighbourhood of each of its regular points and then compute sufficiently many Lie brackets of these generators, checking for linear independence.

This procedure can be implemented in two ways. One is of course that of parameterizing the constraint manifold M and writing the generators of Δ using the chosen local coordinates. The other is that of making all computations using extensions off M of the chosen local generators $X_1, \ldots, X_d$ of Δ. In fact, if $\tilde{X}_1, \ldots, \tilde{X}_d$ are extensions of $X_1, \ldots, X_d$ off M, then

$$[\tilde{X}_a, \tilde{X}_b]_{|M} \;=\; [X_a, X_b] \qquad \forall\, a, b$$

for a known property of related vector fields, see e.g. [35]. We will use both methods in the examples below.

5 The distribution Δ for free and proper actions

A. Determination of Δ. The determination of the distribution Δ requires the determination of the distribution $\mathcal{I}$ as in (10). This is an easy task if the action is proper and free, and we limit ourselves to this case.

As above, we denote by $\mathcal{G}$ the distribution on Q whose fibers $\mathcal{G}_q$ are the tangent spaces $T_q\mathcal{O}_q$ to the orbits $\mathcal{O}_q$ of the action Ψ^Q. For our purposes it is sufficient to determine Δ in Darboux coordinates (q,p). Thus, we identify the tangent spaces $T_{(q,p)}T^*Q$ with $\mathbb{R}^{2n}$ (or with $\mathbb{R}^n \times \mathbb{R}^n$, when we want to stress the individual roles of the q– and p–components of the tangent vectors), we identify the spaces $\mathcal{G}_q = T_q\mathcal{O}_q$ and $\mathcal{D}_q$ with subspaces of $\mathbb{R}^n$ and we identify the spaces $T_{(q,p)}M$, $\Delta_{(q,p)}$ and $\mathcal{I}_{(q,p)}$ with subspaces of $\mathbb{R}^{2n}$. In the sequel, the dot denotes the scalar product in $\mathbb{R}^n$ and, if E is a subspace of $\mathbb{R}^n$, then $E^\perp$ denotes its orthogonal complement in $\mathbb{R}^n$; if $v \in \mathbb{R}^n$, then $v^\perp$ stands for $\langle v\rangle^\perp$. We will distinguish between row and column vectors only when some ambiguity might arise. As before, we denote by $A(q)$ the kinetic matrix, so that the kinetic energy is $T(q,p) = \frac{1}{2}p \cdot A(q)^{-1}p$.

Lemma 1. *In addition to the assumptions of Proposition 4, assume that the action Ψ^Q is free and proper. Then, at each point $(q,p) \in M$,*

$$\mathcal{I}_{(q,p)} \;=\; \left\{(v_q, v_p) \in \mathbb{R}^n \times \mathbb{R}^n \;:\; v_q \in \mathcal{G}_q\,,\; v_p \in {T'_p}^\perp - \frac{v_q \cdot T'_q}{\|T'_p\|^2}\, T'_p\right\} \tag{13}$$

where

$$T'_p \;:=\; \frac{\partial T}{\partial p}(q,p) \;=\; A(q)^{-1}p \qquad \text{and} \qquad T'_q \;:=\; \frac{\partial T}{\partial q}(q,p)\,. \tag{14}$$

Proof A vector $v = (v_q, v_p) \in \mathcal{I}_{(q,p)}$ if and only if $d(T+V)v = 0$ for all $V \in I_G$, namely if and only if $\frac{\partial V}{\partial q} \cdot v_q = 0$ for all $V \in I_G$ and $T'_q \cdot v_q + T'_p \cdot v_p = 0$. If the action is free and proper, then the condition

$$\frac{\partial V}{\partial q} \cdot v_q = 0 \qquad \forall\, V \in I_G$$

is equivalent to $v_q \in \mathcal{G}_q$ (see e.g. Theorem 2.5.10 of [35]). If $p = 0$ then $T'_p = T'_q = 0$ and there are no conditions on v_p; this is consistent with (13) because in this case ${T'_p}^\perp = \mathbb{R}^n$ and $T'_q = 0$. If $p \neq 0$ then $v_p \in {T'_p}^\perp - \|T'_p\|^{-2}(v_q \cdot T'_q)T'_p$. ■

Lemma 2. *Under the hypotheses and with the notation of Lemma 1, at any point $(q,p) \in M$ a vector $(u_q, u_p) \in \mathbb{R}^n \times \mathbb{R}^n$ belongs to $\big[\mathcal{I}_{(q,p)} \cap \overline{\mathcal{D}}_{(q,p)}\big]^\omega$ if and only if*

$$u_q \in \langle T'_p \rangle \qquad \text{and} \qquad u_p \in [\mathcal{D}_q \cap \mathcal{G}_q]^\perp - \frac{u_q \cdot T'_p}{\|T'_p\|^2}\, T'_q \tag{15}$$

with T'_p and T'_q as in (14).

Proof Formulas (4) and (13) imply

$$\mathcal{I}_{(q,p)} \cap \overline{\mathcal{D}}_{(q,p)} \;=\; \left\{ (v_q, v_p) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, v_q \in \mathcal{D}_q \cap \mathcal{G}_q\,,\; v_p \in T'^{\perp}_p - \frac{T'_q \cdot v_q}{\|T'_p\|^2}\, T'_p \right\}. \tag{16}$$

Thus, a vector $(u_q, u_p) \in \mathbb{R}^n \times \mathbb{R}^n$ is symplectically orthogonal to $\mathcal{I}_{(q,p)} \cap \overline{\mathcal{D}}_{(q,p)}$ if and only if $u_q \cdot v_p = u_p \cdot v_q$ for all $v_q \in \mathcal{D}_q \cap \mathcal{G}_q$ and all $v_p \in T'^{\perp}_p - (T'_q \cdot v_q)\|T'_p\|^{-2} T'_p$, that is, if and only if

$$u_q \cdot \left(T'^{\perp}_p - \frac{v_q \cdot T'_q}{\|T'_p\|^2}\, T'_p \right) = u_p \cdot v_q \qquad \forall v_q \in \mathcal{D}_q \cap \mathcal{G}_q\,. \tag{17}$$

When $v_q = 0$ this equality reduces to $u_q \cdot T'^{\perp}_p = 0$, which gives $u_q = \lambda T'_p$ for some $\lambda \in \mathbb{R}$. For this value of u_q, (17) becomes $(\lambda T'_q + u_p) \cdot v_q = 0$ for all $v_q \in \mathcal{D}_q \cap \mathcal{G}_q$. This shows that

$$[\mathcal{I}_{(q,p)} \cap \overline{\mathcal{D}}_{(q,p)}]^\omega \;=\; \{ (u_q, u_p) \,:\, u_q = \lambda T'_p\,,\; u_p \in [\mathcal{D}_q \cap \mathcal{G}_q]^\perp - \lambda T'_q\,,\; \lambda \in \mathbb{R} \}\,.$$

Expressing λ in terms of u_q and T'_p leads to (15). ■

We can now complete the characterization of Δ. Note that, since A is an invertible $n \times n$ matrix and S is a $k \times n$ matrix with rank k, $SA^{-1}S^T$ is an invertible $k \times k$ matrix. Note also that, if the vector subspace $\big[\mathcal{D}_q \cap \overline{\mathcal{G}}_q\big]^\perp$ of $\mathbb{R}^n$ is k–dimensional and has a basis formed by vectors $w_1, \ldots, w_k$ of $\mathbb{R}^n$, then the vector subspace $\big[\mathcal{I}_{(q,p)} \cap \overline{\mathcal{D}}_{(q,p)}\big]^\omega$ of $\mathbb{R}^{2n}$ is $(k+1)$–dimensional and has a basis formed by the vectors $(0, w_1), \ldots, (0, w_k), (T'_p, -T'_q)$ of $\mathbb{R}^{2n}$.

Proposition 5. *Under the hypotheses and with the notation of Lemma 1, at any point $(q,p) \in M$ a vector $(u_q, u_p) \in \mathbb{R}^n \times \mathbb{R}^n$ belongs to $\Delta_{(q,p)}$ if and only if it satisfies (15) and*

$$u_p \;\in\; A(q)\mathcal{D}_q - S(q)^T \big[S(q)A(q)^{-1}S(q)^T\big]^{-1} K(q,p) u_q \tag{18}$$

where $K(q,p)$ is the matrix with components

$$K(q,p)_{ai} \;=\; \frac{\partial}{\partial q_i} \big(S(q)A(q)^{-1}p \big)_a\,, \qquad a = 1, \ldots, k\,, \quad i = 1, \ldots, n\,.$$

Proof Recall that $\Delta_m = [\mathcal{I}_m \cap \overline{\mathcal{D}}_m]^\omega \cap T_m M$. By Lemma 2, $(u_q, u_p) \in [\mathcal{I}_m \cap \overline{\mathcal{D}}_m]^\omega$ if and only if u_q and u_p satisfy (15). Since M is (locally) the zero level set of the k functions $\hat\tau_a = (S(q)A(q)^{-1}p)_a$, a vector $(u_q, u_p) \in \mathbb{R}^n$ is in $T_{(q,p)}M$ if and only if

$$u_q \cdot \frac{\partial \hat\tau_a}{\partial q}(q,p) + u_p \cdot \frac{\partial \hat\tau_a}{\partial p}(q,p) = 0 \qquad \forall a = 1, \ldots, k\,,$$

that is, if and only if

$$S(q)A(q)^{-1} u_p \;=\; -K(q,p) u_q\,. \tag{19}$$

Given that $\mathcal{D}_q = \ker S(q)$ it is immediate to verify that this condition is equivalent to $u_p \in A\mathcal{D} - S^T (SA^{-1}S^T)^{-1} K u_q$. ■

B. Some consequences. We draw here a few consequences from Proposition 5. First, we have:

Corollary 1. *If G acts freely and properly on Q and $\mathcal{G}$ is an over–distribution of $\mathcal{D}$, then Δ has rank one at all points of M, except on the sumbmanifold $p = 0$ of M where it has rank zero.*

Proof We shall prove a little bit more, that is, that if the action is free and proper, then the distribution Δ satisfies

$$\begin{aligned} \dim \Delta_{(q,p)} &= 1 + n - k - \dim(\mathcal{G}_q \cap \mathcal{D}_q) \qquad \text{if } p \neq 0 \\ \dim \Delta_{(q,0)} &= n - k - \dim(\mathcal{G}_q \cap \mathcal{D}_q)\,. \end{aligned}$$

This implies the statement because, if $\mathcal{G}_q \cap \mathcal{D}_q = \mathcal{D}_q$, then $\dim(\mathcal{G}_q \cap \mathcal{D}_q) = n - k$. Let $m = (q, p)$. Note that $\dim \Delta_m = 2n - \dim \Delta_m^\omega$. From (11) and the transversality of $T_m M$ and $\overline{\mathcal{D}}_m^\omega$ it follows that $\Delta_m^\omega = \big(\mathcal{I}_m \cap \overline{\mathcal{D}}_m\big) \oplus (T_m M)^\omega$. Hence $\dim \Delta_m^\omega = \dim(\mathcal{I}_m \cap \overline{\mathcal{D}}_m) + \dim[T_m M]^\omega$, where $\dim[T_m M]^\omega = k$. From (16) it follows that, if $p \neq 0$, then $\dim(\mathcal{I}_m \cap \overline{\mathcal{D}}_m) = \dim(\mathcal{G}_q \cap \mathcal{D}_q) + (n-1)$ because $T_p'^\perp$ is a subspace of $\mathbb{R}^n$ of dimension $n - 1$. If instead $p = 0$, then $T_p'^\perp = \mathbb{R}^n$ and $\dim(\mathcal{I}_m \cap \overline{\mathcal{D}}_m) = \dim(\mathcal{G}_q \cap \mathcal{D}_q) + n$. ■

Hence, if the action Ψ^Q on Q is free and proper, and if the group orbits are sufficiently large, so that $\mathcal{G} \supseteq \mathcal{D}$, then the distribution Δ has rank one at all points except at $p = 0$, where its rank is zero. This implies that Δ is integrable and $\Delta^\infty = \Delta$. Therefore, it follows from the statement at the beginning of the proof of Theorem 1 that, in this case, given any Ψ^{T^*Q}–invariant kinetic energy T on T^*Q, in a neighbourhood of each point of M there is the maximum number $2n - k - 1$ of functionally independent local weakly Noetherian constants of motion. Thus, in this case, which includes the case of transitive actions, our method does not give any new information with respect to the rectification theorem, but the fact that the local constants of motion can be chosen to be weakly Noetherian. This is a sort of weakly Noetherian version of the rectification theorem.

If $\mathcal{G}$ is not an over–distribution of $\mathcal{D}$, instead, the distribution Δ has rank greater than one and may be not integrable. This puts some limitations on the number of (even local) weakly Noetherian constants of motion.

Next, we have:

Corollary 2. *Given Q, M and T, consider two free and proper actions Ψ_1^Q and Ψ_2^Q of two Lie groups G_1 and G_2 on Q, whose tangent lifts preserve T. If the orbits of the Ψ_1^Q–action contain those of the Ψ_2^Q–action, then any weakly Noetherian constant of motion relative to (T, M, Ψ_2^Q) is also a weakly Noetherian constant of motion relative to (T, M, Ψ_1^Q).*

Proof According to (13) and (18), the fact that a vector tangent to M is in Δ depends on the group action only through the tangent space to the group orbits. ■

In particular, if two different group actions have the same orbits in Q (but possibly different orbits in TQ), then they have the same weakly Noetherian constants of motion.

6 Example: the rolling vertical disk.

A. The system. As a first test case for the method, and as an illustration of it, we consider here the system formed by a disk which is constrained to roll without slipping on a horizontal plane, and to stand vertically. This is a well known system, which has been considered e.g. in [16, 8, 13, 23]. We shall consider various classes of forces acting on the disk, with various symmetry groups, so as to show how changing the symmetry group changes the dimension of Δ^∞ and thus the bound on the number of weakly Noetherian constants of motion provided by Theorem 1. We shall then test the optimality of these bounds by comparison to the actual number of *global* weakly Noetherian constants of motion, which in the considered cases can be determined by analyzing the equations of motion.

The holonomic system has configuration manifold $Q = \mathbb{R}^2 \times S^1 \times S^1 \ni (x, y, \theta, \varphi)$, where $(x, y) \in \mathbb{R}^2$ are Cartesian coordinates of the point of contact, φ is the angle between the x–axis and the projection of the disk on the plane, and θ is the angle between a fixed radius of the disk and the vertical. In order to simplify the notation, we assume that mass and radius of the disk are both unitary. The nonholonomic no–slipping constraint imposes that the point of the disk in contact with the plane has zero velocity, that is, $\dot{x} = \dot{\theta}\cos\varphi$ and $\dot{y} = \dot{\theta}\sin\varphi$. The rank–two constraint distribution has fibers

$$\mathcal{D}_{(x,y,\theta,\varphi)} = \big\langle \cos\varphi\,\partial_x + \sin\varphi\,\partial_y + \partial_\theta\,,\ \partial_\varphi \big\rangle$$

that is, $\ker S_{(x,y,\theta,\varphi)}$ for

$$S_{(x,y,\theta,\varphi)} \;=\; \begin{pmatrix} \cos\varphi & \sin\varphi & -1 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \end{pmatrix}.$$

The (Lagrangian) kinetic energy is $T = \frac{1}{2}\big(\dot{x}^2 + \dot{y}^2\big) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2$, where I and J are the appropriate moments of inertia. If we write $h = 1/I$, $k = 1/J$ and if $V(x, y, \varphi, \theta)$ is the potential energy of the forces acting on the disk, then the Hamiltonian is

$$H(x, y, \varphi, \theta, p_x, p_y, p_\theta, p_\varphi) \;=\; \frac{p_x^2 + p_y^2}{2} + \frac{h}{2}p_\theta^2 + \frac{k}{2}p_\varphi^2 + V(x, y, \varphi, \theta)\,.$$

The constraint manifold

$$M \;=\; \big\{(x, y, \theta, \varphi, p_x, p_y, p_\theta, p_\varphi)\,:\, p_x = kp_\theta\cos\varphi\,,\ p_y = kp_\theta\sin\varphi\big\}$$

is six–dimensional. It is diffeomorphic to $\mathbb{R}^4 \times \mathbb{T}^2$ and can be globally parameterized with $(x, y, \theta, \varphi, p_\theta, p_\varphi)$. Using these coordinates, the equations of motion, which can be computed from (2) and (3), are

$$\begin{aligned} \dot{x} \;&=\; hp_\theta\cos\varphi\,, & \dot{y} \;&=\; hp_\theta\sin\varphi\,,\\ \dot{\theta} \;&=\; hp_\theta\,, & \dot{\varphi} \;&=\; kp_\varphi\,,\\ \dot{p}_\theta \;&=\; -\frac{1}{1+h}\big(V'_\theta + V'_x\cos\varphi + V'_y\sin\varphi\big)\,, & \dot{p}_\varphi \;&=\; V'_\varphi\,. \end{aligned} \tag{20}$$

These equations are simple enough to allow us to interpret the results of the forthcoming analysis, where increasingly smaller symmetry groups are considered. For simplicity of exposition, we classify these groups via the corresponding classes of invariant potentials.

B. No forces. If we allow only constant external potentials, then the system is invariant under the group $G = \mathbb{R}^2 \times \mathbb{T}^2$, which acts on Q by translations along the coordinates (x, y, θ, φ).

Since this action is transitive, our method gives the existence of sets of five weakly Noetherian local constants of motion in a neighbourhood of any point of M, see Corollary 1. As we now show, only four of these constants of motion are globally defined; two of them can be chosen to be horizontal momenta and the others horizontal gauge momenta.

In order to see this observe that, for constant potentials, the equations of motion (20) become

$$\dot{x} \;=\; hp_\theta\cos\varphi\,,\quad \dot{y} \;=\; hp_\theta\sin\varphi\,,\quad \dot{\theta} \;=\; hp_\theta\,,\quad \dot{\varphi} \;=\; kp_\varphi\,,\quad \dot{p}_\theta \;=\; 0\,,\quad \dot{p}_\varphi \;=\; 0\,. \tag{21}$$

Thus p_θ and p_φ are constants of motion, and they are obviously horizontal momenta. Observe now that each subsystem (θ, p_θ) and (φ, p_φ) performs uniform rotations on $S^1 \times \mathbb{R}$, with angular frequencies hp_θ and kp_φ respectively. Since φ grows linearly in time and p_θ is constant, integrating the first two equations (21) shows that the projection of the motion in the (x, y)–plane is generically a uniform circular motion, with frequency kp_φ. (The motion is, exceptionally, linear if $p_\varphi = 0$). Thus, motions of the system are generically quasi–periodic on tori of dimension three, with frequencies $hp_\theta, kp_\varphi, kp_\varphi$. Since two of the frequencies are equal the closure of all these orbits is contained in tori of dimension two. But since the ratio $hp_\theta/(kp_\varphi)$ varies continuously, the set of orbits whose closure is a two–dimensional torus forms a dense set. Therefore, there are not more than $6-2=4$ independent constants of motion which are defined in open invariant sets.

A simple computation shows that two other globally defined constants of motion are $kxp_\varphi - hp_\theta \sin\varphi$ and $kyp_\varphi + hp_\theta \cos\varphi$. They are horizontal gauge momenta because, if the action is transitive, then any global constant of motion which is linear in the velocities is a horizontal gauge momentum, see Proposition 4 of [22].

Remark: The fifth local integral gives the "slope" of the orbits on the two–dimensional tori. It can be taken e.g. as $hp_\theta\varphi - kp_\varphi\theta$ and it is not globally defined on the tori with irrational $kp_\theta/(hp_\varphi)$. Hence, it is not a gauge momentum even though it is locally a linear combination of two horizontal momenta.

C. Potentials $V = V(\varphi)$. We allow now potential energies which depend on the angle φ, thus restricting the symmetry group to $G = \mathbb{R}^2 \times S^1$, which acts by translations along x, y and θ. We begin by determining the distribution Δ:

Fact 1 *The fibers of Δ are spanned by the two vector fields*

$$X_1(q,p) \;=\; \partial_{p_\varphi} \tag{22}$$
$$X_2(q,p) \;=\; p_x\partial_x + p_y\partial_y + hp_\theta\partial_\theta + kp_\varphi\partial_\varphi - kp_\varphi p_y\partial_{p_x} + kp_\varphi p_x\partial_{p_y}\,. \tag{23}$$

Proof The fibers of Δ are the subpaces of vectors $(u_q,u_p)\in\mathbb{R}^n\times\mathbb{R}^n$ which satisfy conditions (15) and (18). First note that, at a point $q=(x,y,\theta,\varphi)\in Q$, the distribution of tangent spaces to the orbits of the group action has fiber $\mathcal{G}_q = \big\langle \partial_x, \partial_y, \partial_\theta \big\rangle$ and hence

$$\mathcal{D}_q \cap \mathcal{G}_q \;=\; \big\langle \cos\varphi\partial_x + \sin\varphi\partial_y + \partial_\theta \big\rangle\,.$$

Consider now a point $p=(p_x,p_y,p_\theta,p_\varphi)$ such that $(q,p)\in M$ and fix a vector $u_q\in\mathbb{R}^4$ such that $u_q \in \big\langle T'_p \big\rangle = \langle A^{-1}p\rangle$, that is,

$$u_q = \lambda\left(p_x, p_y, hp_\theta, kp_\varphi\right) \tag{24}$$

for some $\lambda\in\mathbb{R}$. Since the kinetic matrix A is constant, the vector $T'_q := \frac{\partial T}{\partial q} = 0$ and the second condition (15) reduces to the orthogonality of u_p to $\mathcal{D}_q\cap\mathcal{G}_q$, that is,

$$u_p \;=\; \big(\alpha, \beta, -\alpha\cos\varphi - \beta\sin\varphi, \gamma\big) \tag{25}$$

for some $\alpha,\beta,\gamma\in\mathbb{R}$. Simple computations show that, on the points of M,

$$K(q,p) \;=\; \begin{pmatrix} 0 & 0 & 0 & p_y\cos\varphi - p_x\sin\varphi \\ 0 & 0 & 0 & -p_x\cos\varphi - p_y\sin\varphi \end{pmatrix} \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -hp_\theta \end{pmatrix}$$
$$S^T(SA^{-1}S^T)Ku_q \;=\; \lambda\big(hkp_\theta p_\varphi\sin\varphi\,,\; -hkp_\theta p_\varphi\cos\varphi\,,\; 0\,,\; 0\big)^T \;=\; \lambda\big(kp_yp_\varphi\,,\; -kp_xp_\varphi\,,\; 0\,,\; 0\big)^T\,,$$

where the last expressions follow from the constraint equations $hp_\theta\cos\varphi = p_x$ and $hp_\theta\sin\varphi = p_y$. Thus, since $A\mathcal{D}_q = \big\langle \cos\varphi\partial_x + \sin\varphi\partial_y + h^{-1}\partial_\theta, k^{-1}\partial_\varphi \big\rangle$, condition (18) is

$$u_p \;=\; \big(\mu\cos\varphi - \lambda kp_yp_\varphi\,,\; \mu\sin\varphi + \lambda kp_xp_\varphi\,,\; \mu h^{-1}\,,\; \nu k^{-1}\big) \tag{26}$$

for some $\mu, \nu \in \mathbb{R}$. Together, conditions (25) and (26) demand that $\mu = 0$, $\alpha = -\lambda k p_y p_\varphi$ and $\beta = \lambda k p_x p_\varphi$. Thus, we conclude that $(u_q, u_p) \in \Delta_{(q,p)}$ if and only if

$$u_q = \lambda\big(p_x, p_y, hp_\theta, kp_\varphi\big)T \qquad \text{and} \qquad u_p = \big(-\lambda k p_y p_\varphi\,,\, \lambda k p_x p_\varphi\,,\, 0\,,\nu\big)^T$$

with $\lambda, \nu \in \mathbb{R}$. This shows that $\Delta = \langle X_1, X_2\rangle$. ∎

Note that the distribution Δ has obviously rank two at all points of M, except where $p_\theta = p_\varphi = 0$, where it has rank 1. Moreover, it is real analytic.

Fact 2 *The distribution Δ^∞ has rank five at all points of M except where $p_\theta = 0$, where it has rank two.*

Proof The distribution $\Delta^1 := \Delta$ is not integrable because the vector field

$$h^{-1}[X_1, X_2] = \partial_\theta - p_y\partial_{p_x} + p_x\partial_{p_y} =: X_3$$

is linearly independent of X_1 and X_2 in all of M but where $p_\theta = 0$.

We thus investigate the integrability of the distribution $\Delta^2 = \langle X_1, X_2, X_3\rangle$. Note that $[X_1, X_3] = 0$ and

$$[X_2, X_3] = p_y\partial_x - p_x\partial_y =: X_4\,.$$

A simple computation shows that, in M, X_4 is linearly independent of X_1, X_2, X_3 wherever $p_\theta \neq 0$. Therefore, $\Delta^2 = \langle X_1, X_2, X_3\rangle$ is not integrable.

The distribution Δ^3 is generated by X_1, X_2, X_3 and X_4. Note that $[X_1, X_4] = 0$,

$$[X_3, X_4] = p_x\partial_x + p_y\partial_y =: X_5\,,$$

and $[X_2, X_4] = kp_\varphi X_5$. The vector fields X_1, X_2, X_3, X_4, X_5 are linearly independent at all points of M except where $p_\theta = 0$. Thus, Δ^3 is not integrable.

The distribution Δ^4 is generated by X_1, X_2, X_3, X_4 and X_5. Since $[X_1, X_5] = [X_4, X_5] = 0$, $[X_3, X_5] = -X_4$ and $[X_2, X_5] = -kp_\varphi X_4$, Δ^4 is integrable. Thus, $\Delta^\infty = \Delta^4$. Computing the minors of the matrix whose rows are the vector fields X_1, X_2, X_3, X_4, X_5 one sees that this matrix has rank five wherever $p_\theta \neq 0$, and rank two where $p_\theta = 0$. ∎

Since Δ^∞ has rank five in an open dense subset of M and M is six–dimensional we conclude that the system has at most one weakly Noetherian constant of motion. It is immediate to verify that the system has indeed one global weakly Noetherian constant of motion, which is the horizontal momentum p_θ. In fact, for potentials depending only on the angle φ the equations of motion are

$$\dot x = hp_\theta\cos\varphi\,, \quad \dot y = hp_\theta\sin\varphi\,, \quad \dot\theta = hp_\theta\,, \quad \dot\varphi = kp_\varphi\,, \quad \dot p_\theta = 0\,, \quad \dot p_\varphi = -V'(\varphi)\,.$$

If we compare this situation with that of constant potentials of the previous subsection, we note that the (φ, p_φ) subsystem has still one constant of motion, the energy $p_\varphi^2/(2k) + V(\varphi)$, which however now depends on V and is therefore not weakly Noetherian. On the other hand, the nonconstancy of $\dot\varphi$ has the consequence that the (x, y) subsystem might not have a global constant of motion at all—not only a weakly Noetherian one—because of, e.g., the presence of a limit cycle.

D. Other cases. The study of other symmetry groups is computationally very similar to the last one, and we review very quickly a few cases. Technically, changing the group changes the distribution $\mathcal{G}$, which determines the p–components of vectors of Δ through its intersection with $\mathcal{D}$, see (15).

Potentials $V = V(x, y)$. If the group is $G = \mathbb{T}^2$ acting by translations of the two angles θ and φ, then the distribution of tangent spaces to the orbits of the group action has fibers $\mathcal{G}_q = \langle\partial_\varphi, \partial_\theta\rangle$ and

$$\mathcal{D}_q \cap \mathcal{G}_q = \langle\partial_\varphi\rangle\,.$$

Correspondingly, the second condition (15), namely $u_p \in [\mathcal{D}_q \cap \mathcal{G}_p]^\perp$, imposes that the fourth component of u_p is zero. Vectors (u_q, u_p) in $\Delta_{(q,p)}$ are thus given by (24) and

$$u_p \;=\; \big(\mu\cos\varphi - \lambda k p_y p_\varphi \,,\; \mu\sin\varphi + \lambda k p_x p_\varphi \,,\; \mu h^{-1} \,,\; 0\big)$$

for $\lambda, \mu \in \mathbb{R}$. Equivalently, $\Delta = \big\langle X_1, X_2 \big\rangle$ with

$$X_1 \;=\; \cos\varphi\, \partial_{p_x} + \sin\varphi\, \partial_{p_y} + \partial_{p_\theta} \tag{27}$$

and X_2 as in (23). The distribution $\Delta^1 := \Delta$ has rank two in all of $M' = M \setminus \{p = 0\}$ and is not integrable. Its involutive closure is Δ^4, which has rank five in M'. (It is spanned by X_1, X_2, $[X_1, X_2]$, $[X_2, X_3]$ and $[X_2, X_4]$). Therefore, the system has at most one weakly Noetherian constant of motion. A glance at the equations of motion

$$\dot{x} = h p_\theta \cos\varphi \,,\; \dot{y} = h p_\theta \sin\varphi \,,\; \dot{\theta} = h p_\theta \,,\; \dot{\varphi} = k p_\varphi \,,\; \dot{p}_\theta = \frac{h}{1-h}\big(V_x' \cos\varphi + V_y' \sin\varphi\big) \,,\; \dot{p}_\varphi \;=\; 0$$

shows that the horizontal momentum p_φ is a constant of motion (and that there are no other constants of motion, except the energy, unless V has special properties).

Potentials $V = V(\theta, \varphi)$. Finally, we consider the case of potentials which are invariant under translations along x and y. The distribution $\mathcal{G} = \big\langle \partial_\varphi, \partial_\theta \big\rangle$ has trivial intersection with $\mathcal{D}$ and there is no restriction on u_p from the second condition (15). Vectors (u_q, u_p) in $\Delta_{(q,p)}$ are thus given by (24) and

$$u_p \;=\; \big(\mu\cos\varphi - \lambda k p_y p_\varphi \,,\; \mu\sin\varphi + \lambda k p_x p_\varphi \,,\; \mu h^{-1} \,,\; \nu k^{-1}\big)$$

for $\lambda, \mu, \nu \in \mathbb{R}$. Equivalently, $\Delta = \big\langle X_1, X_2, X_3 \big\rangle$ with X_1 as in (27), X_2 as in (23) and $X_3 = \partial_{p_\theta}$. The distribution $\Delta^1 := \Delta$ has now rank three in all of $M \setminus \{p = 0\}$, is not integrable, and its involutive closure is Δ^4, which has has rank six in $M \setminus \{p = 0\}$. Therefore, the system has no weakly Noetherian constants of motion.

7 Example: the ball rolling on a surface of revolution.

A. Generalities. We consider now a second, computationally more challenging example. This is the classical system formed by a heavy homogeneous ball which is constrained to roll without slipping on a smooth surface of revolution with vertical axis [37, 34, 26, 42, 21, 11]. The surface, if not convex, is usually assumed to satisfy a technical condition which amounts to the fact that the ball should be sufficiently small (see below), but it is otherwise arbitrary.

It is classically known that, whatever the surface, the system has three functionally independent constants of motion [37, 26]. One is the energy and the other two have been shown to be horizontal gauge momenta—and hence weakly Noetherian—relatively to a natural action of $S^1 \times \mathrm{SO}(3)$ [5, 36, 38] which corresponds to rotate the ball around its center and to rotate the center around the surface's axis. Dynamically this means that, given the surface, these two functions are constants of motion for all systems obtained by replacing gravity with any other potential which depends only on the height of the center of the ball.

In the case of gravity, additional independent constants of motion may exist for particular choices of the surface. For instance, if the surface is convex, then the system is integrable and there is a total of five functionally independent constants of motion [26, 20]. Known expressions of the two additional constants of motion [20] involve a property of the solutions, the so–called shift or phase [26], whose computation is prohibitive but which presumably depends on gravity. Hence, it is expected that these additional integrals are not weakly Noetherian. Applying our method we will show that this is indeed the case, under the additional hypothesis of real analyticity of the surface. More generally, we will show that there are only two weakly Noetherian constants of motion for any choice of the surface of revolution.

Since the computations involved in the determination of the distributions $\Delta^1, \Delta^2, \dots$ and of their ranks become quickly exceedingly complex to be performed by hand, we have done them with the aid of a symbolic manipulation package (*Mathematica* [41]). Even so, the task is complex. To succeed, we exploit in an essential way two facts. One is the real analyticity of the system, which greatly simplifies the determination of the ranks of the distribution $\Delta^2, \dots$ because, if a real analytic distribution has rank p at one point, then its rank is $\geq p$ in an open dense set. The other is the existence of the two weakly Noetherian constants of motion, which ensures that Δ^∞ cannot have rank greater than $\dim M - 2 = 6$ and allows us to arrest the construction of the distributions $\Delta^2, \Delta^3, \dots$ when rank six is reached.

B. The system. Since Routh [37], this system is usually studied under the hypothesis that the surface of contact $\mathcal{S}_C$ on which the ball rolls is such that the center of the ball moves on a smooth surface $\mathcal{S}$, and it is this latter surface which is regarded as given. Specifically, the embedding of the surface $\mathcal{S}$ in the physical space $\mathbb{R}^3 \ni (x, y, z)$ is given by an equation of the form

$$z \;=\; \mathcal{F}(\sqrt{x^2+y^2})$$

with an even, smooth function $\mathcal{F} : I \to \mathbb{R}$, where I is either the entire real line or an open interval symmetrical to zero. We will call 'profile function' the function $\mathcal{F}$. Note that, under these hypothesis, the surface $\mathcal{S}$ has a smooth minimum or maximum at the origin.

The hypothesis that the center of the ball moves on a smooth surface puts a few conditions on the geometry of the problem, and we will benefit from one of them. Specifically, we will use the fact that, at each point of concavity of the profile function, the radius of curvature of the graph of the profile function must be greater than the radius r of the ball, namely

$$\frac{|\mathcal{F}''(x)|}{1+\mathcal{F}'(x)^2} \leq \frac{1}{r} \qquad \text{at each } x \in I \text{ at which } \mathcal{F}''(x) < 0\,. \tag{28}$$

We will assume that the profile function $\mathcal{F} : I \to \mathbb{R}$ is real analytic. For notational simplicity, we will consider only the case $I = \mathbb{R}$, but all conclusions remain true for I an open interval.

The configuration manifold $\widetilde{Q}$ of the system is diffeomorphic to $\mathbb{R}^2 \times \mathrm{SO}(3) \ni (x, y, R)$, where the two coordinates $(x, y) \in \mathbb{R}^2$ parameterize the position $(x, y, \mathcal{F}(\sqrt{x^2+y^2}))$ of the center of the ball and $R \in \mathrm{SO}(3)$ parameterizes the attitude of the ball. The kinetic energy of the system is invariant under the tangent lift of the action $(a, A).(x, y, R) = (x\cos a - y\sin a, y\cos a + x\sin a, RA)$ of the group $S^1 \times \mathrm{SO}(3) \ni (a, A)$ on Q (see below the expression of the kinetic energy). Potential energies which are invariant under this action depend only on the coordinates (x, y) and are constants on the circles $x^2 + y^2 = \text{const}$. Our aim is to determine the number of weakly Noetherian constants of motion relative to this action.

Since Δ^∞ will be a real analytic distribution, its rank will take its maximum value in an open and dense subset of $\widetilde{Q}$. Therefore, we may freely exclude from $\widetilde{Q}$ any submanifold of positive codimension. Thus, we begin by excluding from $\widetilde{Q}$ the submanifold $(x, y) = (0, 0)$, which contains points where the considered action is not free. Correspondingly, we resort to (suitably rescaled) polar coordinates $(\rho, \gamma) \in \mathbb{R}_+ \times S^1$ to parameterize the position of the center of the ball, with

$$x \;=\; \sqrt{\frac{J}{m}}\,\rho\cos\gamma\,, \qquad y \;=\; \sqrt{\frac{J}{m}}\,\rho\sin\gamma$$

where m is the mass of the ball and J is its moment of inertia relative to any baricentric axis. Furthermore, we use Euler angles $(\varphi, \psi, \vartheta) \in S^1 \times S^1 \times (0, \pi)$ to parameterize $SO(3)$, with the convention of ref. [2]. The coordinates $(\rho, \gamma, \varphi, \psi, \vartheta)$ are defined in an open and dense submanifold

Q of $\widetilde{Q}$, to which we restrict our consideration. This submanifold Q is diffeomorphic to $\mathbb{R}_+ \times S^1 \times S^1 \times S^1 \times (0,\pi) \ni (\rho,\gamma,\varphi,\psi,\vartheta)$, and will be identified with it.

In this way, we are left with a holonomic system with configuration manifold Q. If we define

$$F(\rho) \;:=\; \sqrt{\frac{m}{J}}\,\mathcal{F}\left(\rho\sqrt{\frac{J}{m}}\right) \qquad \text{and} \qquad G(\rho) \;:=\; 1+F'(\rho)^2\,,$$

then the Lagrangian kinetic energy may be written as

$$\frac{J}{2}\left[G(\rho)\big(\dot\rho^2+\rho^2\dot\vartheta^2\big)+\dot\vartheta^2+\dot\varphi^2+\dot\psi^2+2\dot\varphi\dot\psi\cos\vartheta\right].$$

The constraint of rolling without slipping is given by two 1–forms whose kernel defines a rank 3 constraint distribution on Q, see e.g. [37, 26]. The matrix representation of these 1–forms is

$$S(\rho,\gamma,\varphi,\psi,\vartheta) \;=\; \begin{pmatrix} \sqrt{G(\rho)} & 0 & 0 & kc_{\gamma\varphi}s_\vartheta & ks_{\gamma\varphi} \\ 0 & \rho\sqrt{G(\rho)} & kF'(\rho) & k(F'(\rho)c_\vartheta - s_{\gamma\varphi}s_\vartheta) & kc_{\gamma\varphi} \end{pmatrix}$$

where $k := r\sqrt{m/J}$. In order to make formulas more readable, here and in the sequel we use the following shortands: $s_\vartheta = \sin\vartheta$, $c_\vartheta = \cos\vartheta$, $s_{\gamma\varphi} = \sin(\gamma-\varphi)$, $c_{\gamma\varphi} = \cos(\gamma-\varphi)$, $a = p_\psi - p_\varphi c_\vartheta$, $b = p_\varphi - p_\psi c_\vartheta$, $c = p_\vartheta s_\vartheta$, $d = p_\varphi s_\vartheta$, $\alpha = cs_{\gamma\varphi} + ac_{\gamma\varphi}$ and $\beta = cc_{\gamma\varphi} - as_{\gamma\varphi}$.

We note now that, according to formulas (10) and (11), the distribution Δ^∞ is invariant under constant rescalings of the kinetic energy. Therefore, in order to determine this distribution for the problem at hand, we may ignore the overall factor J in the expression of the kinetic energy. Doing so, and passing to the Hamiltonian formulation on T^*Q via the Legendre transformation, the kinetic energy becomes

$$T \;=\; \frac{p_\rho^2}{2G(\rho)} + \frac{p_\gamma^2}{2\rho^2} + \frac{p_\vartheta^2}{2} + \frac{p_\varphi^2 + p_\psi^2 - 2c_\vartheta p_\varphi p_\psi}{2s_\vartheta^2}$$

while the constraint manifold M is the eight–dimensional submanifold of T^*Q given by the equations

$$p_\rho = -k\,\frac{\alpha G^{1/2}}{s_\vartheta}\,, \qquad p_\gamma = k\,\rho\,\frac{\beta + F'd}{s_\vartheta G^{1/2}}\,. \tag{29}$$

Thus, the constraint manifold M can be identified with $T^3\times(0,\pi)\times\mathbb{R}_+\times\mathbb{R}^3$, with global coordinates $(\gamma,\varphi,\psi,\vartheta,\rho,p_\varphi,p_\psi,p_\vartheta)$.

From Section 5 we know that Δ^∞ depends only on T, on S and on the group action. Hence, in the case under study, Δ^∞ depends on the single parameter $k = r\sqrt{m/J} > 0$. We will also write $j = 1 + k^2$. Finally note that, in terms of the rescaled profile function F, condition (28) becomes $|F''(x)| < G(x)/k$ wherever $F''(x) < 0$.

C. The distributions Δ and Δ^∞. We will do all the computations which lead to Δ^∞ using coordinates on $M \subset \mathbb{R}^8$ (rather than extensions to $\mathbb{R}^{10}$). First, we have:

Fact 3 *Assume that the profile function $\mathcal{F}:\mathbb{R}\to\mathbb{R}$ is an even and real analytic function and consider any positive value of the parameter k. Then the distribution Δ has rank 2 in an open dense subset of M, is real analytic, and is spanned by the two vector fields*

$$\begin{aligned} X_1 \;=\; & -\left[k\rho^2G^2s_\vartheta^2s_{\gamma\varphi}\alpha\right]\partial_\rho \;-\; \left[k\rho G^2s_\vartheta^2s_{\gamma\varphi}(\beta+F'd)\right]\partial_\gamma \;+\; \left[b\rho^2G^{5/2}s_\vartheta s_{\gamma\varphi}\right]\partial_\varphi \\ & +\left[a\rho^2G^{5/2}s_\vartheta s_{\gamma\varphi}\right]\partial_\psi \;+\; \left[c\rho^2G^{5/2}s_\vartheta^2\right]\partial_\vartheta \;-\; \left[ab\rho^2G^{5/2}s_{\gamma\varphi}\right]\partial_{p_\vartheta} \\ & - j^{-1}k^3\alpha\rho^2s_\vartheta F''(\beta F'-d)\Big(\left[s_{\gamma\varphi}F'\right]\partial_{p_\varphi} - \left[(s_\vartheta - c_\vartheta s_{\gamma\varphi}F')\right]\partial_{p_\psi}\Big) \\ X_2 \;=\; & c_{\gamma\varphi}s_\vartheta\partial_{p_\psi} + s_{\gamma\varphi}\,\partial_{p_\vartheta}\,. \end{aligned}$$

Proof To determine Δ we follow its description given in Proposition 5 and Lemma 2. First, we determine a set of generators of $[\mathcal{I}\cap\mathcal{G}]^\omega$, see (15). Since the distribution $\mathcal{G}$ on Q is generated by ∂_γ, ∂_φ, ∂_ψ and ∂_ϑ, the fiber $\mathcal{D}_q\cap\mathcal{G}_q$ over a point $q=(\rho,\gamma,\varphi,\psi,\vartheta)\in Q$ is spanned by the two vectors

$$(0,kF',-\rho G^{1/2},0,0)\,,\qquad (0,k(s_\vartheta - s_{\gamma\varphi}c_\vartheta F'),0,\rho s_{\gamma\varphi}G^{1/2},-\rho c_{\gamma\varphi}s_\vartheta G^{1/2})\,.$$

Hence, a basis for the subspace $[\mathcal{D}_q\cap\mathcal{G}_q]^\perp$ of $\mathbb{R}^5$ is formed by the three vectors

$$w^1=(1,0,0,0,0)\,,\quad w^2=(0,0,0,c_{\gamma\varphi}s_\vartheta,s_{\gamma\varphi})\,,\quad w^3=\big(0,\rho s_{\gamma\varphi}G^{1/2},\,ks_{\gamma\varphi}F',\,k(s_{\gamma\varphi}c_\vartheta F'-s_\vartheta),0\big)\,.$$

Fix now a point $(q,p)\in M$. Thus, as noticed just before Proposition 5, fixed any point $(q,p)\in M$, a basis for the subspace $\big[\mathcal{I}_{(q,p)}\cap\overline{\mathcal{D}}_{(q,p)}\big]^\omega$ of $\mathbb{R}^{10}$ is given by the four vectors $(0,w^1)$, $(0,w^2)$, $(0,w^3)$, $(T'_p,-T'_q)$ with T'_p and T'_q as in (13), namely,

$$T'_q \;=\; \Big(-\frac{p_\gamma^2}{\rho^3}-\frac{G'p_\rho^2}{2G^2},0,0,0,\frac{ab}{s_\vartheta^3}\Big)\,,\qquad T'_p \;=\; \Big(\frac{p_\rho}{G},\frac{p_\gamma}{\rho^2},\frac{b}{s_\vartheta^2},\frac{a}{s_\vartheta^2},p_\vartheta\Big)\,.$$

Next, the space $\Delta_{(q,p)}$ consists of those vectors $(u_q,u_p)\in\big[\mathcal{I}_{(q,p)}\cap\overline{\mathcal{D}}_{(q,p)}\big]^\omega$ which satisfy (18), or equivalently (19), with

$$K \;=\; \begin{pmatrix} -\frac{p_\rho G'}{2G^{3/2}} & \frac{k\beta}{s_\vartheta} & -\frac{k\beta}{s_\vartheta} & 0 & \frac{kbc_{\gamma\varphi}}{s_\vartheta^2} \\ \frac{\rho G'-2G}{2\rho^2G^{1/2}}p_\gamma + kF''p_\varphi & -\frac{k\alpha}{s_\vartheta} & \frac{k\alpha}{s_\vartheta} & 0 & -\frac{kbs_{\gamma\varphi}}{s_\vartheta^2}\end{pmatrix}\,.$$

Thus, a vector $(u_q,u_p)\in\mathbb{R}^{10}$ belongs to $\Delta_{(q,p)}$ if and only if it equals $(\lambda_4T'_p,\lambda_1w^1+\lambda_2w^2+\lambda_3w^3-\lambda_4T'_q)$ for some $\lambda_1,\ldots,\lambda_4\in\mathbb{R}$ and satisfies $S\alpha^{-1}u_p=-Ku_q$. Solving these equations (via computer assisted symbolic computation) gives two vectors of $\mathbb{R}^{10}$ which, after dropping their p_γ– and p_ρ–components and substituting expressions (29) for p_ρ and p_γ, reduce to X_1 and X_2. It follows from their expressions that these two vector fields are real analytic in all of M.

In order to prove that Δ has rank 2 in an open subset of M we consider the matrix with columns X_1,X_2. The restriction of one of its 2×2 minors to the submanifold M_1 of M where $\gamma=0$, $\varphi=\pi/4$, $\vartheta=\pi/2$, $p_\psi=0$, $p_\varphi=0$, $p_\vartheta=1$ equals $\frac{1}{2}\rho^2G(\rho)^{5/2}$, which is everywhere nonzero. By analiticity, this minor is non–zero in an open and dense subset of M, and X_1 and X_2 are there linearly independent. ■

Fact 4 *In addition to the hypotheses of Fact 3, assume that the profile function $\mathcal{F}$ satisfies condition (28). Then Δ^∞ has rank 6 in an open and dense subset of M.*

Proof The vector field $X_3:=[X_1,X_2]$ is given by

$$\begin{aligned} X_3 \;=\; & \big[kG^2\rho^2s_{\gamma\varphi}s_\vartheta^3\big]\partial_\rho+\big[G^{5/2}\rho^2c_\vartheta s_\vartheta^2c_{\gamma\varphi}s_{\gamma\varphi}\big]\partial_\varphi-\big[G^{5/2}\rho^2c_{\gamma\varphi}s_{\gamma\varphi}s_\vartheta^2\big]\partial_\psi-\big[G^{5/2}\rho^2s_{\gamma\varphi}^2s_\vartheta^3\big]\partial_\vartheta\\ & -\big[j^{-1}k^3\rho^2s_{\gamma\varphi}s_\vartheta^2(d-\beta F')F'F''\big]\partial_{p_\varphi}\\ & +\rho s_\vartheta^2\big[j^{-1}k^3\rho(s_\vartheta-c_\vartheta s_{\gamma\varphi}F')(d-\beta F')F''+kG^2s_{\gamma\varphi}^2s_\vartheta(\beta+dF')+\rho G^{5/2}s_{\gamma\varphi}(bs_{\gamma\varphi}+cc_{\gamma\varphi}c_\vartheta)\big]\partial_{p_\psi}\\ & -\rho G^2c_{\gamma\varphi}s_{\gamma\varphi}s_\vartheta\big[a\rho G^{1/2}c_\vartheta+ks_\vartheta(\beta+dF')\big]\partial_{p_\vartheta}\,. \end{aligned}$$

The restriction to the submanifold M_1 introduced in the Proof of Fact 3 of one of the 3×3 minors of the matrix with columns X_1,X_2,X_3 equals $\frac{1}{4}\rho^4G(\rho)^5>0$. Hence, Δ is not integrable and we proceed to consider the integrability of $\Delta^2=\langle X_1,X_2,X_3\rangle$.

A computation shows that $[X_2,X_3]=0$. The expression of the vector field $X_4:=k^{-1}[X_1,X_3]$ is not yet extremely long—but since its inspection would not add much to the comprehension we do not report it here. However, the restriction to the submanifold M_1 of one of the 4×4 minors of the matrix with columns X_1, X_2, X_3, X_4 equals

$$\frac{\rho^8 G^7}{16}\big(G^{5/2} + j^{-1}k^3 F'^2 F''\big)\,.$$

Since $G \ge 1$, this quantity is positive at each point at which $F'' \ge 0$. If instead $F''(x) < 0$ then $F''(x) > -G(x)/k$ because of (28). Hence, since $k^2 < j$,

$$G^{5/2} + \frac{k^3}{j}F'^2F'' \;>\; G^2 - \frac{k^2}{j}F'^2 G \;>\; G(G - F'^2) \;=\; G \;\ge\; 1\,.$$

It follows that the distribution Δ^2 is not integrable, and we proceed to consider the integrability of $\Delta^3 = \langle X_1, X_2, X_3, X_4\rangle$.

By the Jacobi identity, the commutation of X_2 with $X_3 = k^{-1}[X_1, X_2]$ implies that also X_2 and X_4 commute. Hence $\Delta^4 = \langle X_1, \ldots, X_6\rangle$ with $X_5 = [X_1, X_4]$ and $X_6 = [X_3, X_4]$. The expressions of X_5 and X_6 are too long to be reported. Proceeding as above, we must now show that at least one of the 6×6 minors of the matrix with columns $X_1, \ldots, X_6$ is nonzero at least at one point.

One of these 6×6 minors is a function which, restricted to the submanifold M_1, equals the function $Z : \mathbb{R}_+ \to \mathbb{R}$ given by

$$Z(\rho) \;=\; \frac{1}{2^{10}kj^2}\rho^{17}G(\rho)^{31/2}Z_1(\rho)Z_2(\rho)$$

with

$$\begin{aligned}
Z_1 \;&=\; 2j\rho G^3 - k^2(G+k^2)G' + k^3\rho G^{1/2}G'F' \\
Z_2 \;&=\; k^3\big(3\rho G^3 + k^2 GG' + k^2\rho F''^2\big)G'F' \\
&\quad +2G^{3/2}\Big(j\rho G^4 + k^4 GG'F'^2 + k^2\rho\big(\tfrac{3}{4}F'^2 + \tfrac{3}{2}j - 1\big)G'^2 - jk^2\rho F''^2 - k^2\rho(G+k^2)GF'F'''\Big)
\end{aligned}$$

We must prove that Z is nonzero in (at least) one point. Since $G \ge 1$, this certainly happens if Z_1 and Z_2 do not identically vanish in a right neighbourhood of $\rho = 0$. Now, since we have assumed that $\mathcal{F}$, and hence F and G, are real analytic functions on the entire real line, the two functions Z_1 and Z_2 are also real analytic on the entire real line. Therefore, in order to prove that they are nonzero in a (right) neighbourhood of $\rho = 0$ it suffices to check that their first derivatives in $\rho = 0$ are nonzero. Observing that $G(0) = 1$, $G'(0) = F'(0) = 0$ and $G''(0) = 2F''(0)^2$ one readily finds

$$Z_2'(0) \;=\; 2jZ_1'(0) \;=\; 4j^2\big(1 - k^2F''(0)^2\big)\,.$$

This is nonzero, unless $kF''(0) = \pm 1$. Since condition (28) implies $F''(0) \ne -1/k$, we have proven that the minor in question does not identically vanish unless $F''(0) = 1/k$.

In order to show that the minor in question does not identically vanish even for this special value of $F''(0)$, we consider the restriction of this minor to the submanifold M_2 of M defined by $\gamma = 0$, $\varphi = \pi/4$, $\vartheta = \pi/2$, $p_\psi = 0$, $p_\varphi = 1$, $p_\vartheta = 0$. This restriction equals

$$\frac{\rho^{15}G(\rho)^{31/2}}{64\sqrt{2}\,kj^2}\,(kF' + \rho G^{1/2})\,Z_3(\rho)\,Z_4(\rho)$$

with

$$\begin{aligned}
Z_3 \;&=\; k^3\rho F'' + 2jkG^2F' + j\rho G^{5/2} \\
Z_4 \;&=\; 12jk^2G^4F'(G^{3/2} + kF'') + 2k^2\rho G^{5/2}\big(3jF'^4 + 5k^2F'^2 - 3\big)F'' + k^3\rho^2G'(3G^3 + k^2F''^2) \\
&\quad +2k\rho G\big[5jG^4F'^2 + k^2\big((2k^2 - 8jG)F'^2 - 3G\big)F''^2 + 2jk^2G^2F'F'''\big] \\
&\quad +2\rho^2G^{3/2}\big[jG^4 + k^2(7k^2 - 4jG)F''^2\big]F' + 2k^2\rho^2G^{5/2}(1 + jF'^2)F'''\,.
\end{aligned}$$

Since $F'(0) = 0$ and $F''(0) = 1/k > 0$, F' is strictly positive in a right neighbourhood of 0 and hence $kF'(\rho) + \rho G^{1/2}(\rho) > 0$ for all $\rho \neq 0$ in that neighbourhood. Proceeding as above, we thus only have to check that both Z_3 and Z_4 have nonzero first derivative in $\rho = 0$ if $F''(0) = 1/k$. In fact,

$$\begin{aligned} Z_3'(0) &= j + k(2 + 3k^2)F''(0) \\ Z_4'(0) &= 6k^2(1 + 2k^2)\big(1 + kF''(0)\big)F''(0)\,. \end{aligned}$$

This proves that, for any value of $k > 0$ and for any (allowed) choice of the profile function, the distribution Δ^4 has rank 6 at some point of M, and hence in an open dense subset of M. In turn, this implies that Δ^∞ has rank at least six in such a subset. As we have already noticed, the existence of two gauge integrals implies that $\operatorname{rank}\Delta^\infty \leq 6$. Hence, $\operatorname{rank}\Delta^\infty = 6$ in an open dense subset of the constraint manifold. ■

Fact 4 implies that, for any (allowed) choice of the profile function, the system has exactly two weakly Noetherian constants of motion, which are the two classically known horizontal gauge momenta.

8 Conclusions, and some perspectives

In this article we have developed a method to produce an estimate on the number of a certain type of constants of motion of nonholonomic systems—the weakly Noetherian ones—and we have shown on examples that this procedure can actually be carried out in practice.

The heart of the method is the fact that the constants of motion of the considered type are first integrals of a certain smooth distribution Δ, and the number of these first integrals is bounded by the corank of the involutive closure Δ^∞ of Δ. The main limitation of this procedure is that it cannot distinguish between local and global constants of motion. Geometrically, this is because integrable distributions give foliations rather than fibrations. Dynamically, this has the consequence that our method gives only an upper bound on the number of global constants of motion, which are the only constants of motion of interest from a dynamical point of view. However, the examples of Sections 6 and 7 indicate that this estimate may be non trivial and informative.

If one is interested to a specific nonholonomic system, with a given symmetry group, then an estimate on the number of weakly Noetherian constants of motion may have different reasons of interest. One of them, of course, is to confirm that all constants of motion of this type (including those of the simplest type—the horizontal gauge momenta) have been determined, or to motivate the search for more of them. This kind of information may be relevant in the study of a basic question in this field: which is—if it exists—the ultimate relationship between symmetries and conservation laws for nonholonomic systems?

It would be of interest, of course, to generalize and/or specialize this approach to other classes of constants of motion of nonholonomic systems. For instance, one might be interested to determine the number of conserved gauge momenta or, even more particularly, the number of horizontal gauge momenta, but new ideas may be necessary for this goal. The extension to more general cases (affine constraints, non–lifted actions) should instead be a rather standard matter.

Even though the method has been taylored to weakly Noetherian constants of motion of nonholonomic systems, it is in principle more general and introduces a new idea in the study of constants of motion of dynamical systems. As pointed out above, the method can be applied to cases in which conservation laws can be regarded as first integrals of some distribution, and this is a typical situation in the Hamiltonian and symplectic world. Just to point out another possible field of application, we thus mention here the search for conservation laws of Hamiltonian systems linked to non–symplectic actions. Non–symplectic actions do not have a momentum map.

However, they may have 'gauge–like' conserved quantities, in a sense which is made precise by the following Proposition:

Proposition 6. *Consider a (not necessarily symplectic) action of a Lie group G on a symplectic manifold P. For each $p \in P$, let $\mathcal{O}_p$ be the G–orbit through p.*

(i) Assume that a function $F : P \to \mathbb{R}$ is such that its Hamiltonian vector field X_F is tangent to the G–orbits, that is,

$$\ker dF(p) \supseteq (T_p\mathcal{O}_p)^\omega \qquad \forall\, p \in P\,. \tag{30}$$

Then, F is a constant of motion of all Hamiltonian systems on P with G–invariant Hamiltonian H.

(ii) Assume, moreover, that G acts freely and properly. Then, a function $F : P \to \mathbb{R}$ is a constant of motion of all Hamiltonian systems on P with G–invariant Hamiltonian H if and only if it satisfies (30).

Proof (i) First note that $X_F(p) \in T_p\mathcal{O}_p$, namely $\langle X_F(p)\rangle \subseteq T_p\mathcal{O}_p$, is equivalent to (30) because $\langle X_F(p)\rangle = (\ker dF(p))^\omega$. If H is G–invariant, then $\ker dH(p) \supseteq T_p\mathcal{O}_p$ and hence

$$\ker dF(p) \;=\; \langle X_F(p)\rangle^\omega \;\supseteq\; (T_p\mathcal{O}_p)^\omega \;\supseteq\; (\ker dH(p))^\omega \;=\; \langle X_H(p)\rangle\,,$$

so that $X_H(F) = 0$. (ii) If F is a constant of motion of all Hamiltonian systems with G–invariant Hamiltonian, then $\ker dF(p) \supseteq \bigcup_{G-\text{invariant } H}(\ker dH(p))^\omega$ and, as we have already remarked in the proof of Lemma 1, for a free and proper action this union equals $(T_p\mathcal{O}_p)^\omega$. ■

By analogy with the nonholonomic case, let us call here 'conserved gauge momentum' any function F which satisfies (30). Part (ii) of the Proposition characterizes conserved gauge momenta as first integrals of the distribution $(T_p\mathcal{O}_p)^\omega$. Without additional properties on the actions, such a distribution need not be integrable and the number of independent (local) conserved gauge momenta can be bounded by computing the corank of the involutive closure of this distribution.

(We note that the polar distribution $(T_p\mathcal{O}_p)^\omega$ is certainly integrable if the action is Hamiltonian, in the sense of [31]. The reason is that in that case there is a momentum map $J : T^*Q \to \mathfrak{g}^*$ with the property that $\ker dJ(p) = (T_p\mathcal{O}_p)^\omega$ and the components of the momentum map are (global) first integrals of $(T_p\mathcal{O}_p)^\omega$. The distribution $(T_p\mathcal{O}_p)^\omega$ is in fact integrable even if the action is only symplectic, because in that case all infinitesimal generators of the action are locally Hamiltonian vector fields [31]; however, these local Hamiltonians provide only local first integrals of $(T_p\mathcal{O}_p)^\omega$).

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Note added in proof. The examples of Section 6 illustrate how the number of weakly Noetherian constants of motion vary with the potential energy, while the kinetic energy is kept fixed. This number may as well depend on the kinetic energy. One of the referees suggested to investigate the case of the Chaplygin sleigh, see [7, 25, 10], where the kinetic energy depends on the position of the center or mass relative to the point of contact. We report here the results of this analysis. The configuration manifold is $Q = \mathbb{R}^2 \times S^1 \ni (x, y, \varphi)$, where (x, y) is the contact point and φ fixes the orientation of the sleigh. On Q, we consider the $\mathbb{R}^2$–action of the translations of the contact point;

its lift leaves in all cases the kinetic energy invariant. If the center of mass of the sleigh coincides with the contact point then the corank of Δ^∞ is one; in this case, the system is known to have the horizontal gauge momentum $p_x/\cos\varphi|_M$. Otherwise, Δ^∞ has full rank; therefore, there are no (even local) weakly Noetherian constant of motion.

References

[1] C. Agostinelli, *Nuova forma sintetica delle equazioni del moto di un sistema anolonomo ed esistenza di un integrale lineare nelle velocità.* Boll. Un. Mat. Ital. **11** (1956), 1–9.

[2] V.I. Arnold, *Mathematical Methods of Classical Mechanics.* Graduate Texts in Mathematics **60** (Springer-Verlag, New York, 1989).

[3] V.I. Arnol'd, V.V. Kozlov and A.I. Neishtadt, *Mathematical aspects of classical and celestial mechanics. Dynamical systems, III. Encyclopaedia Math. Sci.* **3** (Springer, Berlin, 1993).

[4] L. Bates and J. Śniatycki, *Nonholonomic reduction.* Rep. Math. Phys. **32** (1993), 99–115.

[5] L. Bates, H. Graumann and C. MacDonnell, *Examples of gauge conservation laws in nonholonomic systems.* Rep. Math. Phys. **37** (1996), 295–308.

[6] S. Benenti, *A general method for writing the dynamical equations of nonholonomic systems with ideal constraints.* Regul. Chaotic Dyn. **13** (2008), 283–315.

[7] A.M. Bloch, *Nonholonomic Mechanics and Controls.* Interdisciplinary Applied Mathematics **24**, Systems and Control. (Springer–Verlag, New York, 2003).

[8] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, R.M.Murray, *Nonholonomic mechanical systems with symmetry.* Arch. Rational Mech. Anal. **136** (1996), 21–99.

[9] A.V. Borisov and I.S. Mamaev, *On the history of the development of the nonholonomic dynamics.* Regul Chaotic Dyn. **7** (2002), 43-47.

[10] A.V. Borisov and I.S. Mamayev, *The dynamics of a Chaplygin sleigh.* Journal of applied mathematics and mechanics **73** (2009), 156-161.

[11] A.V. Borisov, I.S. Mamaev and A.A. Kilin, *Rolling of a ball on a surface. New integrals and hierarchy of dynamics.* Regul Chaotic Dyn. **7** (2002), 201-219.

[12] F. Bullo and A.D. Lewis, *Geometric control of mechanical systems. Modeling, analysis, and design for simple mechanical control systems.* Texts in Applied Mathematics **49** (Springer-Verlag, New York, 2005).

[13] F. Cantrijn, M. de Leon, D. Martín de Diego and J. Marrero, *Reduction of nonholonomic mechanical systems with symmetries.* Rep. Math. Phys. **42** (1998), 25-45.

[14] J. Cortés Monforte, *Geometric, control and numerical aspects of nonholonomic systems.* Lecture Notes in Mathematics **1793** (Springer-Verlag, Berlin, 2002).

[15] R. Cushman, H. Duistermaat and J. Śniatycki, *Geometry of Nonholonomically Constrained Systems.* To be published by World Scientific, Singapore.

[16] R. Cushman, D. Kemppainen, J. Śniatycki and L. Bates, *Geometry of nonholonomic constraints.* Proceedings of the XXVII Symposium on Mathematical Physics (Toruń, 1994). Rep. Math. Phys. **36** (1995), 275–286.

[17] R. Cushman and J. Śniatycki, *Non–holonomic reduction of symmetries, constraints, and integrability.* Regul. Chaotic Dyn. **12** (2007), 615–621.

[18] M. de Leon, J.C. Marrero and D.M. de Diego, *Mechanical systems with nonlinear constraints.* Int. J. Theor. Phys. **36** (1997), 979-995.

[19] J.J. Duistermaat, *Chaplygin's sphere.* `arXiv:math/0409019`, 2004.

[20] F. Fassò, A, Giacobbe, *Geometry of invariant tori of certain integrable systems with symmetry and an application to a nonholonomic system.* SIGMA Symmetry Integrability Geom. Methods Appl. **3** (2007), Paper 051, 12 pp. (electronic).

[21] F. Fassò, A. Giacobbe and N. Sansonetto, *Periodic flows, rank-two Poisson structures, and nonholonomic mechanics.* Regul. Chaotic Dyn. **10** (2005), 267–284.

[22] F. Fassò, A. Giacobbe and N. Sansonetto, *Gauge conservation laws and the momentum equation in nonholonomic mechanics.* Rep. Math. Phys. **62** (2008), 345-367.

[23] F. Fassò, A. Ramos and N. Sansonetto, *The reaction–annihilator distribution and the nonholonomic Noether theorem for lifted actions.* Regul. Chaotic Dyn. **12**, 579-588 (2007).

[24] F. Fassò and N. Sansonetto, *An elemental overview of the nonholonomic Noether theorem.* To appear in International Journal of Geometric Methods in Modern Physics **6** (2009).

[25] N. Fedorov and D.V. Zenkov, *Dynamics of the discrete Chaplygin sleigh.* .N. Fedorov and D.V. Zenkov, *Dynamics of the discrete Chaplygin sleigh.* Discrete Contin. Dyn. Syst. (2005, supplement volume), 258-267.

[26] J. Hermans, *A symmetric sphere rolling on a surface.* Nonlinearity **8** (1995), 493-515.

[27] Il. Iliev and Khr. Semerdzhiev, *Relations between the first integrals of a nonholonomic mechanical system and of the corresponding system freed of constraints.* J. Appl. Math. Mech. **36** (1972), 381-388.

[28] Il. Iliev, *On first integrals of a nonholonomic mechanical system.* J. Appl. Math. Mech. **39** (1975), 147-150.

[29] A. Isidori, *Nonlinear Control Systems: An Introduction.* Lectures Notes in Control and Information Sciences **72** (Springer–Verlag, Berlin, 1985).

[30] V.V. Kozlov and N.N. Kolesnikov, *On theorems of dynamics.* J. Appl. Math. Mech. **42** (1978), 26–31.

[31] P. Libermann and C.-M. Marle, *Symplectic geometry and analytical mechanics.* (D. Reidel, Dordrecht, 1987).

[32] C.-M. Marle, *Reduction of constrained mechanical systems and stability of relative equilibria.* Comm. Math. Phys. **174** (1995), 295–318.

[33] C.-M. Marle, *On symmetries and constants of motion in Hamiltonian systems with nonholonomic constraints.* In *Classical and quantum integrability (Warsaw, 2001)*, 223–242, Banach Center Publ. **59**, Polish Acad. Sci. (Warsaw, 2003).

[34] Ju.I. Neimark and N.A. Fufaev, *Dynamics of Nonholonomic Systems.* Translations of Mathematical Monographs **33** (AMS, Providence, 1972).

[35] J.P. Ortega and T.S Ratiu, *Momentum maps and Hamiltonian reduction.* Progress in Mathematics **222** (Birkhäuser, Boston, 2004).

[36] A. Ramos, Private communication.

[37] E.J. Routh, *Treatise on the Dynamics of a System of Rigid Bodies (Advanced Part)* (Dover, New York, 1955).

[38] N. Sansonetto, *First integrals in nonholonmic systems.* Ph.D. thesis, University of Padova (2006).

[39] J. Sniatycki, *Nonholonomic Noether theorem and reduction of symmetries.* Rep. Math. Phys. **42** (1998), 5–23.

[40] R.W. Weber, *Hamiltonian systems with constraints and their meaning in mechanics.* Arch. Rational Mech. Anal. **91** (1986), 309–335.

[41] Wolfram Research, Inc., *Mathematica, Version 7.0* (Champaign, IL, 2008).

[42] D.V. Zenkov, *The geometry of the Routh problem.* Journal of Nonlinear Science **5** (1995), 503-519.